Elektromagnetism TFYA13 2012-08-15

1) $\bar F = \bar F_m + \bar F_e = q(\bar v \times \bar B + \bar E) = q(\bar v \times B_0 \hat z - E_0 \hat z) = m\bar a$

Startvillkor: $(t=0)$: $\bar r(0) = \bar 0$ (valt origo)

$\bar v(0) = v_{x0}\hat x + v_{z0}\hat z$

$$\left.\begin{aligned} \bar r &= x\hat x + y\hat y + z\hat z \\ \bar v = \dot{\bar r} &= \dot x\hat x + \dot y\hat y + \dot z\hat z \\ \bar a = \dot{\bar v} = \ddot{\bar r} &= \ddot x\hat x + \ddot y\hat y + \ddot z\hat z \end{aligned}\right\} \Rightarrow \begin{aligned} &m(\ddot x\hat x + \ddot y\hat y + \ddot z\hat z) = \\ &= q(-\dot x B_0\hat y + \dot y B_0\hat x - E_0\hat z) \end{aligned}$$

$\hat y$: $m\ddot y = -qB_0\dot x \Rightarrow \ddot y = -\underbrace{\frac{qB_0}{m}}_{\equiv w}\dot x \Rightarrow \dot y = -wx + C_1,\ t=0 \Rightarrow \underline{\dot y = -wx}$ ①

$\hat x$: $m\ddot x = qB_0\dot y \Rightarrow \ddot x - w\dot y = 0 \overset{①}{\Rightarrow} \ddot x + w^2x = 0 \Rightarrow x = C_1\cos wt + C_2\sin wt$

$t=0 \Rightarrow x = C_2\sin wt \Rightarrow \dot x = C_2 w\cos wt,\ t=0 \Rightarrow \underline{x = \frac{v_{x0}}{w}\sin wt}$

Ger i ①: $\dot y = -v_{x0}\sin wt \Rightarrow y = \frac{v_{x0}}{w}\cos wt + C_1,\ t=0 \Rightarrow \underline{y = \frac{v_{x0}}{w}(\cos wt - 1)}$

$\hat z$: $m\ddot z = -qE_0 \Rightarrow \dot z = \frac{-qE_0}{m}t + C_1,\ t=0 \Rightarrow \dot z = \frac{-qE_0}{m}t + v_{z0} \Rightarrow$

$z = \frac{-qE_0}{2m}t^2 + v_{z0}t + C_1,\ t=0 \Rightarrow \underline{z = \frac{-qE_0}{2m}t^2 + v_{z0}t}$

Projicerad i xy-planet blir rörelsen en cirkel med omloppstid $T = \frac{2\pi}{w}$. Kortast sträcka om den bara går ett varv då den går upp längs z-axeln och vänder. $\Rightarrow$

$0 = \frac{-qE_0}{2m}T^2 + v_{z0}T \Rightarrow \underline{\underline{E_0}} = \frac{2mv_{z0}}{qT} = \frac{mv_{z0}w}{\pi q} = \underline{\underline{\frac{v_{z0}B_0}{\pi}}}$

b) $\underline{\underline{\bar v(T)}} = v_{x0}\cos(wT)\hat x - v_{x0}\sin(wT)\hat y + \left(-\frac{qE_0}{m}T + v_{z0}\right)\hat z = \underline{\underline{v_{x0}\hat x - v_{z0}\hat z}}$

Svar: a) $\underline{\underline{E_0 = \frac{v_{z0}B_0}{\pi}}}$ b) $\underline{\underline{\bar v(T) = v_{x0}\hat x - v_{z0}\hat z}}$

Uppgiften går att lösa mycket snabbare och kortare genom att resonera sig fram till, och motivera, rörelsen och sedan använda kända samband för central rörelse i xy-planet och konstant acceleration i z-led.

2, Använd t.ex. symmetri + Gauss sats eller Poissons ekv. som nedan.

$$\bar\nabla\cdot\bar D=\rho,\quad \bar D=\varepsilon_0\bar E,\quad \bar E=-\bar\nabla V \Rightarrow \bar\nabla^2 V=-\rho/\varepsilon_0$$

Symmetri ⇒ $V(r)$ g° $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)=-\rho/\varepsilon_0 \Rightarrow$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)=\frac{-r\rho_0 a^5}{\varepsilon_0(r^2+a^2)^2} \Rightarrow r^2\frac{\partial V}{\partial r}=\frac{\rho_0 a^5}{2\varepsilon_0(r^2+a^2)}+C'$$

Enligt uppg. ska $\bar E=-\bar\nabla V=-\frac{\partial V}{\partial r}\hat r$ vara ändlig då $r\to 0 \Rightarrow$

$$C=-\frac{\rho_0 a^3}{2\varepsilon_0} \Rightarrow r^2\frac{\partial V}{\partial r}=\frac{\rho_0 a^3}{2\varepsilon_0}\left(\frac{a^2}{r^2+a^2}-1\right)=\frac{-\rho_0 a^3 r^2}{2\varepsilon_0(r^2+a^2)} \Rightarrow$$

$$\frac{\partial V}{\partial r}=\frac{-\rho_0 a^3}{2\varepsilon_0(r^2+a^2)} \Rightarrow V=\frac{-\rho_0 a^3}{2\varepsilon_0}\cdot\frac{1}{a}\arctan\frac{r}{a}+C'$$

$V\to 0$ då $r\to\infty \Rightarrow \underline{\underline{V=\frac{-\rho_0 a^2}{2\varepsilon_0}\left(\arctan\frac{r}{a}-\frac{\pi}{2}\right)}}$

Svar: $\underline{\underline{V(r)=\frac{-\rho_0 a^2}{2\varepsilon_0}\left(\arctan\frac{r}{a}-\frac{\pi}{2}\right)}}$

3, Sfärisk symmetri ⇒ $\bar E=E(r)\hat r$. Ansätt fri laddning Q_a på inre sfären. Gauss sats $\oint_S\bar D\cdot d\bar S=Q_{\text{fri,inne}} \Rightarrow$

$$\bar D=\frac{Q_a}{4\pi r^2}\hat r \Rightarrow \bar E=\frac{Q_a}{4\pi\varepsilon_0 r^2}\hat r$$

Fältet är starkast då $r=a \Rightarrow E_{max}=\frac{Q_a}{4\pi\varepsilon_0 a^2} \Rightarrow$

$$\bar E=\frac{a^2}{r^2}E_{max}\hat r \Rightarrow \bar D=\varepsilon_0\bar E=\varepsilon_0\frac{a^2}{r^2}E_{max}\hat r \Rightarrow$$

$$W_e=\int_{\text{"Hela }R^3\text{"}}\frac{1}{2}\bar E\cdot\bar D\,d\tau=\int_a^b\frac{1}{2}\varepsilon_0\frac{a^4}{r^4}E_{max}^2\,4\pi r^2\,dr=2\pi\varepsilon_0 a^4E_{max}^2\left[\frac{-1}{r}\right]_a^b=$$

$$=2\pi\varepsilon_0E_{max}^2a^4\left(\frac{1}{a}-\frac{1}{b}\right)=2\pi\varepsilon_0E_{max}^2\left(a^3-\frac{a^4}{b}\right)$$

Max då $\frac{dW_e}{da}=2\pi\varepsilon_0E_{max}^2\left(3a^2-\frac{4a^3}{b}\right)=0 \Rightarrow \underline{\underline{a=\frac{3}{4}b}}$

Svar: Radien på inre sfären ska vara $\underline{\underline{a=\frac{3}{4}b}}$

4) Magnetiseringsströmmar: $\bar J_m = \bar\nabla \times \bar M = \bar\nabla \times M\hat z = \bar 0$

$$\bar J_{sm} = \bar M \times \hat n = \begin{cases} M\hat z \times (-\hat z) = \bar 0 & \text{plana undersidan} \\ M\hat z \times \hat r = M\sin\theta\hat\phi & \text{halvsfären} \end{cases}$$

$\uparrow \sin\theta\hat R + \cos\theta\hat z$

$\hat\theta' = -\sin\theta'\hat z + \cancel{\cos\theta\hat R}$ ty $0 \to 2\pi$

$$\underline{\underline{\bar B(\bar r)}} = \underset{ⓧ}{\int} \frac{\mu_0 \bar J \times (\bar r - \bar r')}{4\pi|\bar r - \bar r'|^3}\,d\tau' = \int_0^{2\pi}\int_0^{\pi/2} \frac{\mu_0 M\sin\theta'\overbrace{\hat\phi' \times (\bar 0 - a\hat r')}}{4\pi\, a^3}\, a^2\sin\theta' d\theta' d\phi'$$

$$= \frac{\mu_0 M}{4\pi}\cdot 2\pi \int_0^{\pi/2} \sin\theta'(1-\cos^2\theta')\,d\theta'\hat z = \frac{\mu_0 M}{2}\left[-\cos\theta' + \frac{\cos^3\theta'}{3}\right]_0^{\pi/2}\hat z =$$

$$= \frac{\mu_0 M}{2}\cdot\frac{2}{3}\hat z = \frac{\mu_0 M}{3}\hat z = \underline{\underline{\frac{\mu_0}{3}\bar M}}$$

Svar: Magnetisk flödestätheten i origo: $\bar B = \frac{\mu_0}{3}\bar M$

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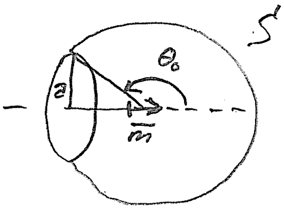
5)



$\bar B = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat r + \sin\theta\hat\theta)$

Magnetiskt flöde: $\bar\Phi = \int_S \bar B\cdot d\bar S = \int_0^{2\pi}\int_0^{\theta_0} \bar B\cdot\hat r\, r^2\sin\theta d\theta d\phi =$

$$= \int_0^{\theta_0} \frac{\mu_0 m}{4\pi r^3} 2\cos\theta\sin\theta\, r^2\cdot 2\pi\, d\theta = \frac{\mu_0 m}{2r}\left[\sin^2\theta\right]_0^{\theta_0} = \frac{\mu_0 m}{2r}\sin^2\theta_0 =$$

$$= \frac{\mu_0 m}{2r}\frac{a^2}{r^2} = \frac{\mu_0 m a^2}{2(a^2+z^2)^{3/2}} \Rightarrow \underline{\varepsilon_{emf}} = -\frac{d\bar\Phi}{dt} = -\frac{d\bar\Phi}{dz}\cdot\frac{dz}{dt} = \underline{\frac{3\mu_0 m a^2 z v}{2(a^2+z^2)^{5/2}}}$$

Sök max: $\frac{d\varepsilon_{emf}}{dz} = \frac{3\mu_0 m a^2 v}{2}\left\{\frac{1}{(a^2+z^2)^{5/2}} - \frac{5z^2}{(a^2+z^2)^{7/2}}\right\} = 0 \Rightarrow$

$\Rightarrow (a^2+z^2) - 5z^2 = 0 \Rightarrow a^2 = 4z^2 \Rightarrow \underline{z = \pm a/2} \Rightarrow$

$$\underline{\underline{\varepsilon_{emf,max}}} = \pm\frac{3\mu_0 m a^2 a v}{4(a^2+(a/2)^2)^{5/2}} = \pm\frac{3\mu_0 m v\cdot 2^5}{4a^2 5^{5/2}} = \underline{\underline{\pm\frac{24\mu_0 m v}{25\sqrt5\, a^2}}}$$

Svar: Maximal inducerad elektromotorisk spänning:

$$\underline{\underline{\varepsilon_{emf} = \frac{24\mu_0 m v}{25\sqrt5\, a^2}}}$$