EXAM IN OPTIMAL CONTROL

ROOM: U14

TIME: April 24, 2019, 14–18

COURSE: TSRT08, Optimal Control

PROVKOD: TEN1

DEPARTMENT: ISY

NUMBER OF EXERCISES: 4

NUMBER OF PAGES (including cover pages): 4

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VISITS: 15:30, 17:00

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APPROVED TOOLS: Formula sheet for the course, printed collections of formulas and tables, calculator.

SOLUTIONS: Linked from the course home page after the examination.

The exam can be inspected and checked out 2019–05–16 at 12.30–13.00 in room 2A:473, B-building, entrance 27, A-corridore to the left.

PRELIMINARY GRADING:	betyg 3	15 points
	betyg 4	23 points
	betyg 5	30 points

All solutions should be well motivated.

Good Luck!

1. (a) Find the control signal $u(t)$ which satisfies the optimal control problem

$$\begin{aligned} \underset{u(\cdot)}{\text{minimize}} \quad & \int_0^T \Big(x_2(t) + u^2(t)\Big)\, dt \\ \text{subject to} \quad & \dot{x}_1(t) = -x_1(t) + u(t), \\ & \dot{x}_2(t) = x_1(t) \\ & x_1(0) = a,\ x_2(0) = 0, \end{aligned}$$

for a fixed $T > 0$, using the PMP. (5p)

(b) Find the extremal to the functional

$$J(y) = \int_0^1 \Big(y^2(t) + 4\dot{y}^2(t)\Big)\, dt,$$

satisfying $y(0) = 1$ and $y(1) = 0$. (5p)

2. Consider the problem

$$\begin{aligned} \underset{u(\cdot)}{\text{minimize}} \quad & \frac{1}{2}\Big(x(T)\Big)^2 \\ \text{subject to} \quad & \dot{x}(t) = u(t), \\ & x(0) = x_0 \text{ given}, \\ & u(t) \in [-1, 1], \text{ for all } t \in \mathbb{R}. \end{aligned}$$

(a) Show by the use of PMP that the optimal controller is given by

$$\mu^*(t, x) = -\operatorname{sign}(x). \qquad (4p)$$

(b) Show that the cost-to-go function $V(t, x)$ that corresponds to the PMP solution above is given by

$$V(t, x) = J^*(t, x) = \frac{1}{2}\Big(\max\Big\{0, |x| - (T - t)\Big\}\Big)^2. \qquad (3p)$$

(c) Show that the cost-to-go function above satisfies the HJBE:

$$-\frac{\partial V}{\partial t}(t, x) = \min_{|u| \le 1}\Big\{f_0(t, x, u) + \frac{\partial V}{\partial x}(t, x)^T f(t, x, u)\Big\}, \quad V(T, x) = \phi(x),$$

for all (t, x). (3p)

3. Assume that we have a vessel whose maximum weight capacity is z and whose cargo is to consist of different quantities of N different items. Let v_k denote the value of the kth type of item, and let w_k denote the weight of the kth type of item.

   (a) Let x_k be the used weight capacity of the vessel after the first $k-1$ items have been loaded and let the control u_k be the quantity of item k to be loaded on the vessel. Formulate the dynamic equation

$$x_{k+1} = f(k, x_k, u_k),$$

   describing the process. (3p)

   (b) Determine the constraint set $U(k, x_k)$ on the control signal u_k. (3p)

   (c) Formulate a DP recursion that solves the problem of finding the most valuable cargo satisfying the maximal weight capacity. Observe that you do *not* need to solve the problem. (4p)

4. Consider the following problem

$$\begin{aligned} \underset{u(\cdot)}{\text{maximize}} \quad & \int_0^T e^{-\beta t}\sqrt{u(t)}\, dt \\ \text{subject to} \quad & \dot{x}(t) = \alpha x(t) - u(t), \\ & x(0) = x_0 > 0, \\ & x(t) \geq 0, \ \text{for all } t \in \mathbb{R}. \end{aligned}$$

   Determine a positive function $\gamma(t)$ such that the value function

$$V(t, x) \triangleq -e^{-\beta t}\sqrt{\gamma(t)x},$$

   satisfies the finite horizon HJBE:

$$-\frac{\partial V}{\partial t}(t, x) = \min_u \left\{ f_0(t, x, u) + \frac{\partial V}{\partial x}(t, x)^T f(t, x, u) \right\}, \quad V(T, x) = \phi(x)$$

   for all (t, x), via the following steps:

   (a) Show by minimizing the right hand side of the HJBE with respect to u that $\mu(t, x) = x/\gamma(t)$ is an optimal control candidate. (4p)

   (b) Determine $\gamma(t)$ so that $\mu(t, x)$ above satisfies the HJBE. (6p)

TSRT08: Optimal Control
Solutions

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1. (a) The Hamiltonian is given by

$$H(t,x,u,\lambda) = x_2 + u^2 + \lambda_1(-x_1 + u) + \lambda_2 x_1.$$

Pointwise minimization is obtained via

$$0 = \frac{\partial H}{\partial u}(t,x,u,\lambda) = 2u + \lambda_1 \quad \Rightarrow \quad u^* = -\frac{1}{2}\lambda_1,$$

since H is strictly convex in u. The adjoint equations are given by

$$\begin{aligned}
\dot{\lambda}_1(t) &= -\frac{\partial H}{\partial x_1}(t, x(t), \mu^*(t, x(t), \lambda(t)), \lambda(t)) = \lambda_1(t) - \lambda_2(t),\\
\dot{\lambda}_2(t) &= -\frac{\partial H}{\partial x_2}(t, x(t), \mu^*(t, x(t), \lambda(t)), \lambda(t)) = -1
\end{aligned}$$

with boundary conditions

$$\lambda(T) = \frac{\partial \phi}{\partial x}(T, x(T)) \quad \Longleftrightarrow \quad \lambda_1(T) = \lambda_2(T) = 0$$

Thus, we get $\lambda_2(t) = T - t$ and

$$\dot{\lambda}_1(t) = \lambda_1(t) + t - T, \quad \lambda_1(T) = 0,$$

which has the solution

$$\lambda_1(t) = -(t-T) - 1 + e^{t-T}$$

and the optimal control is

$$u^*(t) = -\frac{1}{2}\lambda_1(t) = \frac{1}{2}\big(1 + t - T - e^{t-T}\big).$$

(b) Introducing $x = y$ and $u = \dot{y}$, it holds that $\dot{x} = u$ and the Hamiltonian is given by

$$H(t,x,u,\lambda) = x^2 + 4u^2 + \lambda u.$$

The following equations must hold

$$\begin{aligned}
\dot{\lambda} &= -\frac{\partial H}{\partial x}(t,x,u,\lambda) = -2x,\\
0 &= \frac{\partial H}{\partial u}(t,x,u,\lambda) = 8u + \lambda.
\end{aligned}$$

The latter yields $\dot{\lambda} = -8\dot{u} = -8\ddot{y}$, which substituted into the first equation gives

$$-8\ddot{y} = -2x = -2y \quad \Leftrightarrow \quad \ddot{y} - \frac{1}{4}y = 0,$$

which has the solution

$$y(t) = c_1 e^{t/2} + c_2 e^{-t/2},$$

for some constants c_1 and c_2. The boundary constraints $y(0) = 1$ and $y(1) = 0$ yields

$$\begin{pmatrix} 1 & 1 \\ e^{1/2} & e^{-1/2} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

which has the solution

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{e-1} \begin{pmatrix} -1 \\ e \end{pmatrix}.$$

Thus,

$$y(t) = \frac{1}{e-1}(-e^{t/2} + e^{1-t/2}),$$

is the sought after extremal.

2. (a) The Hamiltonian is given by

$$H(t, x, u, \lambda) = \lambda u$$

Pointwise minimization yields

$$\mu(t, x) = \arg\min_{|u| \leq 1} \lambda u = \begin{cases} 1, & \lambda < 0 \\ -1, & \lambda > 0 \\ \tilde{u}, & \lambda = 0 \end{cases} = -\operatorname{sign}(\lambda),$$

where $\tilde{u} \in [-1, 1]$ is arbitrary. The adjoint equation is given by

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x}(t, x, u, \lambda) = 0, \quad \lambda(T) = \frac{\partial \phi}{\partial x}(x(T)) = x(T)$$

which has the solution $\lambda(t) = x(T)$. We now have to cases:

- $x(T) \neq 0$: In this case $\lambda(t) \neq 0$ for all t and we can write

$$\mu(t, x) = -\operatorname{sign}(\lambda) = -\operatorname{sign} x(T) = -\operatorname{sign} x$$

  The last equality holds since x will have the same sign as $x(T)$ during the whole state trajectory.
- $x(T) = 0$: In this case $\lambda = 0$ for all t and we may use any control signal $\tilde{u} \in [-1, 1]$, which obeys the constraint $x(T) = 0$. One such control signal is

$$\mu(t, x) = -\operatorname{sign} x$$

  since this will drive x to zero and stay there.

Consequently, one optimal control is

$$\mu^*(t, x) = -\operatorname{sign}(x).$$

(b) Since $J^*(t, x) = \frac{1}{2}(x^*(T))^2$, we need to find $x^*(T)$. It holds that

$$x(T) - x(t) = \int_t^T \dot{x}(\tau)\, d\tau = \int_t^T u(\tau)\, d\tau,$$

which can be written as

$$x(T) = x(t) - \int_t^T \operatorname{sign}\{x(\tau)\}\, d\tau. \tag{1}$$

There are two cases:

- $x(t) > 0$: In this case the controller will decrease $x(t)$ until, if possible, $x(T) = 0$. Thus, it holds that

$$\begin{aligned} x(T) &= \max\{0, \overbrace{x(t) - (T-t)}^{\text{from (1)}}\} \\ &= \max\{0, |x(t)| - (T-t)\}. \end{aligned}$$

- $x(t) < 0$: In this case the controller will increase $x(t)$ until, if possible, $x(T) = 0$. Thus, it holds that

$$\begin{aligned} x(T) &= \min\{0, \overbrace{x(t) + (T-t)}^{\text{from (1)}}\} = -\max\{0, -x(t) - (T-t)\} \\ &= -\max\{0, |x(t)| - (T-t)\} \end{aligned}$$

Thus, the only difference between the two cases are the sign in front of the max and the optimal cost-to-go function becomes

$$V(t,x) = J^*(t,x) = \frac{1}{2}(x^*(T))^2 = \frac{1}{2}\Big(\max\big\{0, |x| - (T-t)\big\}\Big)^2.$$

(c) The function $V(t,x)$ is differentiable and it holds that

$$\begin{aligned} \frac{\partial V}{\partial t}(t,x) &= \max\big\{0, |x| - (T-t)\big\}, \\ \frac{\partial V}{\partial x}(t,x) &= \text{sign}\,(x) \cdot \max\big\{0, |x| - (T-t)\big\}. \end{aligned}$$

Substituting the above into the HJBE yields

$$-\max\big\{0, |x| - (T-t)\big\} = \min_{|u|\le 1}\big\{\text{sign}\,(x)\cdot u\big\}\max\big\{0, |x| - (T-t)\big\},$$

which can be seen to hold as an identity for all (t,x).

3. (a) With $x_1 = 0$, the dynamic equation may be written as

$$x_{k+1} = x_k + w_k u_k.$$

(b) Since for each item k, we must have $x_k \le z$ for all k and especially

$$x_{k+1} = x_k + w_k u_k \le z \quad \Longleftrightarrow \quad u_k \le (z - x_k)/w_k.$$

Furthermore, it is only possible to ship integer quantities which implies that $u \in \{0, 1, 2, \ldots\}$.

(c) The reward function is given by $f_0(k, x_k, u_k) = v_k u_k$, which yields the DP algorithm

$$\begin{aligned} J(N+1, x) &= 0 \\ J(n, x) &= \max_{\substack{0 \le u_n \le (z - x_n)/w_n \\ u_n \in \{0,1,2,\ldots\}}} \Big\{v_n u_n + J(n+1, x_n + w_n u_n)\Big\}, \quad n = N, N-1, \ldots, 1. \end{aligned}$$

4. It holds that

$$\begin{aligned} \frac{\partial V}{\partial t}(t,x) &= \beta e^{-\beta t}\sqrt{\gamma(t)x} - \frac{1}{2}\frac{\dot{\gamma}(t)x}{\sqrt{\gamma(t)x}}e^{-\beta t} = \frac{xe^{-\beta t}}{2\sqrt{\gamma(t)x}}\Big(2\beta\gamma(t) - \dot{\gamma}(t)\Big) \\ \frac{\partial V}{\partial x}(t,x) &= \frac{1}{2}\frac{\gamma(t)}{\sqrt{\gamma(t)x}}e^{-\beta t}. \end{aligned}$$

Thus,

$$H(t,x,u,\lambda) = f_0(t,x,u) + \frac{\partial V}{\partial x}(t,x)^T f(t,x,u) = -e^{-\beta t}\sqrt{u} + \frac{1}{2}\frac{\gamma(t)}{\sqrt{\gamma(t)x}}e^{-\beta t}(\alpha x - u),$$

which has an extremum at

$$\frac{\partial H}{\partial u}(t,x,u,\lambda) = \frac{e^{-\beta t}}{2}\bigg(-\frac{1}{\sqrt{u}} + \frac{\gamma(t)}{\sqrt{\gamma(t)x}} \bigg) = 0 \quad \Longrightarrow \quad \mu^*(t,x) = \frac{x}{\gamma(t)}.$$

Since

$$\frac{\partial^2 H}{\partial u^2}(t,x,\mu^*(t,x),\lambda) = \frac{e^{-\beta t}}{4(x/\gamma(t))^{3/2}} > 0,$$

it also constitutes a minimum. The Hamiltonian is then given by

$$H(t,x,\mu^*(t,x),\lambda) = \frac{xe^{-\beta t}}{2\sqrt{\gamma(t)x}}\Big(-1 - \alpha\gamma(t)\Big).$$

Finally, for the HJBE to hold, we must have

$$-2\beta\gamma(t) + \dot\gamma(t) = -1 - \alpha\gamma(t) \quad \Longleftrightarrow \quad \dot\gamma(t) + (\alpha - 2\beta)\gamma(t) = -1$$

which has the solution

$$\gamma(t) = \frac{-1}{\alpha - 2\beta} + c_1 e^{-(\alpha-2\beta)t},$$

for some constant c_1. The boundary constraint $V(T,x) = 0$ can be restated as $\gamma(T) = 0$, which implies that

$$c_1 = \frac{1}{\alpha - 2\beta} e^{(\alpha-2\beta)T}.$$

Thus, the function $\gamma(t)$ that makes sure that $V(t,x)$ satisfies the HJBE is given by

$$\gamma(t) = \frac{e^{-(t-T)(\alpha-2\beta)} - 1}{\alpha - 2\beta}.$$