EXAM IN OPTIMAL CONTROL

ROOM: U14, U15

TIME: January 13, 2018, 8–12

COURSE: TSRT08, Optimal Control

PROVKOD: TEN1

DEPARTMENT: ISY

NUMBER OF EXERCISES: 4

NUMBER OF PAGES (including cover pages): 5

RESPONSIBLE TEACHER: Anders Hansson, phone 070–3004401

VISITS: 9:00, 11:00

COURSE ADMINISTRATOR: Ninna Stensgård, phone: 013–282225, ninna.stensgard@liu.se

APPROVED TOOLS: Formula sheet for the course, printed collections of formulas and tables, calculator.

SOLUTIONS: Linked from the course home page after the examination.

The exam can be inspected and checked out 2018-02-01 at 12.30-13.00 in Ljungeln, B-building, entrance 27, A-corridore to the right.

PRELIMINARY GRADING:	betyg 3	15 points
	betyg 4	23 points
	betyg 5	30 points

All solutions should be well motivated.

Good Luck!

1. (a) Solve the optimal control problem

$$\begin{aligned} \underset{u(\cdot)}{\text{minimize}} \quad & \int_0^T \left(x(t) + u^2(t)\right) \, dt \\ \text{subject to} \quad & \dot{x}(t) = x(t) + u(t) + 1, \\ & x(0) = 0. \end{aligned}$$

for a fixed $T > 0$, using the PMP. (5p)

(b) Find the extremal to the functional

$$J(y) = \int_0^1 \left(y^2(t) + \dot{y}^2(t)\right) dt,$$

satisfying $y(0) = 0$ and $y(1) = 1$. (5p)

2. We are interested in text justification which is about making lines in a text of N words have about the same length. More precisely we let the line length $l(i, j)$ be the characters of words i through j including counting blanks inbetween the words. We also let L be the maximum allowed length of a line. Then $f_0(i, j)$, which we call the badness of a line with words i through j, is given by

$$f_0(i, j) = (L - l(i, j))^3, \; 0 \leq l(i, j) \leq L$$

and $f_0(i, j) = \infty$ if $l(i, j) > L$. We are intested in the minimal overall badness defined as:

$$J = \min_{\text{partition of text}} \sum_{k=1}^{K} f_0(i_k, j_k)$$

Here the partition is defined such that $i_1 = 1$, $i_{k+1} = j_k + 1$ and $j_K = N$ for some K, where K is the number of lines. Notice that we do not know beforehand how many lines there will be, and that this will come out of the optimal solution. Introduce also the minimal badness if starting with word i and considering the remaining part of the text:

$$J(i) = \min_{\text{partion of text from word } i} \sum_{k=1}^{K_i} f_0(i_k, j_k)$$

Then it can be shown that

$$J(i) = \min_{i \leq j \leq N} (f_0(i, j) + J(j + 1))$$

with $J(N + 1) = 0$. You should now consider the text “TO BE OR NOT TO BE”. Your task is to compute the optimal partition for this text such that the overall badness is minimized when $L = 6$. What is the resulting overall badness? Moreover, redo your caculations when the text instead is “BE OR NOT TO BE”. (10p)

3. Consider the following optimal control problems:

$$\begin{aligned} \underset{u(\cdot)}{\text{minimize}} \quad & \int_0^{t_f} \left(x^2(t) + u^2(t)\right) dt \\ \text{subject to} \quad & \dot{x}(t) = x(t) + u(t), \\ & x(0) = 1 \end{aligned} \tag{1}$$

$$\begin{aligned} \underset{u(\cdot)}{\text{minimize}} \quad & \int_0^{t_f} \left(x(t) + u^2(t)\right) dt \\ \text{subject to} \quad & \dot{x}(t) = x(t) + u(t) + 1, \\ & x(0) = 0 \end{aligned} \tag{2}$$

$$\begin{aligned} \underset{u(\cdot)}{\text{minimize}} \quad & \int_0^{t_f} \left(x(t) + u^2(t)\right) dt \\ \text{subject to} \quad & \dot{x}(t) = x(t) + u(t) + 1, \\ & x(0) = 0 \\ & x(t_f) = 1. \end{aligned} \tag{3}$$

(a) Suppose you must solve these problems numerically. Describe advantages and disadvantages of (A) the discretization method (constrained nonlinear program), (B) the shooting method (boundary condition iteration), and (C) the gradient method (first order gradient search of the cost function) for solving these three optimal control problems. (5p)

(b) Make comments on if and how the problem (1) can be solved by using HJBE. Note that you do not necessarily have to solve the problems, but your statements must be well motivated. (2p)

(c) Make comments on if and how the problem (3) can be solved by using PMP. Note that you do not necessarily have to solve the problems, but your statements must be well motivated. (3p)

4. Consider the problem of minimizing the integral of the square of the so-called jerk of a trajectory from a current measured position, velocity and acceleration $x(t)$ to the origin, i.e.

$$\min_u \int_t^{t_f} u^2(s)ds$$
$$\text{s.t. } \dot{x}(s) = Ax(s) + Bu(s), \quad x(t_f) = 0$$

Notice that the initial value is $x(t)$. The matrices are

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(a) Use PMP to show that the following TPBVP

$$\dot{x}(s) = Ax(s) - \frac{1}{2}BB^T\lambda(s)$$
$$\dot{\lambda}(s) = -A^T\lambda(s)$$

with boundary conditions $x(t)$ given and $x(t_f) = 0$ is a necessary condition for an optimal trajectory, and that the optimal u is given by

$$u(s) = -\frac{1}{2}B^T\lambda(s)$$

(2 p)

(b) Show that a solution to the TPBVP when the boundary conditions are neglected can be expressed as

$$\begin{aligned}
\lambda_1(s) &= c_1 \\
\lambda_2(s) &= -c_1\bar{s} + c_2 \\
\lambda_3(s) &= \frac{1}{2}c_1\bar{s}^2 - c_2\bar{s} + c_3 \\
x_3(s) &= -\frac{1}{12}c_1\bar{s}^3 + \frac{1}{4}c_2\bar{s}^2 - \frac{1}{2}c_3\bar{s} + c_4 \\
x_2(s) &= -\frac{1}{48}c_1\bar{s}^4 + \frac{1}{12}c_2\bar{s}^3 - \frac{1}{4}c_3\bar{s}^2 + c_4\bar{s} + c_5 \\
x_1(s) &= -\frac{1}{240}c_1\bar{s}^5 + \frac{1}{48}c_2\bar{s}^4 - \frac{1}{12}c_2\bar{s}^3 + \frac{1}{2}c_4\bar{s}^2 + c_5\bar{s} + c_6
\end{aligned}$$

where $\bar{s} = s - t$, and where c_i, $i = 1, 2, \ldots, 6$ are some constants. (2 p)

(c) Show that the boundary conditions implies that

$$c_4 = x_3(t); \quad c_5 = x_2(t); \quad c_6 = x_1(t)$$

and that the remaining constants have to satisfy the following linear system of equations

$$\begin{bmatrix} -\frac{1}{12}\alpha^3 & \frac{1}{4}\alpha^2 & -\frac{1}{2}\alpha \\ -\frac{1}{48}\alpha^4 & \frac{1}{12}\alpha^3 & -\frac{1}{4}\alpha^2 \\ -\frac{1}{240}\alpha^5 & \frac{1}{48}\alpha^4 & -\frac{1}{12}\alpha^3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -x_3(t) \\ -\alpha x_3(t) - x_2(t) \\ -\frac{1}{2}\alpha^2 x_3(t) - \alpha x_2(t) - x_1(t) \end{bmatrix}$$

where $\alpha = t_f - t$. (2 p)

(d) Show that

$$c_3 = \frac{120}{\alpha^3}x_1(t) + \frac{72}{\alpha^2}x_2(t) + \frac{18}{\alpha}x_3(t)$$

Hint: It can be a good idea to define new constants

$$\bar{c}_1 = \alpha^3 c_1; \quad \bar{c}_2 = \alpha^2 c_2; \quad \bar{c}_3 = \alpha c_3$$

when solving the above linear system of equations (2 p)

(e) Express the optimal control $u(t)$ as a feedback from $x(t)$. *Hint:* Notice that $u(t)$ is the same as $u(s)$ for $s = t$. (2 p)

TSRT08: Optimal Control
Solutions

20180113

1. (a) The Hamiltonian is given by

$$H(t, x, u, \lambda) = x + u^2 + \lambda(x + u + 1).$$

Pointwise minimization yields

$$0 = \frac{\partial H}{\partial u}(t, x, u, \lambda) = 2u + \lambda \quad \Rightarrow \quad u^* = -\frac{1}{2}\lambda.$$

The adjoint equation is given by

$$\dot{\lambda}(t) = -\lambda(t) - 1, \quad \lambda(T) = 0$$

which is a first order linear ODE with the solution

$$\lambda(t) = e^{T-t} - 1,$$

and the optimal control is

$$u^*(t) = -\frac{1}{2}\lambda(t) = \frac{1 - e^{T-t}}{2}.$$

(b) Introducing $x = y$ and $u = \dot{y}$, it holds that $\dot{x} = u$ and the Hamiltonian is given by

$$H(t, x, u, \lambda) = x^2 + u^2 + \lambda u.$$

The following equations must hold

$$\dot{\lambda} = -\frac{\partial H}{\partial x}(t, x, u, \lambda) = -2x,$$
$$0 = \frac{\partial H}{\partial u}(t, x, u, \lambda) = 2u + \lambda.$$

The latter yields $\dot{\lambda} = -2\dot{u} = -2\ddot{y}$, which plugged into the first equation gives

$$-2\ddot{y} = -2x = -2y \quad \Leftrightarrow \quad \ddot{y} - y = 0,$$

which has the solution

$$y(t) = c_1 e^t + c_2 e^{-t},$$

for some constants c_1 and c_2. The boundary constraints $y(0) = 0$ and $y(1) = 1$ yields

$$\begin{pmatrix} 1 & 1 \\ e^1 & e^{-1} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

which has the solution

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{e^{-1} - e^1} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Thus,

$$y(t) = \frac{1}{e^{-1} - e^1}(e^{-t} - e^t),$$

is the sought after extremal.

2. (a) TO BE OR NOT TO BE, $N = 6$, $L = 6$.
Stage $k = N + 1 = 7$: $J(7) = 0$.
Stage $k = 6$:

$$J(6) = \min_{6 \leq j \leq 6} \{f_0(6, j) + 0\} = f_0(6, 6) = 64 \tag{1}$$

Stage $k = 5$:

$$\begin{aligned} J(5) &= \min_{5 \leq j \leq 6} \{f_0(5, j) + J(j + 1)\} \\ &= \min\{f_0(5, 5) + J(6), f_0(5, 6) + J(7)\} = \min\{128, 1\} = 1 \end{aligned} \tag{2}$$

optimum for (5,6)
Stage $k = 4$:

$$\begin{aligned} J(4) &= \min_{4 \leq j \leq 6} \{f_0(4, 4) + J(5), f_0(4, 5) + J(6), f_0(4, 6) + J(7)\} \\ &= \min\{27 + 1, 0 + 64, \infty\} = 28 \end{aligned} \tag{3}$$

optimum for (4,4) and (5,6)
Stage $k = 3$:

$$\begin{aligned} J(3) &= \min_{3 \leq j \leq 6} \{f_0(3, 3) + J(4), f_0(3, 4) + J(5), f_0(3, 5) + J(6), f_0(3, 6) + J(7)\} = \\ &= \min\{64 + 28, 0 + 1, \infty, \infty\} = 1 \end{aligned} \tag{4}$$

optimum for (3,4) and (5,6)
Stage $k = 2$:

$$\begin{aligned} J(2) &= \min_{2 \leq j \leq 6} \{f_0(2, 2) + J(3), f_0(2, 3) + J(4), f_0(2, 4) + J(5), f_0(2, 5) + J(6), f_0(2, 6) + J(7)\} \\ &= \min\{64 + 1, 1 + 28, \infty, \infty, \infty\} = 29 \end{aligned} \tag{5}$$

optimum for (2,3), (4) and (5,6)
Stage $k = 1$:

$$\begin{aligned} J(1) = \min_{1 \leq j \leq 6} \{&f_0(1, 1) + J(2), f_0(1, 2) + J(3), f_0(1, 3) + J(4), f_0(1, 4) + J(5), \\ &f_0(1, 5) + J(6), f_(1, 6) + J(7)\} = \min\{64 + 29, 1 + 1, \infty, \infty, \infty, \infty\} = 2 \end{aligned} \tag{6}$$

optimum for (1,2), (3,4) and (5,6)
The minimum overall badness,

$$J^*(1) = 2, \tag{7}$$

achieved for the optimal partition of (1,2), (3,4), and (5,6).

(b) BE OR NOT TO BE, $N = 5$, $L = 6$.
Using the principle of optimality it can be read from (6) that the minimum overall badness,

$$J^*(1) = 29, \tag{8}$$

is achieved for the optimal partition of (1,2), (3), and (4,5).

3. (a)
- The discretization method is straightforward to apply to all problems. There exist many good algorithms for nonlinear optimization. Drawbacks are the large number of variables and constraints, and that the solution may not converge to the solution of the original problem.
- A shooting method is straightforward to apply to all problems, but it is crucial to find a good initial guess of $\lambda(0)$. The transition matrix may sometimes be ill conditioned when using a shooting method, but that is a minor problem for these quite simple problems.
- A gradient method is straightforward to apply to (1) and (2), but (3) requires a slightly more complex gradient algorithm due to the the terminal constraint. Convergence tends to be slow for the gradient methods, but this is a minor problem for these quite simple problems.

(b) Problem (1) is a linear-quadratic problem that is possible to solve analytically with HJBE. Use $V(t,x) = P(t)x^2$, where $P(t)$ is a positive function that can be obtained by solving the Riccati equation. The optimal feedback law is $\mu(t,x) = -P(t)x$.

(c) Note that Problem (3) is the same problem as in exercise 1, but with an additional constraint on the final state. Thus, since S_f is a set with just one point then there is no constraint on $\lambda(t_f)$. Thus, the shape of the control signal can be derived, but with one unknown constant. By substituting the control signal in the dynamic model with the control law $u(t) = -1/2\lambda(t)$ we obtain a linear ODE of order one that is straightforward to solve and by using the initial and final state constraints all constants can be found.

4. (a) The Hamiltonian is given by

$$H(x,u,\lambda) = u^2 + \lambda^T(Ax + Bu)$$

The adjoint equations are

$$\dot{\lambda} = -H_x = -A^T\lambda$$

with no boundary conditions. Since the Hamiltonian is strictly convex in u the minimum is obtained when the gradient with respect to u is zero, i.e.

$$2u + B^T\lambda = 0$$

which has the solution $u = -\frac{1}{2}B^T\lambda$. Substituting this expression into the dynamical equations for x results in

$$\dot{x} = Ax - \frac{1}{2}BB^T\lambda$$

with boundary conitions $x(t)$ given and $x(t_f) = 0$.

(b) Examining the TPBVB one realizes that the equations can be solved recursivley by integration. One starts with the equation $\dot{\lambda}_1(s) = 0$ and take as primitive function $\lambda_1(s) = c_1$ for some constant c_1. Then one consider $\dot{\lambda}_2(s) = -\lambda_1(s)$, and hence one may take $\lambda_2(s) = -c_1(s-t) + c_2$ for some constant c_2. Continuation of this gives the desired result.

(c) The first three conditions immediately follows from expressions for $x(s)$ when s is substituted with t, since then $\bar{s} = 0$. The equations for c_i, $i = 1, 2, 3$ immediately follow from the expressions for $x(s)$ when s is substituted with t_f, since $x(t_f) = 0$, where we also make use of the three first conditions.

(d) After the change of variables and after dividing the first equation with α and the second with α^2 the equations read

$$\begin{bmatrix} -\frac{1}{12} & \frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{48} & \frac{1}{12} & -\frac{1}{4} \\ -\frac{1}{240} & \frac{1}{48} & -\frac{1}{12} \end{bmatrix} \begin{bmatrix} \bar{c}_1 \\ \bar{c}_2 \\ \bar{c}_3 \end{bmatrix} = \begin{bmatrix} -x_3(t) \\ -x_3(t) - \frac{1}{\alpha}x_2(t) \\ -\frac{1}{2}x_3(t) - \frac{1}{\alpha}x_2(t) - \frac{1}{\alpha^2}x_1(t) \end{bmatrix}$$

Several manipulations now lead to

$$\bar{c}_3 = \frac{120}{\alpha^2}x_1(t) + \frac{72}{\alpha}x_2(t) + 18x_3(t)$$

from which the result follows.

(e) The optimal $u(t)$ is given by

$$u(t) = -\frac{1}{2}B^T\lambda(t) = -\frac{1}{2}\lambda_3(t) = -\frac{1}{2}c_3 = -\left(\frac{60}{\alpha^3}x_1(t) + \frac{36}{\alpha^2}x_2(t) + \frac{9}{\alpha}x_3(t)\right)$$

which is a feedback from $x(t)$.