EXAM IN OPTIMAL CONTROL

ROOM: G35

TIME: April 4, 2018, 14–18

COURSE: TSRT08, Optimal Control

PROVKOD: TEN1

DEPARTMENT: ISY

NUMBER OF EXERCISES: 4

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APPROVED TOOLS: Formula sheet for the course, printed collections of formulas and tables, calculator.

SOLUTIONS: Linked from the course home page after the examination.

The exam can be inspected and checked out 2018-04-18 at 12.30-13.00 in Room 2A:573, B-building, entrance 25, A-corridor to the right.

PRELIMINARY GRADING:	betyg 3	15 points
	betyg 4	23 points
	betyg 5	30 points

All solutions should be well motivated.

Good Luck!

1. (a) One wishes to find a control signal u on the interval $0 \le t \le 1$ that takes the system

$$\dot{x} = u$$

from $x(0) = 1$ to $x(1) = 0$ and minimizes the criterion

$$\int_0^1 \frac{u^2}{2} dt$$

Show that the necessary conditions for optimality are satisfied by a constant u-value. Compute this constant value. (5p)

(b) The system

$$\dot{x} = -x + u, \quad |u| \le 1$$

is to be controlled so that $x(1) = 0$ and the criterion

$$J = \int_0^1 |u| \, dt$$

minimized. A possible control is

$$u(t) = \begin{cases} 0 & 0 \le t < 0.5 \\ -1 & 0.5 \le t \le 1 \end{cases}$$

Show that this control satisfies the necessary conditions for optimality for some value of $x(0)$ (It is not necessary to specify the exact value of $x(0)$.) (5p)

2. A businessman operates out of a van that he sets up in one of two locations on each day. If he operates in location i (where $i = 1, 2$) on day k he makes a known and predictable profit denoted r_i^k. However, each time he moves from one location to the other, he pays a setup cost c. The businessman wants to maximize his total profit over N days.

(a) The problem can be formulated as a shortest path problem (SPP) where the node (i, k) represents location i at day k. Let s and e be the start node and the end node, respectively. Further, denote $\bar{i}$ as the location that is not equal to i, i.e. $\bar{1} = 2$ and $\bar{2} = 1$. The costs of all edges are:

- s to $(i, 1)$ with cost $-r_i^1$
- (i, k) to $(i, k+1)$ (i.e. no switch) with cost $-r_i^{k+1}, k = 1, ..., N-1$
- (i, k) to $(\bar{i}, k+1)$ (i.e. switch) with cost $c - r_{\bar{i}}^{k+1}, k = 1, ..., N-1$
- (i, N) to e with cost 0

Write a figure to illustrate the SPP and the definitions of variables and parameters. Write the corresponding dynamic programming algorithm. (Note that you do not have to solve the problem.) (5p)

(b) Suppose he is at location i on day $k-1$ and let

$$R_i^k = r_{\bar{i}}^k - r_i^k.$$

Show that if $R_i^k \leq 0$ it is optimal to stay at location i, while if $R_i^k \geq 2c$ it is optimal to switch. You can use the following lemma.
Lemma: For every $k = 1, 2, ..., N$ it holds:

$$|J(k,i) - J(k,\bar{i})| \leq c$$

where $J(k,i)$ is the optimal cost-to-go function at stage k for state i. (5p)

3. (a) The system

$$\dot{x} = u$$

is controlled to minimize the criterion

$$\int_0^\infty (x^{2m} + u^2)\, dt$$

where m is a positive integer. Derive a control law $u = k(x)$ that minimizes the criterion. (5p)

(b) Consider an analogous discrete time problem. Minimize

$$\sum_0^{N-1} (x(t)^{2m} + u(t)^2) + x(N)^{2m}$$

for the system

$$x(t+1) = x(t) + u(t)$$

Write down the dynamic programming recursion for solving the problem. Explain why it is difficult to solve the problem explicitly if $m > 1$. (5p)

4. We consider a linear system in discrete time:

$$\begin{aligned} x_{k+1} &= Ax_k + v_k \\ y_k &= Cx_k + e_k \end{aligned}$$

where x_k and y_k are n and p dimensional vectors, repectively, and $k = 0, \dots, N-1$. The initial state x_0 is given. We assume that the sequences of vectors v_k and e_k are zero mean identically distributed and independet random vectors with symmetric and postive definite covariance matrices R_1 and R_2, respectively. The likelihood function $L(\bar{x}, \bar{v})$ satisfies

$$\begin{aligned} -\log L(\bar{x}, \bar{v}) = &\frac{1}{2} \sum_{k=0}^{N} (y_k - Cx_k)^T R_2^{-1} (y_k - Cx_k) \\ &+ \frac{1}{2} \sum_{k=0}^{N-1} v_k^T R_1^{-1} v_k + c \end{aligned}$$

where c is some constant, and where $\bar{x} = (x_1, \ldots, x_N)$ and $\bar{v} = (v_0, \ldots, v_{N-1})$. We are interested in solving the maximum likelihood problem of maximizing $L(\bar{x}, \bar{v})$ with respect to $(\bar{x}, \bar{v})$ under the constraint of the dynamic recursion above and for given measurements $\bar{y} = (y_0, \ldots, y_N)$.

Show that the optimal $\bar{x}$ satisfies the following two-point boundary problem for some sequence of Lagrange multipliers λ_k:

$$\begin{aligned} x_{k+1} &= Ax_k - R_1\lambda_{k+1}, \quad k = 0, \ldots, N-1 \\ \lambda_k &= A^T\lambda_{k+1} - C^T R_2^{-1}\left(y_k - Cx_k\right), \quad k = 1, \ldots, N-1 \end{aligned}$$

with boundary conditions x_0 given and $\lambda_N = -C^T R_2^{-1}\left(y_N - Cx_N\right)$. (10 p)

TSRT08: Optimal Control
Solutions

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1. (a) We will solve the problem

$$\min_u H(u, \lambda) \tag{1}$$

where $H(u, \lambda) = \frac{u^2}{2} + \lambda u$ and λ fulfill

$$\dot{\lambda} = -H_x(u^*, \lambda) = 0 \tag{2}$$

$$\lambda(1) = \nu, \qquad \nu \in \mathbb{R} \tag{3}$$

Equation (1) is minimized by $u = -\lambda$. Equation (2) and (3) give that $\lambda = \nu$. This results in $x(t) = -\nu t + C_1$. The boundary conditions $x(0) = 1$ and $x(1) = 0$ give $u(t) = -1$.

(b) The Hamiltonian is

$$H(x, u, \lambda) = |u| + \lambda(u - x)$$

and for the end condition we have

$$\phi(x) = 0, \qquad g(x) = x$$

The multipliers λ shall fulfill

$$\dot{\lambda}(t) = -H_x(x^*, u^*, \lambda) = \lambda(t)$$
$$\lambda(1) = \phi_x(x^*(1)) + \nu g_x(x^*(1)) = \nu, \qquad \nu \in \mathbb{R}$$

This means that $\lambda(t) = \nu e^{t-1}$, this means positive and increasing or negative and decreasing. We will minimize the Hamiltonian with respect to u for each time instance. This is equivalent to minimizing

$$|u| + \lambda u = \underbrace{(\text{sign}(u) + \lambda)}_{\sigma} u$$

If $\sigma > 0$ we want to choose u as the smallest feasible negative value and if $\sigma < 0$ we want to choose u as the largest feasible positive value. Consequently, the following cases minimize the Hamiltonian:

Positive λ

$$\begin{cases} u = -1 \text{ and } \lambda > 1 \\ u = 0 \text{ and } 0 \leq \lambda \leq 1 \end{cases}$$

Negative λ

$$\begin{cases} u = 1 \text{ and } \lambda < -1 \\ u = 0 \text{ and } -1 \leq \lambda \leq 0 \end{cases}$$

Our control candidate contains the control signals $0, -1$ which requires a positive λ. Thus, we have to find a ν such that λ starts with a positive value less than 1 and passes 1 at the time 0.5. This results in

$$1 = \nu e^{-\frac{1}{2}} \quad \Rightarrow \quad \nu = e^{\frac{1}{2}}$$

This ν fulfills the requirements and the optimality conditions are satisfied for the suggested control.

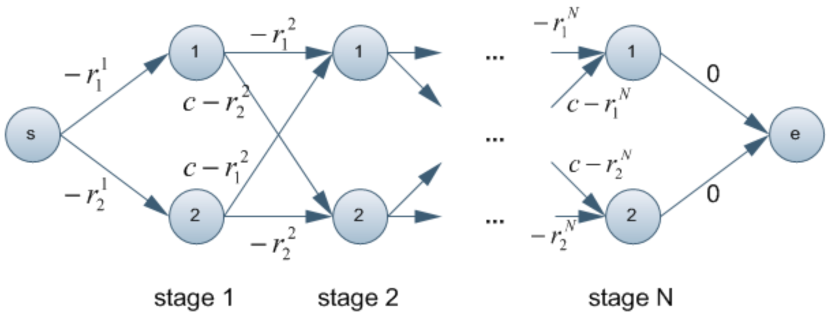

Figure 1: The shortest path problem in exercise 2.

2. (a) The shortest path from s to e maximizes the total profit over N days, see figure 1. The corresponding dynamic programming algorithm is

$$J(N,i) = 0$$
$$J(k,i) = \min\{\underbrace{-r_i^{k+1} + J(k+1,i)}_{\triangleq q_i^{k+1},\ stay},\ \underbrace{c - r_{\bar{i}}^{k+1} + J(k+1,\bar{i})}_{\triangleq q_{\bar{i}}^{k+1},\ switch}\}$$
$$J(0,s) = \min\{-r_1^1 + J(1,1),\ \ -r_2^1 + J(1,2)\}$$

(b) Consider the difference $Q_i^k = q_i^k - q_{\bar{i}}^k$. If $Q_i^k \leq 0$ it is optimal to stay in i, and if $Q_i^k \geq 0$ it is optimal to switch to $\bar{i}$.

$$\begin{aligned} Q_i^k &= q_i^k - q_{\bar{i}}^k \\ &= -r_i^k + J(k,i) - c + r_{\bar{i}}^k - J(k,\bar{i}) \\ &= R_i^k - c + J(k,i) - J(k,\bar{i}) \end{aligned}$$

By using the lemma we have

$$R_i^k - 2c \leq Q_i^k \leq R_i^k.$$

Thus, if $R_i^k \leq 0$ then $Q_i^k \leq 0$ and it is optimal to stay. If $R_i^k \geq 2c$ then $Q_i^k \geq 0$ and it is optimal to switch.

3. (a) The problem gives the following Hamilton-Jacobi-equation

$$0 = \min_u (V_x u + u^2 + x^{2m}). \tag{4}$$

Minimizing with respect to u gives $u = -V_x/2$ and if you plug that into (4) one gets

$$0 = -\frac{V_x^2}{4} + x^{2m} \Rightarrow V_x = \pm 2|x|^m.$$

Since V shall be a value function we require $V(x) > 0$ for $x \neq 0$ and $V(0) = 0$. This means that the sign of V_x shall be

$$V_x = \begin{cases} 2|x|^m & x > 0, \\ -2|x|^m & x < 0, \end{cases}$$

which gives $V(x) = 2|x|^{m+1}/(m+1)$. The optimal feedback is then

$$u = \begin{cases} -|x|^m & x > 0, \\ |x|^m & x < 0. \end{cases}$$

(b) The dynamic programming recursion is given by

$$\begin{aligned} J(N,x) &= \phi(x) \\ J(n,x) &= \min_u \{f_0(n,x,u) + J(n+1, f(n,x,u))\}, \end{aligned}$$

which is our case will be

$$\begin{aligned} J(N,x) &= x^{2m} \\ J(n,x) &= \min_u \{x^{2m} + u^2 + J(n+1, x+u)\}, \end{aligned}$$

For the case $m = 1$ this is a LQ problem (with finite time horizon). This can be solved with a cost-to-go function which is quadratic in x, this means with the ansatz $J(n,x) = \alpha(n)x^2$. However, with $m > 1$, the polynomial order of $J(n,x)$ will inevitably increase as we proceed backward in time which makes it difficult to find an analytical solution.

4. By interpreting v_k as control input, we can specify a discrete-time optimal control problem on the standard form

$$\begin{aligned} \text{minimize} \quad & \phi(x_N) + \sum_{k=0}^{N-1} f_0(k, x_k, v_k) \\ \text{subject to} \quad & x_{k+1} = f(k, x_k, v_k) \\ & x_0 \quad \text{given} \end{aligned}$$

where

$$\begin{aligned} \phi(x) &= \frac{1}{2}(y_N - x)^T R_2^{-1}(y_N - x) + c, \\ f_0(k,x,v) &= \frac{1}{2}(y_k - x)^T R_2^{-1}(y_k - x) + \frac{1}{2} v^T R_1^{-1} v, \\ f(k,x,v) &= Ax + v. \end{aligned}$$

We will use discrete-time PMP to solve the optimal control problem above. We do that by first defining the Hamiltonian, which is given by

$$\begin{aligned} H(k,x,v,\lambda) &= f_0(k,x,v) + \lambda^T f(k,x,v) \\ &= \frac{1}{2}(y_k - x)^T R_2^{-1}(y_k - x) + \frac{1}{2} v^T R_1^{-1} v + \lambda^T (Ax + v) \end{aligned}$$

Since R_1 is positive definite, also R_1^{-1} is positive definite. The quadratic form $v^T R_1^{-1} v$, and consequently also the Hamiltonian H, is therefore strictly convex in v. Pointwise minimization with respect to v then yields

$$\begin{aligned} 0 &= \frac{\partial H}{\partial v}(k,x,v,\lambda) = R_1^{-1} v + \lambda \Rightarrow \\ v &= -R_1 \lambda \end{aligned}$$

and the candidate optimal control is given by $v_k^* = -R_1 \lambda_{k+1}$.

The two-point boundary value problem is now given by

$$\begin{aligned} x_{k+1} &= \frac{\partial H}{\partial \lambda}(k, x_k, v_k^*, \lambda_{k+1}) = Ax_k + v_k^* = Ax_k - R_1 \lambda_{k+1}, \quad k = 0, \ldots, N-1 \\ \lambda_k &= -\frac{\partial H}{\partial x}(k, x_k, v_k^*, \lambda_{k+1}) = -C^T R_2^{-1}(y_k - x_k) + A^T \lambda_{k+1}, \quad k = 1, \ldots, N-1 \end{aligned}$$

with the boundary condition x_0 given and

$$\lambda_N = \frac{\partial \phi}{\partial x}(x_N) = -C^T R_2^{-1}(y_N - x_N).$$