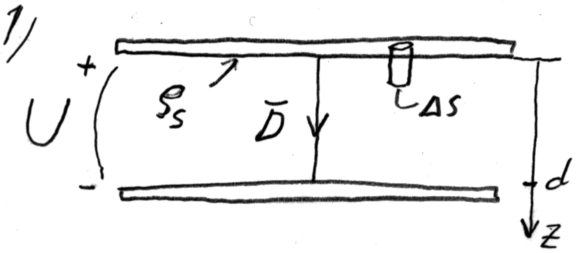
Tentamen i Elektromagnetism TFYA13, 2008-01-12

1)



Ansätt plattarea A

$$\varepsilon_r(z) = \frac{\varepsilon_{r2} - \varepsilon_{r1}}{d} \cdot z + \varepsilon_{r1} \equiv kz + \varepsilon_{r1}$$

Platt symmetri $\Rightarrow \bar{D} = D(z)\hat{z}$

ansätt ytladdningstäthet ϱ_s på övre beläggels under sida.

$\oint_S \bar{D}\cdot d\bar{S} = Q_{fri.innes}$ på Gauss burk $\Rightarrow$

$$\underbrace{D(z)\Delta S}_{} + \underbrace{0}_{\text{mantel};\ \bar{D}\cdot\hat{n}=0} + \underbrace{0}_{\text{lock};\ \text{metall}} = \varrho_s \Delta S \Rightarrow \bar{D} = \varrho_s \hat{z} \Rightarrow$$

$$\bar{E}(z) = \frac{\bar{D}}{\varepsilon_0 \varepsilon_r(z)} = \frac{\varrho_s}{\varepsilon_0(kz + \varepsilon_{r1})}\hat{z}$$

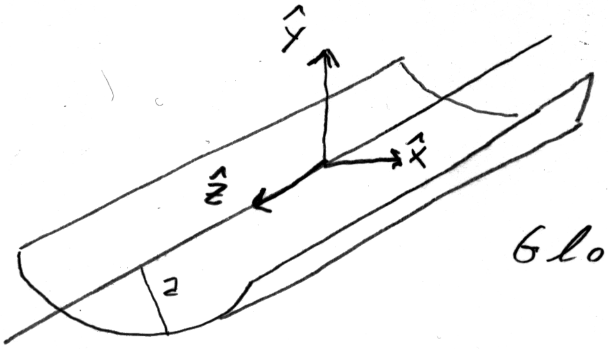
$$U = \int_{Ref}^{Alt} -\bar{E}\cdot d\bar{l} = \int_d^0 \frac{-\varrho_s}{\varepsilon_0(kz+\varepsilon_{r1})}\hat{z}\cdot\hat{z}\,dz = \frac{\varrho_s}{\varepsilon_0 k}\ln\frac{kd+\varepsilon_{r1}}{\varepsilon_{r1}} =$$

$$= \frac{d\varrho_s}{\varepsilon_0(\varepsilon_{r2} - \varepsilon_{r1})}\ln\frac{\varepsilon_{r2}}{\varepsilon_{r1}}$$

$$C = \frac{Q}{U} \Rightarrow \frac{C}{A} = \frac{Q}{UA} = \frac{\varrho_s}{U} = \frac{\varepsilon_0(\varepsilon_{r2} - \varepsilon_{r1})}{d\ln(\varepsilon_{r2}/\varepsilon_{r1})}$$

Svar: Kapacitansen per areaenhet är: $\dfrac{\varepsilon_0(\varepsilon_{r2} - \varepsilon_{r1})}{d\ln(\varepsilon_{r2}/\varepsilon_{r1})}$

2)



$$\bar{E}(\bar{r}) = \int \frac{(\bar{r} + \bar{r}')\,dQ'}{4\pi\varepsilon_0 |\bar{r} - \bar{r}'|^3}$$

(*)

Gloslisk: $\bar{r} = \bar{0}$, $\bar{r}' = a\hat{R}' + z'\hat{z}$

$dQ' = \rho_s\,dS' = \rho_s\,a\,d\phi'\,dz'$

Udda funktion i z över jämt interv. $\Rightarrow \bar{0}$

$$\bar{E}(\bar{r}) = \int_{\pi}^{2\pi}\int_{-l}^{+l} \frac{-a\hat{R}' - z'\hat{z}}{4\pi\varepsilon_0[a^2 + (z')^2]^{3/2}}\rho_s\,a\,dz'\,d\phi' =$$

$\hat{R}' = \cos\phi'\hat{x} + \sin\phi'\hat{y}$

$$= \int_{\pi}^{2\pi} \frac{-a^2\rho_s}{4\pi\varepsilon_0}\left[\frac{z'}{a^2\sqrt{a^2+(z')^2}}\right]_{-l}^{+l}\hat{R}'\,d\phi' = \frac{-\rho_s}{2\pi\varepsilon_0}\frac{l}{\sqrt{a^2+l^2}}\int_{\pi}^{2\pi}\hat{R}'\,d\phi' =$$

$$= \frac{-\rho_s l}{2\pi\varepsilon_0\sqrt{a^2+l^2}}\Big[\sin\phi'\hat{x} - \cos\phi'\hat{y}\Big]_{\pi}^{2\pi} = \frac{\rho_s l}{\pi\varepsilon_0\sqrt{a^2+l^2}}\hat{y}$$

Svar: $\bar{E}(\bar{0}) = \dfrac{\rho_s l}{\pi\varepsilon_0\sqrt{a^2+l^2}}\hat{y}$

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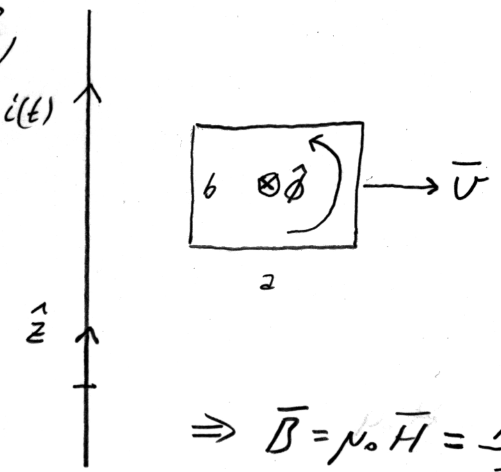
3)



Fält från lång rät ledare fås mha cirkulationssatsen och symmetri

$\oint_C \bar{H}\cdot d\bar{l} = I_{\text{oms. fri}}$, $\bar{H} = H(R)\hat{\phi}$ på

cirkel runt ledaren $\Rightarrow$

$2\pi R\,H(R) = i(t) \Rightarrow \bar{H} = \dfrac{i(t)}{2\pi R}\hat{\phi} \Rightarrow$

$\Rightarrow \bar{B} = \mu_0\bar{H} = \dfrac{\mu_0\,i(t)}{2\pi R}\hat{\phi}$

Flödet genom slingan $\quad \Phi = \int_{S'} \bar{B}\cdot d\bar{S} = \{\hat{n} = -\hat{\phi}\} =$

Ur figur

$$= \int_0^b\int_{R_1}^{R_2} \frac{\mu_0\,i(t)}{2\pi R}\hat{\phi}\cdot(-\hat{\phi})\,dR\,dz = \frac{-\mu_0 b\,i(t)}{2\pi}\ln\frac{R_2}{R_1}$$

Men $\left.\begin{array}{l} R_1 = c + vt \\ R_2 = a + R_1 = a + c + vt \end{array}\right\} \Rightarrow \Phi = \dfrac{-\mu_0 b\,i(t)}{2\pi}\ln\dfrac{a+c+vt}{c+vt}$

3 forts)

$$\mathcal{E} = -\frac{d\Phi}{dt} = \frac{\mu_0 b}{2\pi}\left\{\frac{di}{dt}\ln\frac{a+c+vt}{c+vt} + i(t)\left[\frac{v}{a+c+vt} - \frac{v}{c+vt}\right]\right\} =$$

$$= \frac{\mu_0 b}{2\pi} I_0 \left\{w\cos wt \ln\frac{a+c+vt}{c+vt} + \sin wt\left[\frac{v}{a+c+vt} - \frac{v}{c+vt}\right]\right\}$$

Svar: $\mathcal{E}(t) = \frac{\mu_0 b I_0}{2\pi}\left\{w\cdot\cos wt\cdot\ln\left(\frac{a+c+vt}{c+vt}\right) + \sin wt\left[\frac{v}{a+c+vt} - \frac{v}{c+vt}\right]\right\}$

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4) Entydighetssatsen säger att vi har lösningen om $\bar{\nabla}^2 V = -\rho/\varepsilon_0$ och alla randvillkor är uppfyllda. Dvs i vårt fall att $V(r=a) = \text{konst.}$, $\bar{E} \to E\hat{z}$ då $r \to 0$ och att $\bar{\nabla}^2 V = 0$ (Vi har ju $\rho = 0$)

$V(r=a) = \text{konst} \Rightarrow Aa\cos\theta + \frac{B}{a^2}\cos\theta = \text{konst} \Rightarrow \underline{\underline{A = \frac{-B}{a^3}}}$

$$\bar{\nabla}^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) =$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left[A\cos\theta - 2\frac{B}{r^3}\cos\theta\right]\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left[Ar + \frac{B}{r^2}\right](-\sin\theta)\right) =$$

$$= \frac{1}{r^2}\left(2rA\cos\theta + \frac{2B}{r^2}\cos\theta\right) + \frac{-2\cos\theta}{r^2}\left(Ar + \frac{B}{r^2}\right) = 0 \quad \underline{\text{ok}}$$

$$\bar{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{r} - \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta} = -\left(A\cos\theta - \frac{2B}{r^3}\cos\theta\right)\hat{r} +$$

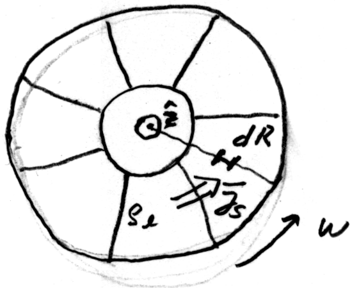
$$+\frac{1}{r}\sin\theta\left[Ar + \frac{B}{r^2}\right]\hat{\theta} \longrightarrow -A\cos\theta\hat{r} + A\sin\theta\hat{\theta} = -A\hat{z} = E_0\hat{z}$$

$\underline{\text{ok}}$

$\therefore A = -E_0 \Rightarrow B = -a^3 A = a^3 E_0$

Svar: Det är en korrekt lösning om $A = -E_0$ och $B = a^3 E_0$

5,



Börja med att ta fram ett uttryck för en approximativ ytströmtäthet. Studera ett litet linjeelement med längd dR längs en radie.

Varje varv passeras det av laddningen $n\varrho_e dR$ dvs i medel per tidsenhet $\frac{w}{2\pi} n\varrho_e dR = J_s dR$

$$\Rightarrow \bar{J}_s = \frac{w n \varrho_e}{2\pi}\hat{\phi} \quad ; \; a < R < b$$

$$\bar{B}(\bar{r}) = \int \frac{\mu_0 \bar{J}_s \times (\bar{r}-\bar{r}')}{4\pi |\bar{r}-\bar{r}'|^3} dS' \quad (*)$$

Gläslista: $\bar{r} = z\hat{z}$, $\bar{r}' = R'\hat{R}'$

$dS' = R' d\phi' dR'$

$\hat{R}' \Rightarrow \bar{0}$ ty $0 \to 2\pi$

$$\bar{B}(z\hat{z}) = \int_a^b \int_0^{2\pi} \frac{\mu_0 \frac{w n \varrho_e}{2\pi} \overbrace{\underbrace{\hat{\phi}' \times (z\hat{z}}_{\hat{R}'} - R'\hat{R}')}^{-\hat{z}}}{4\pi [z^2 + (R')^2]^{3/2}} R' d\phi' dR' =$$

$$= \frac{\mu_0 w n \varrho_e}{2\pi 4\pi} 2\pi \int_a^b \frac{(R')^2 dR'}{[z^2+(R')^2]^{3/2}} \hat{z} =$$

$$= \frac{\mu_0 w n \varrho_e}{4\pi} \left[\frac{-R'}{\sqrt{z^2+(R')^2}} + \ln\left[R' + \sqrt{z^2+(R')^2}\right] \right]_a^b \hat{z} =$$

$$= \frac{\mu_0 w n \varrho_e}{4\pi} \left\{ \frac{a}{\sqrt{a^2+z^2}} - \frac{b}{\sqrt{b^2+z^2}} + \ln \frac{b+\sqrt{b^2+z^2}}{a+\sqrt{a^2+z^2}} \right\} \hat{z}$$

Svar $\bar{B}(r\hat{r}) = \frac{\mu_0 w n \varrho_e}{4\pi} \left\{ \frac{a}{\sqrt{a^2+z^2}} - \frac{b}{\sqrt{b^2+z^2}} + \ln \frac{b+\sqrt{b^2+z^2}}{a+\sqrt{a^2+z^2}} \right\} \hat{z}$