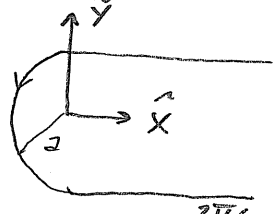
Elektromagnetism, TFYA13, 2011-01-14

1) I



$$\bar{A}(\bar{r}) = \int \frac{\mu_0 I\, d\bar{l}'}{4\pi|\bar{r}-\bar{r}'|}$$

Ordlista: $\begin{cases} \bar{r} = \bar{0}, \quad \bar{r}' = a\hat{R}' \\ |\bar{r}-\bar{r}'| = a \\ d\bar{l}' = a\hat{\phi}' d\phi' \\ \hat{\phi}' = \cos\phi'\hat{y} - \sin\phi'\hat{x} \\ \frac{\pi}{2} < \phi' < \frac{3\pi}{2} \end{cases}$

$$\underline{\underline{\bar{A}(\bar{0})}} = \int_{\pi/2}^{3\pi/2} \frac{\mu_0 I a[\cos\phi'\hat{y} - \sin\phi'\hat{x}]}{4\pi a} d\phi' =$$

$$= \frac{\mu_0 I}{4\pi}\left[\sin\phi'\hat{y} + \cos\phi'\hat{x}\right]_{\pi/2}^{3\pi/2} = \frac{\mu_0 I}{4\pi}(-2\hat{y} + 0\hat{x}) = \underline{\underline{-\frac{\mu_0 I}{2\pi}\hat{y}}}$$

b) $$\underline{\underline{\bar{B}(\bar{r})}} = \int \frac{\mu_0 I\, d\bar{l}' \times (\bar{r}-\bar{r}')}{4\pi|\bar{r}-\bar{r}'|^3} = \int_{\pi/2}^{3\pi/2} \frac{\mu_0 I[a\hat{\phi}' \times (\bar{0} - a\hat{R}')]}{4\pi a^3} d\phi' =$$

$$= \frac{\mu_0 I}{4\pi a}\int_{\pi/2}^{3\pi/2} d\phi'\hat{z} = \frac{\mu_0 I}{4\pi a}\left[\phi'\right]_{\pi/2}^{3\pi/2}\hat{z} = \underline{\underline{\frac{\mu_0 I}{4a}\hat{z}}}$$

Svar a) $\bar{A}(\bar{0}) = -\frac{\mu_0 I}{2\pi}\hat{y}$ b) $\bar{B}(\bar{0}) = \frac{\mu_0 I}{4a}\hat{z}$

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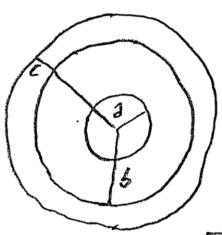
2)



Cylindersym. $\Rightarrow \bar{H} = H(R)\hat{\phi}$

Cirkulationssatsen: $\oint_C \bar{H}\cdot d\bar{l} = I_{fri,oms.}$

$\Rightarrow 2\pi R H(R) = I_{fri,oms}$

$$\Rightarrow \bar{H} = \frac{I_{fri,oms}}{2\pi R}\hat{\phi} = \left\{ I_{fri,oms} = \begin{cases} I\frac{\pi R^2}{\pi a^2} = I\frac{R^2}{a^2} & ; 0 \le R \le a \\ I & ; a \le R \le b \\ I - \frac{\pi R^2 - \pi b^2}{\pi c^2 - \pi b^2} I = I\frac{c^2-R^2}{c^2-b^2} & ; b \le R \le c \\ 0 & ; c \le R \end{cases} \right\} \Rightarrow$$

$$\bar{H} = \begin{cases} \frac{IR}{2\pi a^2}\hat{\phi} & 0 \le R \le a \\ \frac{I}{2\pi R}\hat{\phi} & a \le R \le b \\ \frac{I}{2\pi R}\cdot\frac{c^2-R^2}{c^2-b^2}\hat{\phi} & b \le R \le c \\ \bar{0} & c \le R \end{cases}$$

b) $\bar{B} = \mu_0\mu_r\bar{H} = \mu_0(\bar{H} + \bar{M}) \Rightarrow \bar{M} = (\mu_r - 1)\bar{H}$

$\bar{J}_m = \bar{\nabla}\times\bar{M}$, $\bar{J}_{sm} = \bar{M}\times\hat{n}$; $\bar{\nabla}\times\bar{M} = \frac{1}{R}\frac{\partial}{\partial R}(RM_\phi)\hat{z}$

2 b forts)

$$\underline{\underline{\bar{J}_m}} = \bar{\nabla} \times \bar{M} = \begin{cases} \dfrac{I(\mu_r - 1)}{2\pi a^2 \cdot R} \dfrac{\partial R^2}{\partial R} \hat{z} = \dfrac{(\mu_r - 1) I}{\pi a^2} & 0 \le R < a \\ \dfrac{I}{2\pi R} \dfrac{\partial}{\partial R}\left(\dfrac{R}{R}\right) \hat{z} = \bar{0} & a < R < b \\ \dfrac{I(\mu_r - 1)}{2\pi R} \dfrac{\partial}{\partial R}\left(\dfrac{c^2 - R^2}{c^2 - b^2}\right) \hat{z} = \dfrac{(\mu_r - 1) I}{\pi (c^2 - b^2)} (-\hat{z}) & b < R < c \\ \bar{0} & c < R \end{cases}$$

$$\underline{\underline{\bar{J}_{sm}}} = \begin{cases} \dfrac{(\mu_r - 1) I}{2\pi a} \hat{\phi} \times \hat{R} = \dfrac{-(\mu_r - 1) I}{2\pi a} \hat{z} & R = a \\ \dfrac{(\mu_r - 1) I}{2\pi b} \cdot \dfrac{c^2 - b^2}{c^2 - b^2} \hat{\phi} \times (-\hat{R}) = \dfrac{(\mu_r - 1) I}{2\pi b} \hat{z} & R = b \\ \dfrac{(\mu_r - 1)}{2\pi c} \cdot \dfrac{c^2 - c^2}{c^2 - b^2} \hat{\phi} \times \hat{R} = \bar{0} & R = c \end{cases}$$

Svar: Se ovan.

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3) Ansätt laddningen Q på längden L av den inre ledaren.

Cylinder sym. $\Rightarrow \bar{D} = D(R)\hat{R}$

Gauss sats $\oint \bar{D} \cdot d\bar{S} = Q_{fri,\,innes}$

$\Rightarrow \bar{D} = \dfrac{Q}{2\pi R L} \hat{R} \qquad a < R < c$

$$\Rightarrow \begin{cases} \bar{E}_p = \bar{D}/\varepsilon_0 \varepsilon_r = \dfrac{Q}{2\pi \varepsilon_0 \varepsilon_r R L} \hat{R} & a < R < b \\ \bar{E}_l = \bar{D}/\varepsilon_0 = \dfrac{Q}{2\pi \varepsilon_0 R L} \hat{R} & b < R < c \end{cases}$$

Vi ser att E-fälten ökar med ökande Q och minskande R och att $\bar{E}_l > \bar{E}_p$ för givet Q och R, dvs vi vill ha så mycket luft som möjligt.

Villkor för genomslag i luft $\Rightarrow E_{lmax} \ge \dfrac{Q}{2\pi \varepsilon_0 b L} \Rightarrow \dfrac{Q}{2\pi \varepsilon_0 L} \le b E_{lmax}$

$$\Rightarrow \begin{cases} \bar{E}_p = \dfrac{b E_{lmax}}{\varepsilon_r R} \hat{R} \\ \bar{E}_l = \dfrac{b E_{lmax}}{R} \hat{R} \end{cases}$$

Villkoret för att vi inte ska få genomslag i plexiglaset ger

$\dfrac{b E_{lmax}}{\varepsilon_r a} < E_{pmax} \Rightarrow$

$b \le \varepsilon_r a \dfrac{E_{pmax}}{E_{lmax}} = 20\,\text{mm}$

3 forts) Undersök för vilket b i intervallet $1mm < b < 20mm$ spänningen blir maximal.

$$U = \int_c^a -\bar{E}\cdot d\bar{l} = \int_c^b -\bar{E}_l\cdot\hat{R}\,dR + \int_b^a -\bar{E}_p\cdot\hat{R}\,dR = bE_{lmax}\ln\frac{c}{b} + \frac{bE_{lmax}}{\varepsilon_r}\ln\frac{b}{a}$$

$$\frac{dU}{db} = E_{lmax}\left\{\ln\frac{c}{b} - 1 + \frac{1}{\varepsilon_r}\ln\frac{b}{a} + \frac{1}{\varepsilon_r}\right\} = 0 \Rightarrow \ln b = \frac{\ln a - \varepsilon_r \ln c}{1-\varepsilon_r} - 1$$

$\Rightarrow b \approx 60.4\,mm$ dvs utanför intervallet.

Undersök gränserna.

$b = 1\,mm \Rightarrow U \approx 10{,}2\,kV$, $b = 20\,mm \Rightarrow U \approx 84{,}2\,kV$

Svar: a) Högsta spänningen får man om $b = 20\,mm$.

b) Högsta spänningen blir $84\,kV$

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4) $\bar{P} = \bar{E}\times\bar{H}$ dvs vi behöver $\bar{H}$; $\bar{\nabla}\times\bar{E} = -\frac{\partial\bar{B}}{\partial t}$; $\bar{B} = \mu_0\bar{H}$

$$\bar{\nabla}\times\bar{E} = \frac{\partial E_Y}{\partial x}\hat{z} - \frac{\partial E_Y}{\partial z}\hat{x} = \frac{\pi}{a}E_0\cos\left(\frac{\pi x}{a}\right)\sin(\omega t - k_z z)\hat{z} +$$

$$+ k_z E_0 \sin\left(\frac{\pi x}{a}\right)\cos(\omega t - k_z z)\hat{x} = -\frac{\partial\bar{B}}{\partial t} = -\mu_0\frac{\partial\bar{H}}{\partial t} \Rightarrow$$

$$\bar{H} = \frac{E_0}{\mu_0}\left\{\frac{\pi}{a\omega}\cos\left(\frac{\pi x}{a}\right)\cos(\omega t - k_z z)\hat{z} - \frac{k_z}{\omega}\sin\left(\frac{\pi x}{a}\right)\sin(\omega t - k_z z)\hat{x}\right\} \Rightarrow$$

$$\bar{P} = \bar{E}\times\bar{H} = \frac{E_0^2}{\mu_0\omega}\left\{\frac{\pi}{4a}\sin\frac{2\pi x}{a}\sin[2(\omega t - k_z z)]\hat{x} + k_z\sin^2\left(\frac{\pi x}{a}\right)\sin^2(\omega t - k_z z)\hat{z}\right\}$$

b) Tidsmedelvärde: $\langle\bar{P}\rangle = \frac{1}{T}\int_0^T \bar{P}(t)\,dt$ men $\left\{\begin{array}{l}\langle \sin t\rangle = 0\\ \langle \sin^2 t\rangle = \frac{1}{2}\end{array}\right\}$

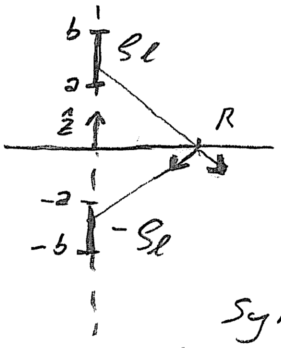
(välkänt / visas lätt) $\Rightarrow \langle\bar{P}\rangle = \frac{1}{2}\frac{E_0^2 k_z}{\mu_0\omega}\sin^2\frac{\pi x}{a}\hat{z}$

c) $$P = \int_0^a\int_0^b \langle\bar{P}\rangle\cdot\hat{z}\,dx\,dy = \frac{1}{2}\frac{E_0^2 k_z}{\mu_0\omega}\int_0^b dy \int_0^a \underbrace{\sin^2\frac{\pi x}{a}}_{\frac{1}{2}\left(1-\cos\frac{2\pi x}{a}\right)} dx =$$

$$= \frac{1}{2}\frac{E_0^2 k_z}{\mu_0\omega}\cdot b\cdot\frac{a}{2} = \frac{abk_z}{4\mu_0\omega}E_0^2$$

Svar: a) Se ovan b) $\langle\bar{P}\rangle = \frac{1}{2}\frac{E_0^2 k_z}{\mu_0\omega}\sin^2\frac{\pi x}{a}\hat{z}$ c) $P = \frac{abk_z}{4\mu_0\omega}E_0^2$

5/ Använd spegelladdningsmetoden och inför en "spegelstav".



Beräkna $\bar E$-fältet i planet på avstånd R från symmetrilinjen.

$\bar r = R\hat R$

$$\bar E(R\hat R) = \int \frac{(\bar r - \bar r')\,dQ'}{4\pi\varepsilon_0 |\bar r - \bar r'|^3} \quad ; \quad \bar r' = z'\hat z \text{ och } -z'\hat z$$

$dQ' = \varrho_\ell dz'$ och $-\varrho_\ell dz'$

Symmetri ger att bara $\hat z$ komponenten överlever.

$$\bar E(R\hat R) = \int_a^b \frac{-2z'\varrho_\ell\,dz'}{4\pi\varepsilon_0 [R^2 + (z')^2]^{3/2}}\hat z = \frac{\varrho_\ell}{2\pi\varepsilon_0}\left[\frac{1}{\sqrt{R^2+(z')^2}}\right]_a^b \hat z =$$

$$= \frac{\varrho_\ell}{2\pi\varepsilon_0}\left\{\frac{1}{\sqrt{R^2+b^2}} - \frac{1}{\sqrt{R^2+a^2}}\right\}\hat z \; ; \; \underline{\varrho_s} = \bar D\cdot\hat n = \varepsilon_0 \bar E\cdot\hat z = \underline{\frac{-\varrho_\ell}{2\pi}\left\{\frac{1}{\sqrt{a^2+R^2}} - \frac{1}{\sqrt{b^2+R^2}}\right\}}$$

b/ Sätt $\varepsilon = b - a$; $Q = \varrho_\ell \varepsilon \Rightarrow \varrho_s = \frac{-Q}{2\pi\varepsilon}\left\{\frac{1}{\sqrt{a^2+R^2}} - \frac{1}{\sqrt{(a+\varepsilon)^2+R^2}}\right\}$

Serieutveckla: $f(\varepsilon) = \frac{1}{\sqrt{(a+\varepsilon)^2+R^2}}$ för små $\varepsilon \Rightarrow$

$$f(\varepsilon) \approx \frac{1}{\sqrt{a^2+R^2}} - \frac{a}{[a^2+R^2]^{3/2}}\varepsilon \Rightarrow \underline{\varrho_s} \approx \frac{-Q}{2\pi\varepsilon}\left\{0 + \frac{a\varepsilon}{[a^2+R^2]^{3/2}}\right\} = \underline{\frac{-aQ}{2\pi[a^2+R^2]^{3/2}}}$$

Svar: a/ $\varrho_s = \frac{-\varrho_\ell}{2\pi}\left\{\frac{1}{\sqrt{a^2+R^2}} - \frac{1}{\sqrt{b^2+R^2}}\right\}$

b/ $\varrho_s \approx \frac{-aQ}{2\pi[a^2+R^2]^{3/2}}$