Tentamen i Elektromagnetism, TFYA13, 2011-05-31

1, Ansätt en cylindersymmetrisk ström i jorden $\bar{J} = J(R)\hat{R}$ i en halvcylinder längs den nedfallna ledaren. $I = \int \bar{J}\cdot d\bar{S} = J(R)\cdot \underbrace{\frac{2\pi RL}{2}}_{\text{Halvcylinder radie } R} = \pi R L J(R)$

$\Rightarrow \bar{J} = \frac{I}{\pi R L}\hat{R} \Rightarrow \bar{E} = \bar{J}/\sigma = \frac{I\hat{R}}{\pi\sigma L R}$

$$U = \int_{Ref}^{Akt} -\bar{E}\cdot d\bar{l} = \int_b^a \frac{-I}{\pi\sigma L R}\hat{R}\cdot\hat{R}\,dR = \frac{I}{\pi\sigma L}\ln b/a \approx 1103\text{ V}$$

Svar: Effektivvärdet av spänningen mellan fåltarna blir $\frac{I}{\pi\sigma L}\ln b/a \approx 1{,}1\text{ kV}$

2, Sfärisk symmetri $\Rightarrow \bar{D} = D(r)\hat{r}$
Ansätt laddning Q på inre sfären
Gauss sats: $\oint_S \bar{D}\cdot d\bar{S} = Q_{fri,innes}$
$\Rightarrow 4\pi r^2 D(r) = Q$

$\therefore \bar{D} = \frac{Q}{4\pi r^2}\hat{r} \Rightarrow \bar{E} = \bar{D}/\varepsilon_0\varepsilon_r = \frac{Q}{4\pi\varepsilon_0\alpha r}\hat{r} \Rightarrow$

$$W_e = \int_{\text{"Hela } R^3\text{"}} \frac{1}{2}\bar{D}\cdot\bar{E}\,d\tau = \underbrace{\int_a^b \frac{1}{2}\frac{Q}{\cancel{4\pi r^2}}\cdot\frac{Q}{4\pi\varepsilon_0\alpha r}\cancel{4\pi r^2}\,dr}_{\text{mellan sfärerna}} + \underset{\substack{\text{allt annat}\\ \bar{E},\bar{D}=\bar{0}}}{0} =$$

$$= \frac{Q^2}{8\pi\varepsilon_0\alpha}\ln b/a$$

$$U = \int_{Ref}^{Akt} -\bar{E}\cdot d\bar{l} = \int_b^a -\frac{Q}{4\pi\varepsilon_0\alpha r}\hat{r}\cdot\hat{r}\,dr = \frac{Q}{4\pi\varepsilon_0\alpha}\ln b/a \Rightarrow$$

$$Q = \frac{4\pi\varepsilon_0\alpha}{\ln(b/a)}U \Rightarrow W_e = \frac{2\pi\varepsilon_0\alpha}{\ln(b/a)}U^2$$

Svar: Arbetet är $W_e = \frac{2\pi\varepsilon_0\alpha}{\ln(b/a)}U^2$

3) Magnetfältet orsakat av strömmen I_0 är cylindersymmetriskt $\Rightarrow \bar{H} = H(R)\hat{\phi}$

Cirkulationssatsen $\oint_C \bar{H}\cdot d\bar{l} = I_{fri\,omsl.}$ $\Rightarrow$

$2\pi R H(R) = I_0 \Rightarrow \bar{H} = \frac{I_0}{2\pi R}\hat{\phi}$ vilket vid den kvadratiska slingan kan skrivas $\bar{H} = \frac{I_0}{2\pi y}(-\hat{x})$

$\Rightarrow \bar{B} = \frac{\mu_0 I_0}{2\pi y}(-\hat{x})$

Flödet genom slingan $\Phi = \int_S \bar{B}\cdot d\bar{S} = \int_S \frac{\mu_0 I_0}{2\pi y}(-\hat{x})(-\hat{x})dS \Rightarrow$

$$\Phi = \frac{\mu_0 I_0}{2\pi} a \int_{y_0}^{y_0+a} \frac{1}{y}dy = \frac{\mu_0 I_0 a}{2\pi} \ln\frac{y_0+a}{y_0}$$

$$\underline{\underline{\mathcal{E}_{emk}}} = -\frac{d\Phi}{dt} = -\frac{d\Phi}{dy}\cdot\frac{dy}{dt} = -\frac{\mu_0 I_0 a}{2\pi}\left(\frac{1}{y_0+a} - \frac{1}{y_0}\right)\cdot(-v) =$$

$$= \underline{\underline{\frac{-\mu_0 I_0 a^2 v}{2\pi y_0(y_0+a)}}}$$

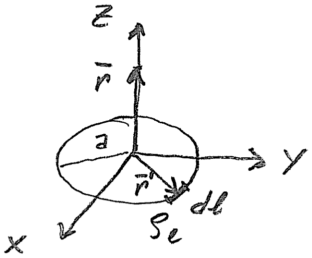
b) Vid kraftjämvikt är effektutvecklingen i slingan, $\mathcal{E}_{emk}^2/R_\Omega$ lika stor som minskningen i potentiell energi per tidsenhet, mgv. $\Rightarrow$

$$\left(\frac{\mu_0 I_0 a^2 v}{2\pi y_0(y_0+a)}\right)^2 \frac{1}{R_\Omega} = mgv \Rightarrow v = mgR_\Omega\left(\frac{2\pi y_0(y_0+a)}{\mu_0 I_0 a^2}\right)^2$$

Svar a) $\underline{\underline{\mathcal{E}_{emk} = \frac{-\mu_0 I_0 a^2 v}{2\pi y_0(y_0+a)}}}$

b) $\underline{\underline{v = mgR_\Omega\left(\frac{2\pi y_0(y_0+a)}{\mu_0 I_0 a^2}\right)^2}}$

4)



$$\bar{E}(\bar{r}) = \oint_{C} \frac{\rho_l(\bar{r}')(\bar{r}-\bar{r}')}{4\pi\varepsilon_0|\bar{r}-\bar{r}'|^3}\,dl' \;; \quad \begin{cases} \bar{r} = z\hat{z} \\ \bar{r}' = a\hat{R} \\ dl' = a\,d\phi' \\ \hat{R} = \cos\phi'\hat{x} + \sin\phi'\hat{y} \\ \rho_l = \rho_0 \sin\phi' \end{cases}$$

$$\sin^2\phi' = \tfrac{1}{2}(1-\cos 2\phi')$$

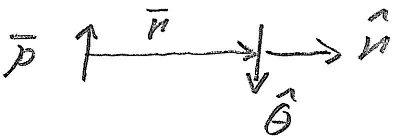
$$\underline{\underline{\bar{E}(z\hat{z})}} = \int_0^{2\pi} \frac{\rho_0 \sin\phi'(z\hat{z} - a\cos\phi'\hat{x} - a\sin\phi'\hat{y})}{4\pi\varepsilon_0[a^2+z^2]^{3/2}}\,a\,d\phi' =$$

$$= \frac{a\rho_0}{4\pi\varepsilon_0[a^2+z^2]^{3/2}}\left[-z\cos\phi'\hat{z} + \frac{a}{2}\cos^2\phi'\hat{x} - \frac{a}{2}\left(\phi' - \frac{\sin 2\phi'}{2}\right)\hat{y}\right]_0^{2\pi} =$$

$$= \frac{a\rho_0}{4\pi\varepsilon_0[a^2+z^2]^{3/2}}\left[0\hat{z} + 0\hat{x} - \frac{a}{2}\cdot 2\pi\hat{y}\right] = \underline{\underline{\frac{\rho_0 a^2}{4\varepsilon_0[a^2+z^2]^{3/2}}(-\hat{y})}}$$

b) Fältet på stort avstånd från en el. dipol ges av $\bar{E}_{\substack{d \\ \uparrow \\ \text{dipol}}} = \frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$

I vårt fall är fältet vinkelrätt mot vektorn från dipolen till fältpunkten, dvs $\theta = 90°$.



Jämför koord. syst. $\Rightarrow \begin{cases} (-\hat{y}) = \hat{\theta} \\ \bar{p} = p\hat{y} \\ r = z \end{cases}$

$$\left.\begin{aligned} &\Rightarrow \bar{E}_d = \frac{p}{4\pi\varepsilon_0 z^3}(\bar{0} + (-\hat{y})) = \frac{p}{4\pi\varepsilon_0 z^3}(-\hat{y}) \\ &\bar{E}(z\hat{z}, |z| \gg a) \approx \frac{\rho_0\pi a^2}{4\pi\varepsilon_0 z^3}(-\hat{y}) \end{aligned}\right\} \Rightarrow p = \rho_0\pi a^2$$

$$\underline{\underline{\bar{p} = \rho_0\pi a^2\hat{y}}}$$

Svar: a) $\underline{\underline{\bar{E}(z\hat{z}) = \frac{\rho_0 a^2}{4\varepsilon_0[a^2+z^2]^{3/2}}(-\hat{y})}}$

b) Motsvarande el. dipol är $\bar{p} = \rho_0\pi a^2\hat{y}$

(Jämför med en mer generell definition av el. dipol:
$\bar{p} \equiv \int \rho(\bar{r})\bar{r}\,d\tau = \int \rho_l(\bar{r})\bar{r}\,dl = \rho_0\pi a^2\hat{y}$)

5) a) Ta divergensen av ekvation (2):

$$\bar{\nabla}\cdot[\bar{\nabla}\times\bar{E}] = \bar{\nabla}\cdot(-\mu_0\bar{J}_m) - \bar{\nabla}\cdot\left(\frac{\partial\bar{B}}{\partial t}\right) \Rightarrow$$

$$0 = -\mu_0\bar{\nabla}\cdot\bar{J}_m - \frac{\partial}{\partial t}(\bar{\nabla}\cdot\bar{B}) \overset{(3)}{\Longrightarrow}$$

$$0 = -\mu_0\bar{\nabla}\cdot\bar{J}_m - \frac{\partial}{\partial t}(\mu_0\rho_m) \Rightarrow \underline{\underline{\bar{\nabla}\cdot\bar{J}_m + \frac{\partial\rho_m}{\partial t} = 0}}$$

b) $\bar{\nabla}\cdot\bar{E}' = \frac{1}{\varepsilon_0}\rho_e'$ ska visas. $\underline{\underline{\bar{\nabla}\cdot\bar{E}'}} = \{(5)\} =$

$$= \bar{\nabla}\cdot(\bar{E}\cos\alpha + c\bar{B}\sin\alpha) = (\bar{\nabla}\cdot\bar{E})\cos\alpha + c(\bar{\nabla}\cdot\bar{B})\sin\alpha =$$

$$= \{(1)\ o\ (3)\} = \frac{1}{\varepsilon_0}\rho_e\cos\alpha + c\mu_0\rho_m\sin\alpha\cdot\frac{\varepsilon_0}{\varepsilon_0} = \left\{c^2 = \frac{1}{\varepsilon_0\mu_0}\right\} =$$

$$= \frac{1}{\varepsilon_0}\left\{\rho_e\cos\alpha + \frac{c}{c^2}\rho_m\sin\alpha\right\}\cdot\frac{c}{c} = \{(9)\} = \frac{1}{\varepsilon_0 c}\cdot c\rho_e' = \underline{\underline{\rho_e'/\varepsilon_0}}$$

$\underline{\underline{\bar{\nabla}\times\bar{E}'}} = -\mu_0\bar{J}_m' - \frac{\partial\bar{B}'}{\partial t}$ ska visas. $\bar{\nabla}\times\bar{E}' = \{(5)\} =$

$$= \bar{\nabla}\times(\bar{E}\cos\alpha + c\bar{B}\sin\alpha) = (\bar{\nabla}\times\bar{E})\cos\alpha + c(\bar{\nabla}\times\bar{B})\sin\alpha =$$

$$= \{(2)\ o\ (4)\} = -\mu_0\bar{J}_m\cos\alpha - \frac{\partial\bar{B}}{\partial t}\cos\alpha + c\mu_0\bar{J}_e\sin\alpha +$$

$$+ c\varepsilon_0\mu_0\frac{\partial\bar{E}}{\partial t}\sin\alpha = -\mu_0(\bar{J}_m\cos\alpha - c\bar{J}_e\sin\alpha) -$$

$$- \frac{\partial}{\partial t}\left(\bar{B}\cos\alpha - \frac{c}{c^2}\bar{E}\sin\alpha\right)\cdot\frac{c}{c} = \{(12)\ o\ (6)\} =$$

$$= -\mu_0\bar{J}_m' - \frac{\partial}{\partial t}\left(\frac{c\bar{B}'}{c}\right) = \underline{\underline{-\mu_0\bar{J}_m' - \frac{\partial\bar{B}'}{\partial t}}}$$

<u>Svar: a) Se bevis ovan</u>

<u>b) Se bevis ovan</u>