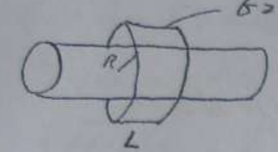
Elektromagnetism TFYA13, 2010-01-12

1/ Då cylindern är lång kan vi anta cylinder symmetri

$\Rightarrow \bar{D} = D(R)\hat{R}$ för $0 \le R \le 10a$.

Gauss sats: $\oint_S \bar{D}\, d\bar{S} = Q_{fri.innes}$



$$\Rightarrow 2\pi R L\, D(R) + \underset{\text{lock}}{0} + \underset{\text{botten}}{0} = Q_{fri.innes}$$

$$Q_{fri.innes} = \begin{cases} \pi R^2 L \rho_0 & 0 \le R \le a \\ \pi a^2 L \rho_0 & a \le R \end{cases} \Rightarrow \bar{D} = \begin{cases} \dfrac{\rho_0 R}{2}\hat{R} & 0 \le R \le a \\ \dfrac{\rho_0 a^2}{2R}\hat{R} & a \le R \end{cases}$$

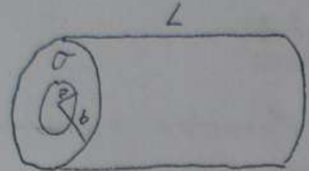
$$\underline{\underline{V(10a)}} = \int_{Ref}^{Alt} -\bar{E}\cdot d\bar{l} = \left\{ \bar{D} = \varepsilon_0 \varepsilon_r \bar{E} \right\} = \int_0^a -\frac{\rho_0 R}{2\varepsilon_0 [1+\alpha(R/a)^2]}\hat{R}\cdot\hat{R}\, dR +$$

$$+ \int_a^{10a} -\frac{\rho_0 a^2}{2\varepsilon_0 R}\hat{R}\cdot\hat{R}\, dR = \frac{-\rho_0 a^2}{2\varepsilon_0\, 2\alpha}\left[\ln[1+\alpha(R/a)^2]\right]_0^a -$$

$$- \frac{\rho_0 a^2}{2\varepsilon_0}\left[\ln R\right]_a^{10a} = \underline{\underline{\frac{-\rho_0 a^2}{4\varepsilon_0 \alpha}\ln(1+\alpha) - \frac{\rho_0 a^2}{2\varepsilon_0}\ln 10}}$$

Svar: $\underline{\underline{V(10a) = \frac{-\rho_0 a^2}{4\varepsilon_0 \alpha}\left[\ln(1+\alpha) + 2\alpha \ln 10\right]}}$

2)



Ansätt en ström I från inre till yttre cylindern.

Symmetri $\Rightarrow \bar{J} = J(R)\hat{R}$

$I = \int \bar{J} \cdot d\bar{S}$ på cylinder med radie R, $a < R < b \Rightarrow$

$$I = 2\pi R L J(R) \Rightarrow \bar{J} = \frac{I}{2\pi L R}\hat{R}$$

$$U = \int_{Ref}^{Alt} -\bar{E}\cdot d\bar{l} = \int_b^a -\bar{J}/\sigma \cdot d\bar{l} = \int_b^a \frac{-I}{2\pi\sigma L R}\cdot \hat{R}\cdot\hat{R}\, dR =$$

$$= \frac{I}{2\pi\sigma L}\ln(b/a) \Rightarrow \bar{J} = \frac{\sigma U}{R\ln(b/a)}\hat{R}$$

Vi ser att $|\bar{J}|$ är max då R är min, dvs. a.

Söker därför maximum för $f(a) = a\ln(b/a)$

$$\frac{df}{da} = \ln(b/a) - 1 = 0 \Rightarrow \frac{b}{a} = e \Rightarrow \underline{\underline{a = b/e}}$$

kontroll: $\frac{d^2f}{da^2} = -\frac{1}{a} < 0 \Rightarrow$ maximum.

Svar: Den maximala strömtätheten blir som minst om $a = b/e$.

$$\bar R_+ = \bar b + \bar a \Rightarrow |\bar R_+| = \sqrt{(\bar b+\bar a)\cdot(\bar b+\bar a)} = \sqrt{\bar b\cdot\bar b + \bar a\cdot\bar a + 2\bar a\cdot\bar b} = \sqrt{b^2+a^2+2ab\cos\omega t}$$

$$\bar R_- = \bar b - \bar a \Rightarrow |\bar R_-| = \sqrt{(\bar b-\bar a)\cdot(\bar b-\bar a)} = \sqrt{\bar b\cdot\bar b + \bar a\cdot\bar a - 2\bar a\cdot\bar b} = \sqrt{b^2+a^2-2ab\cos\omega t}$$

Fältet orsakat av strömmen fås mha cirkulationssatsen och symmetri.

$$\left.\begin{array}{l} \text{Symmetri} \Rightarrow \bar H = H(R)\hat\phi \\ \oint \bar H\cdot d\bar l = I_{t.i.oms.} \end{array}\right\} \Rightarrow 2\pi R\, H(R) = I \Rightarrow \bar H = \frac{I}{2\pi R}\hat\phi$$

$$\Rightarrow \bar B = \mu_0 \bar H = \frac{\mu_0 I}{2\pi R}\hat\phi$$

Flödet genom slingan fås som

$\Phi = \int \bar B\cdot d\bar S$ men tack vare att $\oint \bar B\cdot d\bar S = 0$ kan vi räkna som om slingan hela tiden har ytnormal $\hat\phi$ men med närmsta parallella sidan på avstånd R_- och den längst ifrån på R_+.

$$\Phi = \int_0^{2a}\int_{R_-}^{R_+} \frac{\mu_0 I}{2\pi R}\hat\phi\cdot\hat\phi\, dR\, dz = \frac{2a\mu_0 I}{2\pi}\ln\frac{R_+}{R_-} =$$

$$= \frac{a\mu_0 I}{\pi}\ln\sqrt{\frac{a^2+b^2+2ab\cos\omega t}{a^2+b^2-2ab\cos\omega t}} = \frac{a\mu_0 I}{2\pi}\ln\frac{a^2+b^2+2ab\cos\omega t}{a^2+b^2-2ab\cos\omega t}$$

$$\varepsilon_{emk} = -\frac{d\Phi}{dt} = -\frac{a\mu_0 I}{2\pi}\left[\frac{-2ab\omega\sin\omega t}{a^2+b^2+2ab\cos\omega t} - \frac{2ab\omega\sin\omega t}{a^2+b^2-2ab\cos\omega t}\right]$$

$$= \frac{a^2 b\omega\mu_0 I}{\pi}\left[\frac{1}{a^2+b^2+2ab\cos\omega t} + \frac{1}{a^2+b^2-2ab\cos\omega t}\right]\sin\omega t$$

Svar: $\varepsilon_{emk} = \frac{a^2 b\omega\mu_0 I}{\pi}\left[\frac{1}{a^2+b^2+2ab\cos\omega t} + \frac{1}{a^2+b^2-2ab\cos\omega t}\right]\sin\omega t$

a) Använd att $\bar{\nabla} \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$

$$\bar{\nabla} \times \bar{E} = \frac{\partial E_R}{\partial z}\hat{\phi} = \frac{V_0}{R\ln b/a} \cdot k[-\sin(kz-\omega t)]\hat{\phi} = -\frac{\partial \bar{B}}{\partial t} \Rightarrow$$

$$\underline{\underline{\bar{B} = \frac{kV_0\cos(kz-\omega t)}{\omega R\ln(b/a)}\hat{\phi}}}$$

b) Använd att $\bar{B} = \mu_0\mu_r\bar{H}$, $\bar{D} = \varepsilon_0\varepsilon_r\bar{E}$, $\bar{J} = \bar{0}$, $\bar{\nabla}\times\bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$

$$\Rightarrow \bar{\nabla}\times\bar{B} = \mu_0\mu_r\varepsilon_0\varepsilon_r\frac{\partial \bar{E}}{\partial t}$$

$$\bar{\nabla}\times\bar{B} = -\frac{\partial B_\phi}{\partial z}\hat{R} = \frac{k^2V_0\sin(kz-\omega t)}{\omega R\ln(b/a)}\hat{R} = \mu_0\mu_r\varepsilon_0\varepsilon_r\frac{\partial \bar{E}}{\partial t} \Rightarrow$$

$$\bar{E} = \frac{k^2V_0\cos(kz-\omega t)}{\mu_0\mu_r\varepsilon_0\varepsilon_r\omega^2R\ln(b/a)}\hat{R} \overset{\text{ska vara}}{=} \frac{V_0}{R\ln(b/a)}\cos(kz-\omega t)\hat{R} \Rightarrow$$

$$\frac{\omega^2}{k^2} = \frac{1}{\mu_0\mu_r\varepsilon_0\varepsilon_r} \Rightarrow \underline{\underline{\omega/k = \pm\sqrt{1/\mu_0\mu_r\varepsilon_0\varepsilon_r}}}$$

c) $$\bar{P} = \bar{E}\times\bar{H} = \left[\frac{V_0}{R\ln(b/a)}\cos(kz-\omega t)\right]^2\frac{k}{\omega}\cdot\frac{1}{\mu_0\mu_r}\overbrace{\hat{R}\times\hat{\phi}}^{\hat{z}}$$

$$\langle\bar{P}\rangle = \frac{1}{T}\int_0^T \bar{P}(t)\,dt \quad \text{Men} \quad \frac{1}{T}\int_0^T\cos^2\omega t\,dt = \frac{1}{T}\int_0^T\frac{1}{2}(1+\cos 2\omega t)\,dt =$$

$$= \frac{1}{2} \quad \text{dvs} \quad \langle\bar{P}\rangle = \frac{1}{2}\left[\frac{V_0}{R\ln(b/a)}\right]^2\frac{k}{\omega}\frac{1}{\mu_0\mu_r}\hat{z}$$

$$\underline{\underline{\langle P\rangle}} = \int_S\langle\bar{P}\rangle\cdot d\bar{S} = \int_a^b\int_0^{2\pi}\frac{1}{2}\left[\frac{V_0}{R\ln(b/a)}\right]^2\frac{k}{\omega}\frac{1}{\mu_0\mu_r}\hat{z}\cdot\hat{z}R\,d\phi\,dR =$$

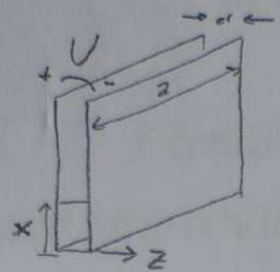
$$= \frac{\pi V_0^2}{\ln(b/a)}\frac{k}{\omega}\frac{1}{\mu_0\mu_r} = \underline{\underline{\frac{\pi V_0^2}{\ln(b/a)}\cdot\sqrt{\frac{\varepsilon_0\varepsilon_r}{\mu_0\mu_r}}}}$$

Svar a) $$\underline{\underline{\bar{B} = \frac{kV_0\cos(kz-\omega t)}{\omega R\ln(b/a)}\hat{\phi}}}$$

b) $$\underline{\underline{\omega/k = \pm\frac{1}{\sqrt{\varepsilon_0\varepsilon_r\mu_0\mu_r}}}}$$

c) $$\underline{\underline{\langle P\rangle = \frac{\pi V_0^2}{\ln(b/a)}\cdot\sqrt{\frac{\varepsilon_0\varepsilon_r}{\mu_0\mu_r}}}}$$

5)



Beräkna $\bar{E}$ o $\bar{D}$-fälten mellan plattorna.

$U = \int -\bar{E}\cdot d\bar{l}$ mellan plattorna och plattsymmetri $\Rightarrow \bar{E} = \frac{U}{d}\hat{z}$ i hela området mellan plattorna.

$\bar{D}_l = \varepsilon_0 \bar{E} = \varepsilon_0 \frac{U}{d}\hat{z}$ i luft

$\bar{D}_v = \varepsilon_0 \varepsilon_r \bar{E} = \varepsilon_0 \varepsilon_r \frac{U}{d}\hat{z}$ i vatten

$$W_e \underset{\text{"Hela } R^3\text{"}}{=} \int \frac{1}{2}\bar{E}\cdot\bar{D}\,d\tau = \underbrace{\frac{1}{2}\varepsilon_0\varepsilon_r\left(\frac{U}{d}\right)^2\cdot a\,d\,x}_{\text{Vatten}} + \underbrace{\frac{1}{2}\varepsilon_0\left(\frac{U}{d}\right)^2 a\,d(a-x)}_{\text{Luft}} + \underset{\text{Resten}}{W_0} =$$

$$= \frac{1}{2}\varepsilon_0 \frac{U^2}{d} a(\varepsilon_r x + a - x)$$

$$\underline{\underline{\bar{F}_e}} = +\bar{\nabla} W_e = \frac{\partial W_e}{\partial x}\hat{x} = \underline{\underline{\frac{1}{2}\varepsilon_0 \frac{U^2}{d} a(\varepsilon_r - 1)\hat{x}}}$$

Ty "V = konst"

b) Kraftjämvikt $\bar{F}_e + \bar{F}_g = \bar{0}$; $\bar{F}_g = h a d \rho g(-\hat{x}) \Rightarrow$

$$\frac{1}{2}\varepsilon_0 \frac{U^2}{d} a(\varepsilon_r - 1) = h a d \rho g \Rightarrow \underline{\underline{h = \frac{1}{2}\frac{\varepsilon_0(\varepsilon_r - 1)}{\rho g}\frac{U^2}{d^2} \approx 3{,}62\text{ cm}}}$$

Svar a) Vattnet sugs upp med kraften $\frac{1}{2}\varepsilon_0 \frac{U^2}{d} a(\varepsilon_r - 1)$

b) Höjden blir $h \approx \frac{1}{2}\frac{\varepsilon_0(\varepsilon_r - 1)}{\rho g}\left(\frac{U}{d}\right)^2 \approx 36$ mm