EXAM IN OPTIMAL CONTROL

ROOM: R36, R37

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COURSE: TSRT08, Optimal Control

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APPROVED TOOLS: Formula sheet for the course, printed collections of formulas and tables, calculator.

SOLUTIONS: Linked from the course home page after the examination.

The exam can be inspected and checked out 2019-02-7 at 12.30-13.00 in Ljungelng.

PRELIMINARY GRADING:	betyg 3	15 points
	betyg 4	23 points
	betyg 5	30 points

All solutions should be well motivated.

Good Luck!

1. We are interested in computing optimal transportation routes in a circular city. The cost for transportation per unit length is given by a function $g(r)$ that only depends on the radial distance r to the city center. This means that the total cost for transportation from a point P_1 to a point P_2 is given by

$$\int_{P_1}^{P_2} g(r)ds$$

where s represents the arc length along the path of integration. In polar coordinates (θ, r) the total cost reads

$$\int_{P_1}^{P_2} g(r)\sqrt{1+(r\dot{\theta})^2}dr$$

where $\theta = \theta(r)$, and $\dot{\theta} = d\theta/dr$.

(a) Formulate the problem of computing an optimal path as an optimal control problem (2p)

(b) For the case of $g(r) = \alpha/r$ for some positive α show that any optimal path satisfies the equation $\theta = a \log r + b$ for some constants a and b. (5p)

(c) Show that if the initial point and the final point are at the same distance from the origin, then the optimal path is a circle segment. You may use the claim in (b). (3p)

2. Consider the following optimal control problems:

$$\begin{aligned} \underset{u(\cdot)}{\text{minimize}} \quad & \int_0^{t_f} \left(x^2(t)+u^2(t)\right) dt \\ \text{subject to} \quad & \dot{x}(t) = x(t)+u(t), \\ & x(0) = 1 \end{aligned} \tag{1}$$

$$\begin{aligned} \underset{u(\cdot)}{\text{minimize}} \quad & \int_0^{t_f} \left(x(t)+u^2(t)\right) dt \\ \text{subject to} \quad & \dot{x}(t) = x(t)+u(t)+1, \\ & x(0) = 0 \end{aligned} \tag{2}$$

$$\begin{aligned} \underset{u(\cdot)}{\text{minimize}} \quad & \int_0^{t_f} \left(x(t)+u^2(t)\right) dt \\ \text{subject to} \quad & \dot{x}(t) = x(t)+u(t)+1, \\ & x(0) = 0 \\ & x(t_f) = 1. \end{aligned} \tag{3}$$

(a) Suppose you must solve these problems numerically. Describe advantages and disadvantages of (A) the discretization method (constrained nonlinear program), (B) the shooting method (boundary condition iteration), and (C) the gradient method (first order gradient search of the cost function) for solving these three optimal control problems. (5p)

(b) Make comments on if and how the problem (1) can be solved by using HJBE. Note that you do not necessarily have to solve the problems, but your statements must be well motivated. (2p)

(c) Make comments on if and how the problem (3) can be solved by using PMP. Note that you do not necessarily have to solve the problems, but your statements must be well motivated. (3p)

3. Consider a constant-power rocket

$$\begin{aligned}\dot{x}_1 &= u \\ \dot{x}_2 &= u^2\end{aligned}$$

where x_1 is velocity and x_2 is inversely proportional to the mass of the rocket, while u is the acceleration caused by the thrust. We assume that $|u| \leq 1$. Given the initial condition $x(0) = x_0$ it is desirable to minimize the final time t_f, such that $x(t_f) = x_f$, where x_f is given.

(a) Formulate necessary conditions for optimality using PMP. (2p)

(b) Show that the candidates for the optimal control signal u using PMP are $u^\star(t) = c \; \forall t$ for some constant $c \in [-1, 1]$, and $u^\star(t) \in \{-1, 1\}$, i.e. bang-bang control. (4p)

(c) Compute the regions for the initial value x_0 for which the candidates in (b) are feasible with respect to the final value x_f. Notice that you need to compute the regions for all different values of x_f. (4p)

4. We consider a discrete-time hidden Markov model with state $x(k)$ and output $y(k)$ at time k. We define the probability

$$p(\bar{x}) = P\{x(0) = x_0, \ldots, x(N-1) = x_{N-1}, y(0) = y_0, \ldots, y(N-1) = y_{N-1}\}$$

where $\bar{x} = (x_0, \ldots, x_{N-1}) \in X^N$, $\bar{y} = (y_0, \ldots, y_{N-1}) \in Y^N$, and where X and Y are some sets to be defined in more detail later. We assume that the sequence of observations $\bar{y}$ is given, and we are interested in estimating the sate sequence $\bar{x}$ based on the observation of $\bar{y}$. We are going to do this by maximizing $p(\bar{x})$, which is known as maximum likelihood estimation. A key feature of a hidden Markov model is that it satisfies what is called the Markov property, i.e.

$$\begin{aligned} p(\bar{x}) = {} & P\{y(N-1) = y_{N-1} | x(N-1) = x_{N-1}\} \\ & \times P\{x(N-1) = x_{N-1} | x(N-2) = x_{N-2}\} \\ & \times P\{y(N-2) = y_{N-2} | x(N-2) = x_{N-2}\} \\ & \times P\{x(N-2) = x_{N-2} | x(N-3) = x_{N-3}\} \\ & \vdots \\ & \times P\{y(1) = y_1 | x(1) = x_1\} P\{x(1) = x_1 | x(0) = x_0\} \\ & \times P\{y(0) = y_0 | x(0) = x_0\} P\{x(0) = x_0\} \end{aligned}$$

(a) Let $V(0, x) = P\{y(0) = y_0 | x(0) = x\} P\{x(0) = x\}$ and define the recursion

$$V(k, x) = P\{y(k) = y_k | x(k) = x\} \max_{u \in X} P\{x(k) = x | x(k-1) = u\} V(k-1, u)$$

for $k = 1, \ldots, N-1$, and let $V(N) = \max_{x \in X} V(N-1, x)$. Show that

$$\max_{\bar{x} \in X^N} p(\bar{x}) = V(N)$$

and that the optimal $\bar{x}$ is such that x_{k-1} is the maximizing u in iteration k above. *Hint:* It is a good idea to consider $\log p(\bar{x})$. (5p)

(b) Consider a primitive clinic in a village. People in the village have the property that they are either healthy or have a fever. They can only tell if they have a fever by asking the doctor in the clinic. The doctor makes a diagnosis of fever by asking patients how they feel. Villagers only answer that they feel normal, dizzy or cold. This defines a hidden Markov model with $X = \{\alpha_1, \alpha_2\}$ and $Y = \{\beta_1, \beta_2, \beta_3\}$, where α_1 =healthy, α_2 =fever, β_1 =normal, β_2 =cold, and β_3 =dizzy. Introduce the notation:

$$\begin{aligned} a_{ij} &= P\{x(k+1) = \alpha_j | x(k) = \alpha_i\} \\ b_{ij} &= P\{y(k) = \beta_j | x(k) = \alpha_i\} \end{aligned}$$

Also let the matrices A and B be defined such that element (i, j) of the matrix is equal to a_{ij} and b_{ij}, respectively. Consider the case when

$$A = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}; \quad B = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.1 & 0.3 & 0.6 \end{bmatrix}$$

and assume that $P\{x(0) = \alpha_1\} = 0.6$ and $P\{x(0) = \alpha_2\} = 0.4$. The doctor has for a patient observed the first day normal, the second day cold, and the third day dizzy. What is the most likely value of the condition for the patient, i.e. the most likely value of $\bar{x}$? Use the recursion in (a). (5p)

TSRT08: Optimal Control
Solutions

20190118

1. (a) By introducing the control $u = \dot{\theta}$, we can specify the optimal control problem

$$
\begin{aligned}
\underset{u}{\text{minimize}} \quad & \int_{r_1}^{r_2} g(r)\sqrt{1+(r\cdot u(r))^2}dr \\
\text{subject to} \quad & \dot{\theta} = u \\
& \theta(r_1) = \theta_1 \\
& \theta(r_2) = \theta_2
\end{aligned}
$$

where r_1, θ_1 and r_2, θ_2 are the polar coordinates of the points P_1 and P_2, respectively.

(b) With $g(r) = \alpha/r$, the Hamiltonian is given by

$$H(r,\theta,u,\lambda) = \frac{\alpha}{r}\sqrt{1+(r\cdot u)^2} + \lambda\cdot u$$

Further we have that

$$
\begin{aligned}
\frac{\partial H}{\partial u}(r,\theta,u,\lambda) &= \alpha\frac{r\cdot u}{\sqrt{1+(r\cdot u)^2}} + \lambda \\
\frac{\partial^2 H}{\partial u^2}(r,\theta,u,\lambda) &= \alpha\frac{r}{(1+(r\cdot u)^2)^{3/2}}
\end{aligned}
$$

Since $\alpha > 0$ and $r > 0$ we have that $\frac{\partial^2 H}{\partial u^2} > 0\ \forall u$. Hence, $H(r,\theta,u,\lambda)$ is strictly convex in u. Therefor, pointwise minimization yields

$$0 = \frac{\partial H}{\partial u}(r,\theta^*(r),u^*(r),\lambda) = \alpha\frac{r\cdot u^*(r)}{\sqrt{1+(r\cdot u^*(r))^2}} + \lambda(r)$$

The adjoint equation is given by

$$\dot{\lambda} = -\frac{\partial H}{\partial \theta}(r,\theta,u,\lambda) = 0$$

without final constraint on $\lambda(r_2)$ since we do have a final constraint on $\theta(r_2)$. This equation has the solution

$$\lambda(r) = c$$

for some constant c and the optimal control is

$$\alpha\frac{r\cdot u^*(r)}{\sqrt{1+(r\cdot u^*(r))^2}} = -c$$

This requires $r\cdot u^*(r)$ to be constant, which can be written as

$$ru^*(r) = a \quad \Rightarrow \quad u^*(r) = \frac{a}{r}$$

for some constant a. This gives the optimal path

$$\dot{\theta} = \frac{a}{r} \quad \Rightarrow \quad \theta = a\log r + b$$

which we were supposed to show.

(c) Reformulate the optimal path as a function of theta

$$r(\theta) = e^{\frac{\theta - b}{a}} = Be^{A\theta}$$

where $A = 1/a$ and $B = e^{-b/a}$. We now require that the initial and final point shall have the same radius r_0, such that $r(\theta_1) = r_0$ and $r(\theta_2) = r_0$. This gives

$$r_0 = Be^{A\theta_1}, \quad r_0 = Be^{A\theta_2} \quad \Rightarrow \quad A = 0, \quad B = r_0$$

which gives

$$r(\theta) = r_0, \quad \text{for} \quad \theta_1 \leq \theta \leq \theta_2$$

This corresponds to a path with constant radius, i.e. a circle segment.

2. (a)
   - The discretization method is straightforward to apply to all problems. There exist many good algorithms for nonlinear optimization. Drawbacks are the large number of variables and constraints, and that the solution may not converge to the solution of the original problem.
   - A shooting method is straightforward to apply to all problems, but it is crucial to find a good initial guess of $\lambda(0)$. The transition matrix may sometimes be ill conditioned when using a shooting method, but that is a minor problem for these quite simple problems.
   - A gradient method is straightforward to apply to (1) and (2), but (3) requires a slightly more complex gradient algorithm due to the the terminal constraint. Convergence tends to be slow for the gradient methods, but this is a minor problem for these quite simple problems.

   (b) Problem (1) is a linear-quadratic problem that is possible to solve analytically with HJBE. Use $V(t,x) = P(t)x^2$, where $P(t)$ is a positive function that can be obtained by solving the Riccati equation. The optimal feedback law is $\mu(t,x) = -P(t)x$.

   (c) The problem can be solved by defining the Hamiltonian, minimize with respect to u and solve the adjoint equations. Since S_f is a set with just one point then there is no constraint on $\lambda(t_f)$. Therefor, a unknown constant will remain in the equation for λ and hence also for u. By substituting the control signal in the dynamic model with this control law we obtain a linear ODE of order one that is straightforward to solve and by using the initial and final state constraints all constants can be found.

3. (a) The optimal control problem can be stated as

$$\begin{aligned} \text{minimize} \quad & \int_0^{t_f} 1dt \\ \text{subject to} \quad & \dot{x_1} = u \\ & \dot{x}_2 = u^2 \\ & x(0) = x_0 \\ & x(t_f) = x_f \end{aligned}$$

The Hamiltonian is given by

$$H(t, x, u, \lambda) = \lambda_0 + \lambda_1 u + \lambda_2 u^2$$

Pointwise minimization yields

$$\mu(t, x, \lambda) = \arg\min_{|u| \leq 1} H(t, x, u, \lambda) = \arg\min_{|u| \leq 1} \lambda_1 u + \lambda_2 u^2$$

and a candidate for optimal control is

$$u^*(t) = \mu(t, x(t), \lambda(t))$$

The adjoint equations are given by

$$\dot{\lambda}_1 = 0, \quad \dot{\lambda}_2 = 0$$

Since we have $x(t_f) = x_f$, we do not have any boundary conditions on $\lambda(t_f)$.
Finally, since the system is autonomous, we also have the constraint

$$H(t, x^*(t), u^*(t), \lambda(t)) = 0 \quad \Rightarrow \quad \lambda_0 + \lambda_1(t) u^*(t) + \lambda_2 {u^*}^2(t) = 0 \tag{1}$$

(b) The solution of the adjoint equations is given by

$$\lambda_1(t) = c_1, \quad \lambda_2(t) = c_2$$

for some constants c_1 and c_2. We now consider four different cases

Case	Condition	Extremal control
(i)	$c_2 > 0$	$u^*(t) = c,\ c \in [-1, 1]$
(ii)	$c_1 \neq 0,\ c_2 \leq 0$	$u^*(t) = c,\ c = -1$ or 1
(iii)	$c_1 = 0,\ c_2 < 0$	$u^*(t) = -1$ or 1
(iv)	$c_1 = 0,\ c_2 = 0$	Not feasible

Case (i), $c_2 > 0$: In this case the optimal value is either at the u where $\frac{\partial H}{\partial u} = 0$ if feasible, or at any of the two boundaries -1 or 1. We have that

$$0 = \frac{\partial H}{\partial u} = c_1 + 2c_2 u \quad \Rightarrow \quad u = \frac{-c_1}{2c_2}$$

Further, since

$$0 = \frac{\partial^2 H}{\partial u^2} = 2c_2 > 0$$

this is indeed a minimum point. This gives the unique minimizing argument

$$\mu(t, x, \lambda) = \arg\min_{|u| \leq 1} c_1 u + c_2 u^2 = \begin{cases} -1, & \frac{-c_1}{2c_2} < -1 \\ \frac{-c_1}{2c_2}, & -1 \leq \frac{-c_1}{2c_2} \leq 1 \\ 1, & \frac{-c_1}{2c_2} > 1 \end{cases}$$

Case (ii), $c_1 \neq 0$, $c_2 \leq 0$: In this case either -1 or 1 will be optimal depending on the sign of c_1. We know that $\frac{\partial H}{\partial u} = 0$ is not a candidate since $\frac{\partial^2 H}{\partial u} = 2c_2 \leq 0$ and hence does not correspond to a minimum point. This gives the unique minimizing argument

$$\mu(t, x, \lambda) = \arg\min_{|u| \leq 1} c_1 u + c_2 u^2 = \begin{cases} -1 & \text{if} \quad c_1 > 0 \\ 1 & \text{if} \quad c_1 < 0 \end{cases}$$

Case (iii), $c_1 = 0$, $c_2 < 0$: In this case we do not have a unique minimizing argument, since both -1, 1 achieve the same minimal value

$$\mu(t, x, \lambda) = \arg\min_{|u| \leq 1} c_2 u^2 = -1 \text{ or } 1$$

Case (iv), $c_1 = 0$, $c_2 = 0$: With $c_1 = c_2 = 0$, eq. (1) gives that $\lambda_0 = 0$. However, $\tilde{\lambda} = [\lambda_1,\ \lambda_2,\ \lambda_3]^T$ has to be non-zero. Hence, this solution is not feasible.

To summarize, all the feasible cases can be combined into just two cases, where Case (A) corresponds to Case (i) and (ii), and Case (B) corresponds to Case (iii)

$$\begin{aligned} &u^*(t) = c, \quad c \in [-1,\ 1] && \text{Case (A)} \\ &u^*(t) = -1 \text{ or } 1 && \text{Case (B)} \end{aligned}$$

For Case (A), $u^*(t)$ will be constant, and for Case (B) we will have a bang-bang controller which can switch between -1 and 1 an arbitrary number of times.

(c) We consider the two cases that we derived in the previous question

Case (A) The trajectory with the constant control $u^*(t) = c$ will be

$$\begin{aligned} x_1(t) &= ct + x_{1,0} \\ x_2(t) &= c^2 t + x_{2,0} \end{aligned}$$

Together with the final constraint $x(t_f) = x_f$, this gives

$$x_1(t_f) = ct_f + x_{1,0} = x_{1,f}$$
$$x_2(t_f) = c^2 t_f + x_{2,0} = x_{2,f}$$

By solving for c and t_f we get

$$c = \frac{x_{2,f} - x_{2,0}}{x_{1,f} - x_{1,0}}, \qquad t_f = \frac{(x_{1,f} - x_{1,0})^2}{x_{2,f} - x_{2,0}}$$

Since we require that $|c| \le 1$ and $t_f > 0$, this control is feasible if

$$x_{2,f} - x_{2,0} > 0, \quad \text{and} \quad |x_{1,f} - x_{1,0}| \ge |x_{2,f} - x_{2,0}|$$

Case (B) The trajectory with the bang-bang control where $u^*(t) = -1$ or 1 will be

$$x_1(t) = I(t) + x_{1,0}$$
$$x_2(t) = t + x_{2,0}$$

where

$$I(t) = \int_0^t u^*(s)ds \quad \Rightarrow \quad |I(t)| \le \int_0^t |u^*(s)|ds \int_0^t 1ds = t$$

which is fulfilled with equality if $u^*(t)$ is constant (zero switches). Together with the final constraint $x(t_f) = x_f$, this gives

$$x_1(t_f) = I(t_f) + x_{1,0} = x_{1,f}$$
$$x_2(t_f) = t_f + x_{2,0} = x_{2,f}$$

By solving for $I(t_f)$ and t_f we get

$$I(t_f) = x_{1,f} - x_{1,0}, \qquad t_f = x_{2,f} - x_{2,0}$$

Since we require that $|I(t_f)| \le t_f$ and $t_f > 0$, this control is feasible if

$$x_{2,f} - x_{2,0} > 0, \quad \text{and} \quad |x_{1,f} - x_{1,0}| \le |x_{2,f} - x_{2,0}|$$

For this case, the candidate for optimal control is not unique. Any feasible bang-bang control is a valid candidate, see Figure 1.

To summarize, a candidate for optimal control exist only if $x_{2,f} - x_{2,0} > 0$. Further if $|x_{1,f} - x_{1,0}| \ge |x_{2,f} - x_{2,0}|$ a constant control $u^*(t) = \frac{x_{2,f} - x_{2,0}}{x_{1,f} - x_{1,0}}$ is a valid candidate for optimal control, otherwise any feasible bang-bang is a valid candidate, see Figure 1.

4. (a) In order to formulate the maximum likelihood estimation problem as an discrete-time optimal control problem, we choose to maximize $\log p(\bar{x})$ instead of maximizing $p(\bar{x})$, which we can do since $\log(\cdot)$ is a strictly increasing function. From the question description we have

$$\log p(\bar{x}) = \log P\{y(0) = y_0 | x(0) = x_0\} + \log P\{x(0) = x_0\} + \sum_{k=1}^{N-1} \log P\{y(k) = y_k | x(k) = x_k\} + \log P\{x(k) = x_k | x(k-1) = x_{k-1}\}$$

By introducing

$$\begin{aligned} J(\bar{x}) &= \log p(\bar{x}) \\ u_k &= x_{k-1} \\ \bar{u} &= (u_1, \dots u_N) = \bar{x} \\ \phi(x) &= \log P\{y(0) = y_0 | x(0) = x\} + \log P\{x(0) = x\} \\ f_0(k, x, u) &= \log P\{y(k) = y_k | x(k) = x\} + \log P\{x(k) = x | x(k-1) = u\} \\ f(k, x, u) &= u \end{aligned}$$

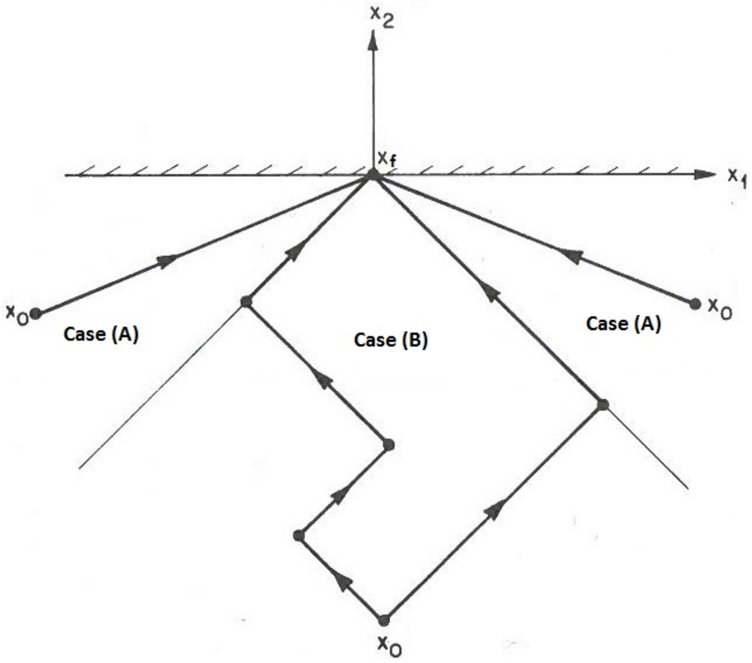

Figure 1: Initial state regions

we can formulate the maximization problem as the following optimal control problem

$$\begin{aligned} \underset{\bar{u}}{\text{maximize}} \quad & J(\bar{u}) = \phi(x_0) + \sum_{k=1}^{N-1} f_0(k, x_k, u_k) \\ \text{subject to} \quad & x_{k-1} = f(k, x_k, u_k) \end{aligned}$$

The solution to the problem can be found with dynamic programming. Since the dynamics is running backwards in time, the dynamics programming recursions will run in reverse order.

$$\begin{aligned} J(0, x) &= \phi(x) \\ J(k, x) &= \max_{u \in X} \{ f_0(k, x, u) + J(k-1, f(k, x, u)) \}, \quad k = 1, \ldots, N-1 \end{aligned}$$

At stage $N-1$ we have optimized for all $u_1, \ldots, u_{N-1}$ except u_N. We then have

$$J(N-1, x_{N-1}) = \max_{(u_1, \ldots, u_{N-1}) \in X^{N-1}} J(\bar{u})$$

where $x_{N-1} = u_N$. Hence

$$\max_{\bar{x} \in X^N} J(\bar{x}) = \max_{\bar{u} \in X^N} J(\bar{u}) = \max_x J(N-1, x)$$

By identifying

$$\log V(k, x) = J(k, x) \quad \text{and} \quad \log V(\bar{x}) = J(\bar{x})$$

we get the following recursions

$$\begin{aligned} V(0, x) &= P\{y(0) = y_0 | x(0) = x\} P\{x(0) = x\} \\ V(k, x) &= P\{y(k) = y_k | x(k) = x\} \max_{u \in X} \big[P\{x(k) = x | x(k-1) = u\} V(k-1, u) \big], \quad k = 1, \ldots, N-1 \end{aligned}$$

$$\max_{\bar{x} \in X^N} V(\bar{x}) = \max_{x \in X} V(N-1, x)$$

where the optimal solution is given by

$$x_{N-1}^* = \arg\max_{u \in X} V(N-1, x)$$
$$x_{k-1}^* = u_k^* = \mu(k, x_k^*), \quad \text{if} \quad k < N$$

with

$$\mu(k, x) = \arg\max_{u \in X} \big[P\{x(k) = x | x(k-1) = u\} V(k-1, u)\big]$$

(b) We have the optimization problem on the form from the previous question with $N = 3$.

Stage $k = 0$: The first day, the doctor is making the observation $y_0 = \beta_1$, which gives

$$\begin{aligned} V(0, \alpha_i) &= P\{y(0) = \beta_1 | x(0) = \alpha_i\} P\{x(0) = \alpha_i\} \\ &= b_{i1} P\{x(0) = \alpha_i\} \end{aligned}$$

This gives

$$V(0, \alpha_1) = b_{11} * 0.6 = 0.5 * 0.6 = 0.3$$
$$V(0, \alpha_2) = b_{21} * 0.4 = 0.1 * 0.4 = 0.04$$

x	$V(0, x)$	u^*
α_1	0.3	-
α_2	0.04	-

Stage $k = 1$: The second day, the doctor is making the observation $y_1 = \beta_2$, which gives

$$\begin{aligned} V(1, \alpha_i) &= P\{y(1) = \beta_2 | x(1) = \alpha_i\} \max_{u \in \{\alpha_i, \alpha_2\}} \big[P\{x(1) = \alpha_i | x(0) = u\} V(0, u)\big] \\ &= b_{i2} \max_{i \in \{1,2\}} \{a_{1i} V(0, \alpha_1), a_{2i} V(0, \alpha_2)\} \end{aligned}$$

This gives

$$V(1, \alpha_1) = b_{12} * \max\{a_{11} V(0, \alpha_1), a_{21} V(0, \alpha_2)\} = 0.4 * \max\{0.7 * 0.3, 0.4 * 0.04\} = 0.4 * 0.7 * 0.3 = 0.084$$
$$V(1, \alpha_2) = b_{22} * \max\{a_{12} V(0, \alpha_1), a_{22} V(0, \alpha_2)\} = 0.3 * \max\{0.3 * 0.3, 0.6 * 0.04\} = 0.3 * 0.3 * 0.3 = 0.027$$

x	$V(0, x)$	u^*
α_1	0.084	α_1
α_2	0.027	α_1

Stage $k = 2$: The third day, the doctor is making the observation $y_2 = \beta_3$. In a similar fashion as in the previous step we get

$$V(2, \alpha_i) = b_{i3} \max\{a_{1i} V(0, \alpha_1), a_{2i} V(0, \alpha_2)\}$$

This gives

$$V(2, \alpha_1) = b_{13} * \max\{a_{11} V(1, \alpha_1), a_{21} V(2, \alpha_2)\} = 0.1 * \max\{0.7 * 0.084, 0.4 * 0.027\} = 0.1 * 0.7 * 0.084$$
$$V(2, \alpha_2) = b_{23} * \max\{a_{12} V(1, \alpha_1), a_{22} V(2, \alpha_2)\} = 0.6 * \max\{0.3 * 0.084, 0.6 * 0.027\} = 0.6 * 0.3 * 0.084$$

x	$V(2, x)$	u^*
α_1	0.07*0.084	α_1
α_2	0.18*0.084	α_1

Since $V(2, \alpha_2) > V(2, \alpha_1)$ we have that

$$x_2^* = \underset{x \in \{\alpha_1, \alpha_2\}}{\arg\max} \; V(2, x) = \alpha_2$$

To summarize, the system evolves as follows:

k	x_k^*	u_k^*	$V(k, x_k^*)$
2	α_2	α_1	0.18*0.084
1	α_1	α_1	0.084
0	α_1	-	0.3

Thus, the most likely condition for the patient is that he/she was healthy the first and second day, $x_0^* = x_1^* = \alpha_1$, and had fever the third day, $x_2^* = \alpha_2$.