Tentamen i Elektromagnetism, TFYA13, 2011-08-17

1) Ansätt en ström I i positiv $\hat{z}$-riktning.

$$\left.\begin{array}{l} \text{Symmetri m.m.} \Rightarrow \bar{H} = H(R)\,\hat{\phi} \\ \text{Cirkulationssatsen } \oint_C \bar{H}\cdot d\bar{l} = I_{omsl.fri} \\ C\text{: Cirkel med radie } R,\ 0<R<a, \text{ runt } z\text{-axeln} \end{array}\right\} \Rightarrow$$

$$\int_0^{2\pi} H(R)\,\hat{\phi}\cdot\hat{\phi}\,R\,d\phi = 2\pi R\,H(R) = I_{omsl.fri} = I\,\frac{\pi R^2}{\pi a^2} = I\,R^2/a^2 \Rightarrow$$

$$\left.\begin{array}{l} \bar{H} = \dfrac{IR}{2\pi a^2}\,\hat{\phi} \\ \bar{B} = \mu_0\mu_r\bar{H} = \mu_0(\bar{H}+\bar{M}) \Rightarrow \bar{M} = (\mu_r - 1)\bar{H} \end{array}\right\} \Rightarrow \bar{M} = \frac{(\mu_r-1)IR}{2\pi a^2}\,\hat{\phi}$$

$$\bar{J}_{sm} = \bar{M}\times\hat{n} = \frac{(\mu_r-1)I a}{2\pi a^2}\,\hat{\phi}\times\hat{R} = -\frac{(\mu_r-1)I}{2\pi a}\,\hat{z} = J_{smo}\,\hat{z}$$

$$\therefore\ J_{smo} = \frac{-(\mu_r-1)I}{2\pi a} \Rightarrow \underline{\underline{I = \frac{-2\pi a}{(\mu_r-1)}\,J_{smo}}}$$

<u>Svar: Strömmen som går i ledaren är $\dfrac{2\pi a}{(\mu_r-1)}J_{smo}$ i negativ $\hat{z}$-riktning</u>

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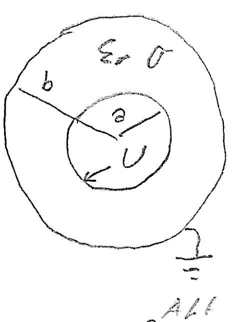
2) Sfär



$$\left.\begin{array}{l} \text{Ansätt en ström } I \text{ från inre sfären mot yttre. Symmetri} \Rightarrow \bar{J} = J(r)\,\hat{r}. \\ \oint_S \bar{J}\cdot d\bar{S} = I,\ S = \text{sfär med radie } a<r<b \end{array}\right\} \Rightarrow$$

$$4\pi r^2 J(r) = I \Rightarrow \bar{J} = \frac{I}{4\pi r^2}\,\hat{r} \Rightarrow \bar{E} = \bar{J}/\sigma = \frac{I}{4\pi\sigma r^2}\,\hat{r}$$

$$U = \int_{Ref}^{Alt} -\bar{E}\cdot d\bar{l} = \int_b^a \frac{-I}{4\pi\sigma r^2}\,\hat{r}\cdot\hat{r}\,dr = \frac{I}{4\pi\sigma}\left(\frac{1}{a}-\frac{1}{b}\right) \Rightarrow \bar{E} = \frac{abU}{(b-a)r^2}\,\hat{r}$$

$$\bar{D} = \varepsilon_0\varepsilon_r\bar{E} = \frac{\varepsilon_0 k r\, abU}{(b-a)r^2}\,\hat{r}\ ;\ \rho_{fri} = \bar{\nabla}\cdot\bar{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 D_r\right) + 0 + 0 \Rightarrow$$

$$\underline{\rho_{fri}} = \frac{\varepsilon_0 k\, abU}{(b-a)r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{r}\right) = \underline{\underline{\frac{\varepsilon_0 k\, abU}{(b-a)r^2}\ ;\ a<r<b}}$$

<u>Svar: Laddningstätheten är $\rho_{fri} = \dfrac{\varepsilon_0 k\, abU}{(b-a)r^2}$; $a<r<b$</u>

3) $\bar{B} = \frac{V_0}{r}\frac{k}{\omega}\sin\theta\cos(\omega t - kr)\hat{\theta}$

$\bar{\nabla}\times\bar{H} = \bar{J} + \frac{\partial\bar{D}}{\partial t}$; $\bar{J} = \bar{0}$; $\bar{B} = \mu_0\bar{H}$; $\bar{D} = \varepsilon_0\bar{E} \Rightarrow \bar{\nabla}\times\bar{B} = \varepsilon_0\mu_0\frac{\partial\bar{E}}{\partial t}$

$\bar{\nabla}\times\bar{B} = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)\hat{\phi} = \frac{V_0 k^2}{r\omega}\sin\theta\sin(\omega t - kr)\hat{\phi} = \varepsilon_0\mu_0\frac{\partial\bar{E}}{\partial t} \Rightarrow$

$\bar{E} = \frac{-V_0 k^2}{\varepsilon_0\mu_0 r\omega^2}\sin\theta\cos(\omega t - kr)\hat{\phi} + \bar{E}_0$ tidsober. sätt till $\bar{0}$

$\bar{P} = \bar{E}\times\bar{H} = \bar{E}\times\bar{B}/\mu_0 = \frac{V_0^2 k^3}{\varepsilon_0\mu_0^2\omega^3 r^2}\sin^2\theta\cos^2(\omega t - kr)\overbrace{(-\hat{\phi})\times\hat{\theta}}^{\hat{r}}$

Momentant utstrålad effekt $P(t) = \oint_S \bar{P}(t)\,d\bar{S}$

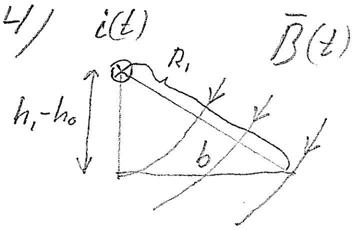
$$P(t) = \int_0^{2\pi}\int_0^{\pi}\frac{V_0^2 k^3}{\varepsilon_0\mu_0^2\omega^3\cancel{r^2}}\underbrace{\sin^2\theta}_{1-\cos^2\theta}\cos^2(\omega t - kr)\,\hat{r}\cdot\hat{r}\,\cancel{r^2}\sin\theta\,d\theta\,d\phi =$$

$$= \frac{V_0^2 k^3}{\varepsilon_0\mu_0^2\omega^3}\,2\pi\underbrace{\left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^{\pi}}_{4/3}\cos^2(\omega t - kr) =$$

$$= \frac{8\pi V_0^2 k^3}{3\varepsilon_0\mu_0^2\omega^3}\cos^2(\omega t - kr) = \frac{8\pi V_0^2}{3\mu_0 c_0}\cos^2(\omega t - kr)$$

Svar: Den momentant utstrålade effekten är

$$P(t) = \frac{8\pi}{3}\frac{V_0^2}{\mu_0 c_0}\cos^2(\omega t - kr)$$

4) 

Inför en z-axel längs den strömförande ledaren. Symmetri m.m. $\Rightarrow \bar{H} = H(R)\hat{\phi}$
Cirkulationssatsen $\oint_C \bar{H}\cdot d\bar{l} = I_{oms,fri}$ $\Rightarrow$

$$\bar{H} = \frac{i(t)}{2\pi R}\hat{\phi} \Rightarrow \bar{B} = \frac{\mu_0 i(t)}{2\pi R}\hat{\phi}$$

Magnetiska flödet genom kabagen Φ ges med stöd av figuren ovan av:

$$\Phi = \int_S\int \bar{B}\cdot d\bar{S} = \int_0^a\int_{h_1-h_0}^{R_1} \frac{\mu_0 i(t)}{2\pi R}\hat{\phi}\cdot\hat{\phi}\, dR\, dz = \frac{a\mu_0 i(t)}{2\pi}\ln\frac{\sqrt{b^2+(h_1-h_0)^2}}{h_1-h_0}$$

$$\mathcal{E}_{emk} = -\frac{d\Phi}{dt} = \frac{-\omega a\mu_0 i_0\cos\omega t}{2\pi}\ln\frac{\sqrt{b^2+(h_1-h_0)^2}}{h_1-h_0}$$

Beräkna resistansen i kabagen, R_Ω.

Antag en ström $I \Rightarrow J = I/S_0 \Rightarrow E = J/\sigma = \frac{I}{\sigma S_0} \Rightarrow$

$$U = 2(a+b)E = \frac{2(a+b)}{\sigma S_0}I \Rightarrow R_\Omega = \frac{U}{I} = \frac{2(a+b)}{\sigma S_0}$$

Effekten i kabagen: $P(t) = \mathcal{E}^2/R_\Omega$

Tidsmedelvärde: $P = \langle P(t)\rangle = \frac{1}{T}\int_0^T P(t)\,dt =$

$$= \frac{\sigma S_0}{2(a+b)}\left(\frac{\omega a\mu_0 i_0}{2\pi}\ln\frac{\sqrt{b^2+(h_1-h_0)^2}}{\sqrt{(h_1-h_0)^2}}\right)^2 \underbrace{\frac{1}{T}\int_0^T \underbrace{\cos^2\omega t}_{\frac{1}{2}[1+\cos(2\omega t)]}\,dt}_{1/2} =$$

$$= \frac{\sigma S_0}{4(a+b)}\left(\frac{\omega a\mu_0 i_0}{4\pi}\ln\left[1+\left(\frac{b}{h_1-h_0}\right)^2\right]\right)^2 \approx 0.22\text{ W}$$

Svar: Effektförlusten blir 0,22 W och är därmed ointressant att debitera.

5) Beräkna vilka polarisationsladdningar $\bar{P}$ ger upphov till.

På övre ytan $\varrho_{sp\ddot{o}} = \bar{P}\cdot\hat{n} = P\hat{z}\cdot\hat{z} = P$

undre ytan $\varrho_{spu} = \bar{P}\cdot\hat{n} = P\hat{z}\cdot(-\hat{z}) = -P$

mantelytan $\varrho_{spm} = \bar{P}\cdot\hat{n} = P\hat{z}\cdot\hat{R} = 0$

inuti $\varrho_p = -\bar{\nabla}\cdot\bar{P} = 0$ ty $\bar{P}$ är konstant.

Ytladdningstätheterna $\varrho_{sp\ddot{o}}$ och ϱ_{spu} ger upphov till ytströmtätheter då pucken roterar,

"$\bar{J}_s = \varrho_s\bar{v}$" (Jfr $\bar{J} \equiv nq\bar{v} = \varrho\bar{v}$) $\Rightarrow \bar{J}_{s\ddot{o}} = \varrho_{sp\ddot{o}} R\omega\hat{\phi} = PR\omega\hat{\phi}$,

$\bar{J}_{su} = -PR\omega\hat{\phi}$

a) Magnetisk vektorpotential $\bar{A}(\bar{r}) = \int \frac{\mu_0 \bar{J}_s(\bar{r}')\,dS'}{4\pi|\bar{r}-\bar{r}'|}$

$$\bar{A}(z\hat{z}) = \int_0^a\int_0^{2\pi} \frac{\mu_0 PR'\omega\hat{\phi}'}{4\pi\sqrt{R'^2+(z-h)^2}} - \frac{\mu_0 PR'\omega\hat{\phi}'}{4\pi\sqrt{R'^2+(z+h)^2}}\,R'd\phi'dR'$$

$$= \bar{0} \quad \text{(Inses lätt pga symmetrin.)}$$

b)
$$\bar{B}(z\hat{z}) = \int \frac{\mu_0\bar{J}_s\times(\bar{r}-\bar{r}')}{4\pi|\bar{r}-\bar{r}'|^3}\,dS' = \int_{\text{övre ytan}} \frac{\mu_0\bar{J}_{sp\ddot{o}}\times[-R'\hat{R}'+(z-h)\hat{z}]}{4\pi(R'^2+(z-h)^2)^{3/2}}\,dS' +$$

$$+\int_{\text{undre ytan}} \frac{\mu_0\bar{J}_{spu}\times[-R'\hat{R}'+(z+h)\hat{z}]}{4\pi(R'^2+(z+h)^2)^{3/2}}\,dS' =$$

$\Rightarrow \bar{0}$ ty $0 \to 2\pi$ $\quad\Rightarrow \bar{0}$ ty $0 \to 2\pi$

$$= \frac{\mu_0 P\omega}{4\pi}\int_0^a\int_0^{2\pi}\left\{\frac{R'[R'\hat{z}+(z-h)\hat{R}']}{(R'^2+(z-h)^2)^{3/2}} - \frac{R'[R'\hat{z}+(z+h)\hat{R}']}{(R'^2+(z+h)^2)^{3/2}}\right\}R'd\phi'dR' =$$

$$= 2\pi\frac{\mu_0 P\omega}{4\pi}\int_0^a\left\{\frac{R'^3}{[R'^2+(z-h)^2]^{3/2}} - \frac{R'^3}{(R'^2+(z+h)^2)^{3/2}}\right\}dR'\hat{z} \Rightarrow$$

$$\underline{\underline{\bar{B} = \frac{\mu_0 P\omega}{2}\left\{2\sqrt{a^2+(z-h)^2} - \frac{a^2}{\sqrt{a^2+(z-h)^2}} - 2|z-h| - \right.}}$$

$$\underline{\underline{\left. - 2\sqrt{a^2+(z+h)^2} + \frac{a^2}{\sqrt{a^2+(z+h)^2}} + 2|z+h|\right\}\hat{z}}}$$

Svar: a) $\bar{A} = \bar{0}$; b) Se ovan.