EXAM IN OPTIMAL CONTROL

ROOM: U14

TIME: April 19, 2017, 14–18

COURSE: TSRT08, Optimal Control

PROVKOD: TEN1

DEPARTMENT: ISY

NUMBER OF EXERCISES: 4

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APPROVED TOOLS: Formula sheet for the course, printed collections of formulas and tables, calculator.

SOLUTIONS: Linked from the course home page after the examination.

The exam can be inspected and checked out 2017-05-04 at 12.30-13.00 in the examiners office, 2A:573, B-building, entrance 25, A-corridor to the right.

PRELIMINARY GRADING:	betyg 3	15 points
	betyg 4	23 points
	betyg 5	30 points

All solutions should be well motivated.

Good Luck!

1. We are interested in computing optimal transportation routes in a circular city. The cost for transportation per unit length is given by a function $g(r)$ that only depends on the radial distance r to the city center. This means that the total cost for transportation from a point P_1 to a point P_2 is given by

$$\int_{P_1}^{P_2} g(r)ds$$

where s represents the arc length along the path of integration. In polar coordinates (θ, r) the total cost reads

$$\int_{P_1}^{P_2} g(r)\sqrt{1+(r\dot{\theta})^2}dr$$

where $\theta = \theta(r)$, and $\dot{\theta} = d\theta/dr$.

   (a) Formulate the problem of computing an optimal path as an optimal control problem (2p)

   (b) For the case of $g(r) = \alpha/r$ for some positive α show that any optimal path satisfies the equation $\theta = a \log r + b$ for some constants a and b. (5p)

   (c) Show that if the initial point and the final point are at the same distance from the origin, then the optimal path is a circle segment. You may use the claim in (b). (3p)

2. Consider the double integrator

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u, \quad |u| \le 1$$

with the following criterion to be minimized

$$\int_0^\infty |x_1|dt$$

   (a) Write down the Hamilton-Jacobi-Bellman equation for the optimal cost $V(x)$. (2p)

   (b) Calculate the optimal control as a function of V. (2p)

   (c) Show that

$$V = x_1x_2 + \frac{x_2^3}{3} + C(2x_1 + x_2^2)^{3/2}$$

   solves the Hamilton-Jacobi-Bellman equation in the region $x_1 > 0$, $x_2 > 0$. (C is a positive constant). What value does this give for u when $x_1 > 0$, $x_2 > 0$? (6p)

3. We consider a one degree of freedom robot joint with constant joint stiffness K and damping D. The dynamics is described by

$$M\frac{d^2q}{dt^2} + D\frac{dq}{dt} + Kq = K\theta + D\frac{d\theta}{dt}$$

where q is the link position, θ is the motor position, and M is the link inertia.

(a) Show that with $x_1 = \theta - q$, $x_2 = \dot{q}$ and $u = \dot{\theta}$ the dynamics can equivalently be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_2 + u \\ \omega_0^2 x_1 + 2\zeta\omega_0(-x_2 + u) \end{bmatrix}$$

where $\omega_0^2 = K/M$ and $\zeta = D/(2\omega_0 M)$. (2p)

(b) We are interested in maximizing the final link velocity $\dot{q}(t_f)$, where the final time t_f is fixed. The control signal u is constrained such that $\underline{u}(t) \leq u(t) \leq \overline{u}(t)$, $t \in [0, t_f]$. Form the Hamiltonian for this optimal control problem and write down the adjoint equations. (2p)

(c) Show that the optimal control is given by

$$u^*(t) = \begin{cases} \overline{u}(t), \ \sigma(t) < 0 \\ \underline{u}(t), \ \sigma(t) > 0 \\ \text{arbitrary}, \ \sigma(t) = 0 \end{cases}$$

where $\sigma(t) = \lambda_1(t) + 2\zeta\omega_0\lambda_2(t)$, and where $\lambda^T = \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix}$ satisfies the adjoint equations. (2p)

(d) Consider the special case of $\zeta = 0$ which gives the switching function

$$\sigma(t) = -\omega_0 \sin(\omega(t_f - t)).$$

Discuss how many switches there will be. (2p)

(e) Consider the special case of $\zeta = 1$ which gives the switching function

$$\sigma(t) = \omega_0^2 e^{-\omega(t_f - t)} \left(t_f - t - \frac{2}{\omega_0} \right).$$

Discuss how many switches there will be. (2p)

4. (a) Describe advantages and disadvantages of the five different computational algorithms that are described in the lecture notes. (5p)

(b) Solve the following problem

$$\begin{aligned} \underset{u(\cdot)}{\text{minimize}} \quad & -\int_0^\infty e^{-\beta t}\sqrt{u(t)}\, dt \\ \text{subject to} \quad & \dot{x}(t) = \alpha x(t) - u(t), \\ & x(0) = x_0 > 0, \end{aligned}$$

where we assume that $2\beta > \alpha$ and $x(t) > 0$ for all $t \geq 0$. *Hint:* Try the value function $V(t, x) = -e^{-\beta t}\sqrt{cx}$ where $c > 0$ is a constant. Use Theorem 2 in the formula sheet with $t_f = \infty$. (5p)

TSRT08: Optimal Control
Solutions

2017-04-19

1. (a) By introducing the control $u = \dot{\theta}$, we can specify the optimal control problem

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & \int_{r_1}^{r_2} g(r)\sqrt{1 + (r \cdot u(r))^2} dr \\ \text{subject to} \quad & \dot{\theta} = u \\ & \theta(r_1) = \theta_1 \\ & \theta(r_2) = \theta_2 \end{aligned}$$

where r_1, θ_1 and r_2, θ_2 are the polar coordinates of the points P_1 and P_2, respectively.

(b) With $g(r) = \alpha/r$, the Hamiltonian is given by

$$H(r, \theta, u, \lambda) = \frac{\alpha}{r}\sqrt{1 + (r \cdot u)^2} + \lambda \cdot u$$

Further we have that

$$\begin{aligned} \frac{\partial H}{\partial u}(r, \theta, u, \lambda) &= \alpha \frac{r \cdot u}{\sqrt{1 + (r \cdot u)^2}} + \lambda \\ \frac{\partial^2 H}{\partial u^2}(r, \theta, u, \lambda) &= \alpha \frac{r}{(1 + (r \cdot u)^2)^{3/2}} \end{aligned}$$

Since $\alpha > 0$ and $r > 0$ we have that $\frac{\partial^2 H}{\partial u^2} > 0 \ \forall u$. Hence, $H(r, \theta, u, \lambda)$ is strictly convex in u. Therefor, pointwise minimization yields

$$0 = \frac{\partial H}{\partial u}(r, \theta^*(r), u^*(r), \lambda) = \alpha \frac{r \cdot u^*(r)}{\sqrt{1 + (r \cdot u^*(r))^2}} + \lambda(r)$$

The adjoint equation is given by

$$\dot{\lambda} = -\frac{\partial H}{\partial \theta}(r, \theta, u, \lambda) = 0$$

without final constraint on $\lambda(r_2)$ since we do have a final constraint on $\theta(r_2)$. This equation has the solution

$$\lambda(r) = c$$

for some constant c and the optimal control is

$$\alpha \frac{r \cdot u^*(r)}{\sqrt{1 + (r \cdot u^*(r))^2}} = -c$$

This requires $r \cdot u^*(r)$ to be constant, which can be written as

$$ru^*(r) = a \quad \Rightarrow \quad u^*(r) = \frac{a}{r}$$

for some constant a. This gives the optimal path

$$\dot{\theta} = \frac{a}{r} \quad \Rightarrow \quad \theta = a \log r + b$$

which we were supposed to show.

(c) Reformulate the optimal path as a function of theta

$$r(\theta) = e^{\frac{\theta - b}{a}} = Be^{A\theta}$$

where $A = 1/a$ and $B = e^{-b/a}$. We now require that the initial and final point shall have the same radius r_0, such that $r(\theta_1) = r_0$ and $r(\theta_2) = r_0$. This gives

$$r_0 = Be^{A\theta_1}, \quad r_0 = Be^{A\theta_2} \quad \Rightarrow \quad A = 0, \quad B = r_0$$

which gives

$$r(\theta) = r_0, \quad \text{for} \quad \theta_1 \leq \theta \leq \theta_2$$

This corresponds to a path with constant radius, i.e. a circle segment.

2. (a) The Hamiltonian is given by

$$\begin{aligned} H(x, u, \lambda) &\triangleq f_0(x, u) + \lambda^T f(x, u) \\ &= |x_1| + \lambda_1 x_2 + \lambda_2 u \end{aligned}$$

Point-wise optimization yields

$$\begin{aligned} \tilde{\mu}(x, \lambda) &\triangleq \arg\min_{|u| \leq 1} H(x, u, \lambda) \\ &= \arg\min_{|u| \leq 1} \{|x_1| + \lambda_1 x_2 + \lambda_2 u\} = -\text{sign}\lambda_2, \end{aligned}$$

The Hamilton-Jacobi-Bellman equation is now given by

$$0 = H(x, \tilde{\mu}(x, V_x), V_x).$$

In our case this is equivalent to

$$0 = |x_1| + V_{x_1} x_2 - V_{x_2} \text{sign} V_{x_2}, \tag{1}$$

(b) The optimal control is given by

$$u^*(t) \triangleq \tilde{\mu}(x(t), V_x(x(t))) = -\text{sign}\{V_{x_2}(x(t))\}$$

(c) With the proposed function V we get

$$V_{x_1} = x_2 + 3C(2x_1 + x_2^2)^{1/2}$$
$$V_{x_2} = x_1 + x_2^2 + 3C(2x_1 + x_2^2)^{1/2} x_2$$

In the region $x_1 > 0$ and $x_2 > 0$ we then also have that $V_{x_1} > 0$ and $V_{x_2} > 0$. Inserted in (1) this gives

$$x_1 + (x_2 + 3C(2x_1 + x_2^2)^{1/2})x_2 - x_1 - x_2^2 - 3C(2x_1 + x_2^2)^{1/2} x_2 = 0$$

Thus, in this region this solves the Hamilton-Jacobi-Bellman equation and the corresponding optimal control will be

$$u^*(t) = -\text{sign} V_{x_2} = -1 \tag{2}$$

3. (a) By introducing $x_1 = \theta - q$, $x_2 = \dot{q}$, $u = \dot{\theta}$ and redefining $x \triangleq (x_1, x_2)^T$, the problem at hand can be written on standard form as

$$\dot{x}_1(t) = \dot{\theta} - \dot{q} = -x_2 + u,$$
$$\dot{x}_2(t) = \ddot{q} = \frac{1}{M}\left(K(\theta - q) + D(\dot{\theta} - \dot{q})\right) = \frac{K}{M} x_1 + \frac{D}{M}\dot{x}_1$$

By introducing $\omega_0^2 = K/M$ and $\zeta = D/(2\omega_0 M)$ we have

$$\dot{x}_1 = -x_2 + u,$$
$$\dot{x}_2 = \omega_0^2 x_1 + 2\zeta\omega_0(-x_2 + u).$$

(b) Considering $\phi(T, x(T)) = -x_1(T)$ and $f_0(t, x, u) = 0$, the Hamiltonian is given by

$$\begin{aligned} H(t, x, u, \lambda) &\triangleq f_0(t, x, u) + \lambda^T f(t, x, u) \\ &= \lambda_1(-x_2 + u) + \lambda_2(\omega_0^2 x_1 + 2\zeta\omega_0(-x_2 + u)). \end{aligned}$$

The adjoint equations are

$$\dot{\lambda}_1(t) = -\frac{\partial H}{\partial x_1} = -\lambda_2\omega_0^2$$
$$\dot{\lambda}_2(t) = -\frac{\partial H}{\partial x_2} = \lambda_1 + 2\zeta\omega_0\lambda_2.$$

(c) Pointwise minimization yields

$$\begin{aligned} \tilde{\mu}(t, x, \lambda) &\triangleq \underset{\underline{u} \le u \le \bar{u}}{\arg\min}\, H(t, x, u, \lambda) \\ &= \underset{\underline{u} \le u \le \bar{u}}{\arg\min}\, (\lambda_1 + 2\zeta\omega_0\lambda_2)u \\ &= \begin{cases} \bar{u}, & \sigma < 0 \\ \underline{u}, & \sigma > 0 \\ \tilde{u}, & \sigma = 0 \end{cases}, \end{aligned}$$

where $\tilde{u} \in [\underline{u}, \bar{u}]$ is arbitrary. Thus, the optimal control is expressed by

$$u^*(t) \triangleq \tilde{\mu}(t, x(t), \lambda(t)) = \begin{cases} \bar{u}, & \sigma(t) < 0 \\ \underline{u}, & \sigma(t) > 0 \\ \tilde{u}, & \sigma(t) = 0 \end{cases},$$

where the switching function is given by

$$\sigma(t) = \lambda_1(t) + 2\zeta\omega_0\lambda_2(t).$$

(d) Since the switching function never remains at zero for finite time, the optimal control, u^*, always takes its minimum or maximum values periodically, depending on the sign of $\sigma(t)$(known as bang-bang control). The number of times σ becomes zero, determines the number of switches, and $\sigma = 0$ when

$$\omega_0(t_f - t) = k\pi, k = 0, 1, 2, ... \Rightarrow t = t_f - \frac{k\pi}{\omega_0}$$

The number of switches is the largest integer, k, for which

$$t_f - \frac{k\pi}{\omega_0} > 0 \Rightarrow k < \frac{t_f\omega_0}{\pi}.$$

(e) The switching function crosses the time axis only once at $t' = t_f - \frac{2}{\omega_0}$, and switches sign from positive to negative. Therefore, if $t_f > \frac{2}{\omega_0}$, u^* starts from its minimum value and at t' switches to its maximum value. If $t_f \leq \frac{2}{\omega_0}$, there will be no switches.

4. (a) See Chapter 10 "Computational Algorithms" in the course compendium.

(b) The Hamiltonian is given by

$$H(t, x, u, \lambda) = -e^{-\beta t}u^{1/2} + \lambda(\alpha x - u)$$

The pointwise minimum is determined by

$$0 = \frac{\partial H}{\partial u}H(t, x, u, \lambda) = -\frac{1}{2}e^{-\beta t}u^{-1/2} - \lambda \Rightarrow \sqrt{u^*} = -\frac{1}{2}e^{-\beta t}\lambda^{-1}$$

where $\lambda < 0$ has to be satisfied. The second derivative yields

$$\frac{\partial^2 H}{\partial u^2}H(t, x, u, \lambda) = \frac{1}{4}e^{-\beta t}u^{-3/2} > 0$$

since $u > 0$. Hence the solution is a minimum. Plugging the optimal control input u^* into the Hamiltonian yields

$$\begin{aligned} H(t, x, u^*, \lambda) &= -e^{-\beta t}\left(-\frac{1}{2}e^{-\beta t}\lambda^{-1}\right) + \lambda\alpha x - \lambda\left(-\frac{1}{2}e^{-\beta t}\lambda^{-1}\right)^2 \\ &= \frac{1}{2}e^{-2\beta t}\lambda^{-1} + \lambda\alpha x - \frac{1}{4}e^{-2\beta t}\lambda^{-1} = \\ &= \frac{1}{4}e^{-2\beta t}\lambda^{-1} + \lambda\alpha x \end{aligned}$$

The HJBE becomes

$$-V_t = \frac{1}{4}e^{-2\beta t}V_x^{-1} + V_x \alpha x$$

The guess $V(t,x) = -e^{-\beta t}\sqrt{cx}$, yields

$$V_t = \beta e^{-\beta t}(cx)^{1/2}, \quad V_x = -\frac{1}{2}e^{-\beta t}(c/x)^{1/2}$$

which plugged into the HJBE yields

$$\begin{aligned}
-\beta e^{-\beta t}(cx)^{1/2} &= \frac{1}{4}e^{-2\beta t}\left(-\frac{1}{2}e^{-\beta t}(c/x)^{1/2}\right)^{-1} + \left(-\frac{1}{2}e^{-\beta t}(c/x)^{1/2}\right)\alpha x \Leftrightarrow \\
\beta c^{1/2}x^{1/2}e^{-\beta t} &= \frac{1}{2}c^{-1/2}x^{1/2}e^{-\beta t} + \frac{1}{2}\alpha c^{1/2}x^{1/2}e^{-\beta t}
\end{aligned}$$

This equation has to hold for all x and t, we thus get

$$\begin{aligned}
\beta c^{1/2} &= \frac{1}{2}c^{-1/2} + \frac{1}{2}\alpha c^{1/2} \Leftrightarrow \\
1 &= (2\beta - \alpha)c \Leftrightarrow \\
c &= \frac{1}{2\beta - \alpha} > 0
\end{aligned}$$

which follows from the assumption $2\beta > \alpha$. Finally the optimal control is given by

$$u^* = \frac{1}{4}e^{-2\beta t}\left(V_x\right)^{-2} = \frac{1}{4}e^{-2\beta t}\left(-\frac{1}{2}e^{-\beta t}(c/x)^{1/2}\right)^{-2} = \frac{x}{c} = (2\beta - \alpha)x$$