LINKÖPINGS UNIVERSITET
Matematiska institutionen
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Tentamen i Diskret Matematik, TATA82, TEN1, 2019–08–22, kl 08–13.

Inga hjälpmedel. Ej räknedosa. Fullständiga motiveringar krävs.

För betyg 3 behövs 9 poäng, för betyg 4 12 poäng och 16 poäng för betyg 5.

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1. Visa med induktionsprincipen att $2 + 6 + \ldots + (n+1)(n+2) = \dfrac{n^3 + 6n^2 + 11n + 6}{3}$, för alla $n \geq 0$.

2. Visa med grafteori följande påstående: '$2n$ professorer sitter vid ett runt bord, var och en har högst $n-1$ fiender bland de andra professorerna. Då kan varje professor sitta mellan två icke-fiender och kan hälsa artigt genom att skaka hand med bredvidsittande professorer'.

3. I ett lopp deltar 10 lag med 5 löpare var. Ett resultat består av endast första, andra och tredje löpare som kommer i mål. Hur många olika resultat är det möjliga om alla 3 första platserna tas av olika lag. Svara med heltal.

4. (a) Vi vet om ett tal x mellan 2000 och 3000 att x är kongruent med 102^{459963} mod 1001. Bestäm talet. (2p)

   (b) Visa att $9 \mid (1^3 + 2^3 + \ldots + 99^3)$. (1p)

5. Lös systemet av rekursiva ekvationer

$$\begin{cases} a_{n+1} = 4a_n - b_n + 2(3)^n \\ b_{n+1} = a_n + 2b_n + (3)^n \end{cases} \quad \text{med} \quad a_0 = 0, \quad b_0 = 0.$$

6. Hur många funktioner $f : A \to \{0,1\}$, där $A = \{(x_1, x_2, x_3, x_4);\ x_i = 0, 1\}$ är mängden av binära följder av längd 4, uppfyller att $f(0,0,0,0) \neq 0$, $f(0,1,0,1) \neq 0$ och $f(1,1,1,1) \neq 1$?

7. Betrakta mängden M av uppåt-höger (U-H) vägar från punkten $A(0,0)$ till punkten $B(25,25)$ i en 25×25 gitter, dvs $M = \{p = (j_i),\ i = 1, \ldots 50; j_i = H \text{ eller } j_i = U\}$. Vi definierar en relation Q på M genom $p_1 Q p_2$ om p_1 och p_2 har lika många hopp till höger.

   (a) Visa att Q är en ekvivalensrelation. (2p)

   (b) Bestäm antalet ekvivalensklasser. (1p)

Written Examination in Discrete Mathematics TATA82, TEN1, 2019–08-22, kl 08–13.

No calculator.

For grade 3 are required 9 points, 12 points for grade 4 and 16 for grade 5.

Complete motivations required.

1. Show using mathematical induction that $2+6+\ldots+(n+1)(n+2) = \dfrac{n^3+6n^2+11n+6}{3}$, for all $n \geq 0$.

2. Show using graphs the following: '$2n$ professors are sitting at a round table, each of them has at most $n-1$ academic enemies among the other professors. So each professor can sit between two non-enemies and politely greet the professors sitting to both sides by shaking hands'.

3. In a race participate 10 teams with 5 runners in each team. A result consists only of the first, second and third runners that arrive to the goal. How many different results are possible if the first, second and third places are taken by different teams. Your answer shall be an integer.

4. (a) We know of a number x between 2000 and 3000 that x is congruent with 102^{459963} mod 1001. Determine the number. (2p)

   (b) Show that $9\,|\,(1^3+2^3+\ldots+99^3)$ (1p)

5. Solve the system of recurrence equations:

$$\begin{cases} a_{n+1} = 4a_n - b_n + 2(3)^n \\ b_{n+1} = a_n + 2b_n + (3)^n \end{cases} \quad \text{with} \quad a_0 = 0, \quad b_0 = 0.$$

6. How many functions $f : A \to \{0,1\}$, where $A = \{(x_1,x_2,x_3,x_4);\ x_i = 0,1\}$ is the set of binary sequences of length 4, satisfy that $f(0,0,0,0) \neq 0$, $f(0,1,0,1) \neq 0$ and $f(1,1,1,1) \neq 1$?

7. Consider the set M up-right (U-H) paths from the point $A(0,0)$ to the point $B(25,25)$ in a 25×25 grid, i.e. $M = \{p = (j_i)\, i = 1,\ldots 50; j_i = H \text{ or } j_i = U\}$. We define a relation Q on M by $p_1 Q p_2$ if p_1 and p_2 have the same number of steps to the right.

   (a) Show that Q is an equivalence relation. (2p)

   (b) Determine the number of equivalence classes. (1p)

Svar TATA82 Discrete Mathematics 22/8 2019

1) Show with Math Ind. $2+6+\dots+(n+1)(n+2)=\frac{n^3+6n^2+11n+6}{3}$ for all $n\geq 0$

With Math Ind. we show first

i) Claim is true for $n=0$: $LH_0 = 1\times 2 = \frac{6}{3} = RH_0$

ii) Assume that $\sum_{k=0}^{p}(k+1)(k+2) = \frac{p^3+6p^2+11p+6}{3}$ for some $p\geq 0$

and show for $p+1$ the claim

$$LH_{p+1} = \sum_{k=0}^{p+1}(k+1)(k+2) = \sum_{k=0}^{p}(k+1)(k+2)+(p+2)(p+3) \overset{M.I}{=}$$

$$\overset{M.I}{=} \frac{p^3+6p^2+11p+6}{3}+(p+2)(p+3) =$$

$$= \frac{(p+2)(p+3)}{3}(p+1+3) = \frac{(p+2)(p+3)(p+4)}{3} = \frac{p^3+9p^2+26p+24}{3}$$

$$= \frac{(p+1)^3+6(p+1)^2+11(p+1)+6}{3} = RH_{p+1}.$$ As required.

2) Consider the graph G whose nodes are the professors and whose edges are a pair of professors that are non-enemies. Question 2 is to ask if such a graph is Hamiltonian and the answer is Yes, since for any node P $\deg(P)\geq n+1$, and $|V(G)|=2n$ $\deg(P)\geq\frac{|V(G)|}{2}$ and the graph is Hamiltonian.

3) First the possibilities for the appearance of three of ten teams in order: $10\times 9\times 8 = 720$ followed by the number of the team that comes first: 5^3

Totally $720\times 5^3 = 25\times 3600$ different results $= 90000$

4b) Show that $9 \mid (1^3+2^3+\dots+99^3)$

For number t $1\leq t\leq 99$ s.t $t\equiv 1 \bmod 3$

$t^3 \equiv 1 \mod 9$

For t, $1 \le t \le 99$ s.t $t \equiv 2 \mod 3$, $t^3 \equiv -1 \mod 9$

And for t, $1 \le t \le 99$ $t \equiv 0 \mod 3$, $t^3 \equiv 0 \mod 9$

So $1^3 + 2^3 + \dots + 99^3 \equiv 33(1) + 33(-1) + 33(0) \equiv 0 \mod 9$

that is $9 \mid (1^3 + \dots + 99^3)$

4a) $2000 \le x \le 3000$ $\quad x \equiv 102^{459963} \mod 1001$

$1001 = 7 \times 11 \times 13$, $(7,11) = (7,13) = (11,13) = 1$

With Chinese Remainder th.

$102^{459963} \equiv b_1 \mod 7$, $102^{459963} \equiv b_2 \mod 11$, $102^{459963} \equiv b_3 \mod 13$

$102 \equiv 4 \mod 7$, $4^3 \equiv 1 \mod 7$; $102 \equiv 3 \mod 11$, $3^{10} \equiv 1 \mod 11$; $102 \equiv -2 \mod 13$

$4^{459963} \equiv (4^3)^{153321} \equiv 1 \mod 7$; $3^{459963} \equiv 1^{45996} \cdot 5 \equiv 5 \mod 11$ $\quad (-2)^{459963} \equiv (1)^{38330} \cdot 5 \mod 13$

$b_1 = 1$, $\quad b_2 = 5$ $\quad b_3 \equiv 5$

$x \equiv 1 \times 143 \times 5 + 5 \times 91 \times 4 + 5 \times 77 \times (-1) \equiv 2150 \mod 1001$

$143 x_1 \equiv 1 \mod 7$, $x_1 \equiv 5$ | $91 x_2 \equiv 1 \mod 11$, $x_2 \equiv 4$ | $77 x_3 \equiv 1 \mod 13$, $x_3 \equiv -1 \equiv 12 \mod 13$

$\boxed{x = 2150}$

5) Solve $\begin{cases} a_{n+1} = 4a_n - b_n + 2(3)^n \\ b_{n+1} = a_n + 2b_n + (3)^n \end{cases}$ $\quad a_0 = b_0 = 0$

$\begin{cases} a_n = b_{n+1} - 2b_n - (3)^n \\ a_{n+1} = b_{n+2} - 2b_{n+1} - (3)^{n+1} \end{cases}$; $b_{n+2} - 2b_{n+1} - (3)^{n+1} = 4b_{n+1} - 8b_n - 4(3)^n - \cancel{b_n} + 2(3)^n$

$b_{n+2} - 6b_{n+1} + 9b_n = (3)^n \quad b_0 = 0, b_1 = 1$

$b_n = b_n^{(h)} + b_n^{(p)}$. For $b_n^{(h)}$. Car. Eq. $r^2 - 6r + 9 = 0$

$r = 3$, $m = 2$ $\quad b_n^{(h)} = (B_1 n + B_2)(3)^n$

So $b_n^{(p)} = Bn^2(3)^n$; $B(n+2)^2(3)^2 - 6B(n+1)^2(3) + 9Bn^2 = 1$

$B = \frac{1}{18}$ $\qquad \underline{b_n = \left(\frac{n^2}{18} + B_1 n + B_2\right)(3)^n}$

IC $\begin{cases} b_0 = 0 = B_2 \\ b_1 = 1 = \frac{1}{6} + 3B_1 \end{cases}$, $B_1 = \frac{5}{18}$

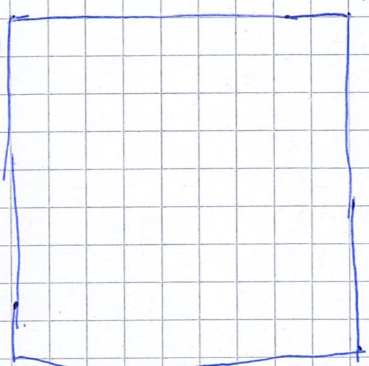
$b_n = \frac{n^2 + 5n}{18}(3)^n$

$a_n = \frac{(n+1)^2 + 5(n+1)}{18}(3)^{n+1} - \frac{n^2 + 5n}{9}(3)^n - (3)^n$

$= (3)^n \left(\frac{n^2 + 11n}{18}\right)$

6) $A = \{(x_1, x_2, x_3, x_4) ; \; x_i = 0, 1 \; i = 1, \ldots, 4\}$ has 16 elements

A function $f : A \to \{0, 1\}$ assigns two values to each sequence, except for $(0,0,0,0)$, that $f(0,0,0,0) = 1$; for $(0,1,0,1)$ that $f(0,1,0,1) = 1$, and $(1,1,1,1)$ that $f(1,1,1,1) = 0$. So two values for 13 sequences gives 2^{13} functions

7) 

B(25, 25)

A(0,0)

$M = \{p = (j_i) ; \; i = 1, \ldots, 50 ; \; j_i = H \text{ or } j_i = U\}$

Define Q, on M as $p_1 Q p_2$ if p_1 and p_2 has the number of steps to the right

a) Q is an equivalence relation since

i) Q is reflexive, for any sequence p, p has as many steps to the right as itself

ii) Q is symmetric: if p_2 has as many steps to the right as p_1 then p_1 has as many steps to the right as p_2

iii) Q is transitive if p_1 has as many steps to the right as p_2 and p_2 has as many steps to the right as p_3 then p_1 has as many steps to the right as p_3.

b) There is one class of equivalence: each sequence has exactly 25 steps to the right