Written Examination in Discrete Mathematics TATA82, TEN1, 2019–06–04, kl 14–19.
No calculator.
For grade 3 are required 9 points, 12 points for grade 4 and 16 for grade 5.
Complete motivations required.

1. Show with mathematical induction that $0+3+8+\ldots+(n^2-1)=\dfrac{2n^3+3n^2-5n}{6}$, för alla $n \geq 1$.

2. Consider a planar, connected, simple graph G. We know about G that its nodes have degree 4 or 6; that the number of nodes with degree 4 is thrice (3 times) the number of nodes of degree 6, and that the number of regions is 1.5 times the number of nodes. Determine the number of nodes, edges and regions in the graph and give a planar representation of the graph.

3. A shop prepares present bags using 10 types of chocolate tablets: A, B, C, D, E, F, G, H, I, J (at most one tablet of each kind in each bag). The answer must be an integer.

   (a) How many different present bags contain at least 6 types of chocolate?

   (b) How many different present bags contain 6 types of chocolate if they contain chocolate A?

   (c) How many different present bags do not contain chocolates A, B neither C?

4. In a cryptosystem we identify each letter by a number mod29: $A=1, B=2, C=3, D=4, E=5, F=6, G=7, H=8, I=9, J=10, K=11, L=12, M=13, N=14, O=15, P=16, Q=17, R=18, S=19, T=20, U=21, V=22, W=23, X=24, Y=25, Z=26$, Å $=27$, Ä $=28$, Ö $=0$. We encrypt each letter/number $x \in \mathbb{Z}_{29}$ by the function $\mathcal{K}(x) \equiv \alpha x^2+\beta x+\kappa \bmod 29$. Determine α, β and κ mod29 if the encryption of HEJ becomes POR.

5. Solve the recurrence equation $a_n - 13a_{n-2} + 36a_{n-4} = 25n(2)^n$, $n \geq 4, a_0 = 1, a_1 = 5, a_2 = 10, a_3 = 16$.

6. In one study on mutations of genes A, B, C och D within a population was shown that 30% has gene A mutated, 30% gene B, 40% gene C, and 25% gene D. One could show also that 25% has genes A and B mutated; 25% genes A och C; 15% genes A and D ; 25% genes B and C; 12% genes B and D; 15% genes C and D. One observed also that 15% has genes A, B and C mutated; 8% genes A, B and D; 10% genes A, C and D; and 6% genes B, C and D. Finally has 4% of the population has all four genes mutated.

   (a) Which percentage of the population has none of these genes mutated?

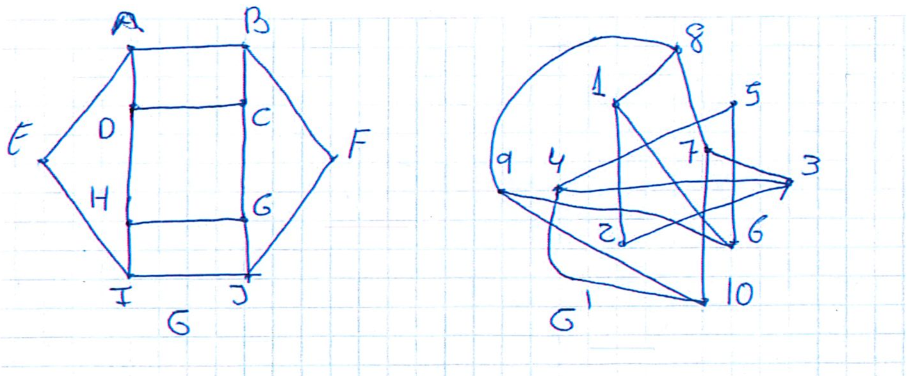
   (b) Which percentage of the population has genes A och D mutated but nor B neither C?

   (c) Which percentage of the population that has gene A mutated has also gene D mutated, but not C or B?

7. Consider the set A of all simple graphs with 10 nodes. We define a relation $\mathcal{R}$ on A as $G\mathcal{R}G'$ if G and G' are isomorphic.

   (a) Show that $\mathcal{R}$ is an equivalence relation (2p)

   (b) Show that the graphs below belong to the same equivalence class (1p)



Matematiska institutionen
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Tentamen i Diskret Matematik, TATA82, TEN1, 2019–06–04, kl 14–19.

Inga hjälpmedel. Ej räknedosa. Fullständiga motiveringar krävs.

För betyg 3 behövs 9 poäng, för betyg 4 12 poäng och 16 poäng för betyg 5.

1. Visa med induktionsprincipen att $0+3+8+\ldots+(n^2-1)=\dfrac{2n^3+3n^2-5n}{6}$, för alla $n \geq 1$.

2. Vi vet om en enkel, sammanhängande planär graf G att noderna har gradtal 4 eller 6; att det finns tre gånger så många noder med gradtal 4 som noder med gradtal 6. Vi vet också att antal regioner är 1,5 gånger antal noder. Bestäm antal noder, kanter och regioner i grafen och ange en planär representation av grafen.

3. En butik gör påsar av 10 sorters chokladaskar A, B, C, D, E, F, G, H, I, J som presenter till kunderna. Varje påse innehåller högst en chokladask av varje sort. Svara med heltal.

   (a) Hur många olika present-påsar finns det som innehåller minst 6 sorters choklad?

   (b) Hur många olika present-påsar finns det som innehåller 6 sorters choklad varav en är A?

   (c) Hur många olika present-påsar finns det som inte innehåller sorterna A, B eller C?

4. För att kryptera identifierar vi varje bokstav i svenska alfabetet med ett tal mod29: $A=1, B=2, C=3, D=4, E=5, F=6, G=7, H=8, I=9, J=10, K=11, L=12, M=13, N=14, O=15, P=16, Q=17, R=18, S=19, T=20, U=21, V=22, W=23, X=24, Y=25, Z=26$, Å $=27$, Ä $=28$, Ö $=0$. Vi krypterar varje bokstav/tal $x \in \mathbb{Z}_{29}$ genom funktionen $\mathcal{K}(x) \equiv \alpha x^2+\beta x+\kappa \bmod 29$. Bestäm α, β and κ mod29 om krypteringen av HEJ blir POR.

5. Lös den rekursiva ekvationen $a_n-13a_{n-2}+36a_{n-4}=25n(2)^n$, $n \geq 4, a_0=1, a_1=5, a_2=10, a_3=16$.

6. I en undersökning om mutationer i gener A, B, C och D ho sen population visade det sig att 30% har gen A muterad, 30% gen B, 40% gen C och 25% gen D. Vidare vet man att 25% har gener A och B muterade; 25% gener A och C; 15% har gener A och D muterade; 25% gener B och C; 12% gener B och D; 15% gener C och D. Det framgår också att 15% har gener A, B och C muterade; 8% gener A, B och D; 10% har gener A, C och D muterade; och 6% gener B, C och D. Slutligen har 4% av populationen alla fyra generna muterade.

   (a) Vilken andel av population har ingen av dessa gener muterade?

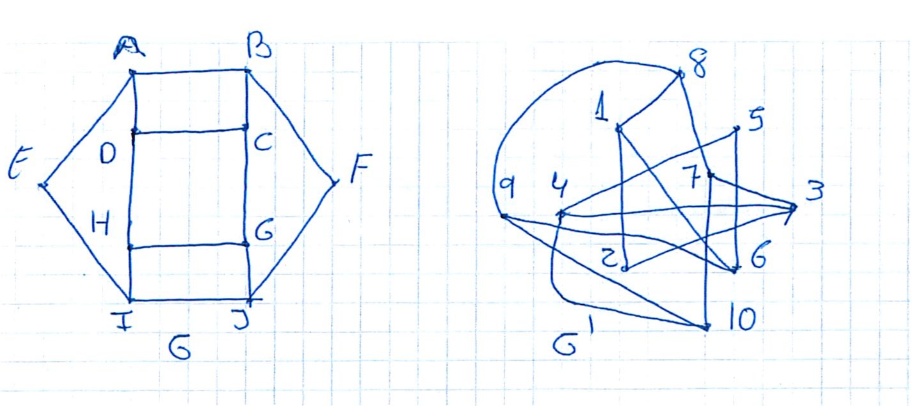
   (b) Vilken andel av populationen har gener A och D muterade men varken B eller C?

   (c) Vilken andel av den population som har gen A muterad har också gen D muterad, men inte C eller B?

7. Betrakta mängden A av alla enkla grafer med 10 noder. Vi definierar en relation $\mathcal{R}$ på A som $G\mathcal{R}G'$ om G och G' är isomorfa.

   (a) Visa att $\mathcal{R}$ är en ekvivalensrelation. (2p)

   (b) Visa att graferna nedan hör till en och samma ekvivalensklass. (1p)



SVAR TATA82 Discrete Mathematics 4/6 2019

1) Show that $\sum_{k=1}^{n} k^2-1 = \frac{2n^3+3n^2-5n}{6}$, for all $n \geq 1$

With Mathematical Induction, we need to show

i) formula true for $n=1$ $LH_1 = 1^2-1=0 = \frac{0}{6} = RH_1$ true

ii) Assume that $\sum_1^p k^2-1 = \frac{2p^3+3p^2-5p}{6}$ for some $p \geq 1$

and controll for $p+1$

$$LH_{p+1} = \sum_1^{p+1} k^2-1 = \sum_1^{p} k^2-1 + p(p+2) \overset{\text{Assup}}{=} \frac{2p^3+3p^2-5p}{6} + p(p+2)$$

$$= \frac{p(p-1)(2p+5)}{6} + p(p+2) = \frac{p}{6}\left[(p-1)(2p+5)+6(p+2)\right]$$

$$= \frac{p}{6}\left[2p^2+9p+7\right] = \frac{p(p+1)(2p+7)}{6} = \frac{(p+1-1)(p+1)(2(p+1)+5)}{6}$$

$= RH_{p+1}$, as desired.

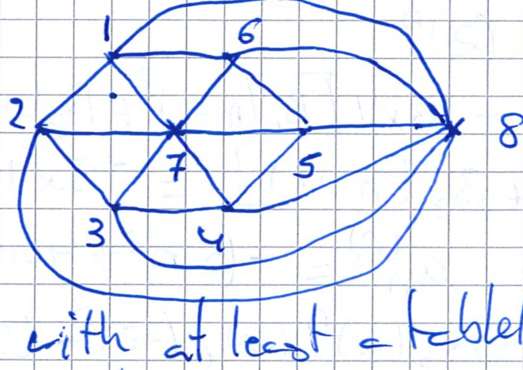
2) G planar, simple, connected. No of nodes n

x nodes of deg 6, $3x$ nodes of deg 4 $\quad n = x+3x = 4x$

No. of faces $f = \frac{3}{2}n = 6x$

Equations $\begin{cases} 12x+6x = 2e \\ n-e+f=2 \end{cases}$ $\begin{cases} e = 9x \\ 4x-9x+6x=2, \ x=2 \end{cases}$

$n=8$, $e=18$, $f=12$ A planar representation:



3 a) Bags with at least a tablet and no more than one tablet of each type of chocolate

a) We have 5 cases
- i) we choose 6 out of 10 $\binom{10}{6}$
- ii) we choose 7 out of 10 $\binom{10}{7}$
- iii) we choose 8 out of 10 $\binom{10}{8}$
- iv) we choose 9 out of 10 $\binom{10}{9}$
- v) All ten types $\binom{10}{10}$

In total $\binom{10}{6}+\binom{10}{7}+\binom{10}{8}+\binom{10}{9}+\binom{10}{11} = 210+120+45+10+1 =$

$= 386$ bags

b) If they contain 6 kinds of chocolate, one of them A, one chooses 5 out of the other nine types:

$$\binom{9}{5} = 126$$

c) All the other 7 types may be or not: $\underbrace{2^7}$ $- \overset{\text{empty bag}}{1} = 127$

4) We encrypt
$$\begin{cases} k(8) \equiv \alpha 64 + \beta 8 + k \equiv 16 \mod 29 \\ k(5) \equiv \alpha 25 + \beta 5 + k \equiv 15 \mod 29 \\ k(10) \equiv \alpha 100 + \beta 10 + k \equiv 18 \mod 29 \end{cases}$$

$$\sim \begin{cases} 6\alpha + 8\beta + k \equiv 16 \mod 29 \\ -4\alpha + 5\beta + k \equiv 15 \mod 29 \\ 13\alpha + 10\beta + k \equiv 18 \mod 29 \end{cases} \sim \begin{cases} 6\alpha + 8\beta + k \equiv 16 \mod 29 \\ 10\alpha + 3\beta \equiv 1 \mod 29 \\ 7\alpha + 2\beta \equiv 2 \mod 29 \end{cases}$$

$$\boxed{k \equiv 9 \mod 29}$$

$$\sim \begin{cases} 6\alpha + 8\beta + k \equiv 16 \mod 29 \\ 10\alpha + 3\beta \equiv 1 \mod 29 \\ \underline{\alpha \equiv 6 - 2 \equiv 4 \mod 29} \end{cases}, \quad 3\beta \equiv 18 \mod 29 ; \boxed{\beta \equiv -13 \equiv 16}$$

$$\underline{k(x) \equiv 4x^2 + 16x + 9 \mod 29}$$

5) Solve $a_n - 13a_{n-2} + 36a_{n-4} = 25n(2)^n$, $n \geq 4$

$a_0 = 1$, $a_1 = 5$, $a_2 = 10$, $a_3 = 16$.

$a_n = a_n^{(h)} + a_n^{(p)}$ C. Eq $r^4 - 13r^2 + 36 = (r-3)(r+3)(r-2)(r+2)$

$a_n^{(h)} = A_1(3)^n + A_2(-3)^n + A_3(2)^n + A_4(-2)^n$

$a_n^{(p)} = (B_1 n^2 + B_2 n)(2)^n$

Controll: $(B_1 n^2 + B_2 n)16 - 13 \times 4 [B_1(n-2)^2 + B_2(n-2)] + 36[B_1(n-4)^2 + B_2(n-4)] = 25 \times 16 n$. We get

$B_1 = -5$, $B_2 = -46$ $\quad a_n^{(p)} = -(5n^2 + 46n)(2)^n$

$a_n = A_1(3)^n + A_2(-3)^n + A_3(2)^n + A_4(-2)^n - (5n^2 + 46n)(2)^n$

$$\text{IC} \begin{cases} a_0 = 1 = A_1 + A_2 + A_3 + A_4 \\ a_1 = 5 = 3A_1 - 3A_2 + 2A_3 - 2A_4 - 102 \\ a_2 = 10 = 9A_1 + 9A_2 + 4A_3 + 4A_4 - 448 \\ a_3 = 16 = 27A_1 - 27A_2 + 8A_3 - 8A_4 - 1464 \end{cases} \Longrightarrow \quad A_3 = -\frac{283}{4}, \quad A_4 = -\frac{381}{20}$$

$A_2 = 62$, $A_1 = \frac{144}{5}$

$$\underline{a_n = \frac{144}{5}(3)^n + 62(-3)^n - (2)^n\left(5n^2 + 46n + \frac{283}{4}\right) - \frac{381}{20}(-2)^n}$$

6) Consider the sets, with a total population $\mathcal{U}$, $|\mathcal{U}|=100$

$A=\{x\in\mathcal{U}$, gene A mutated$\}$ $|A|=30$

$B=\{x\in\mathcal{U}$, gene B mutated$\}$ $|B|=30$

$C=\{x\in\mathcal{U}$, gene C mutated$\}$ $|C|=40$

$D=\{x\in\mathcal{U}$, gene D mutated$\}$ $|D|=25$

$|A\cap B|=25$, $|A\cap C|=25$, $|A\cap D|=15$, $|B\cap C|=25$, $|B\cap D|=12$,

$|C\cap D|=15$, $|A\cap B\cap C|=15$, $|A\cap B\cap D|=8$, $|A\cap C\cap D|=10$

$|B\cap C\cap D|=6$, $|A\cap B\cap C\cap D|=4$

a) We want $|\overline{A\cup B\cup C\cup D}|=|\mathcal{U}|-|A\cup B\cup C\cup D|$

$=100-[30+30+40+25-25-25-15-25-12-15+$

$+15+8+10+6-4]=100-[43]=57$. Answer $\underline{57\%}$

b) We want $|(A\cap D)\setminus(B\cup C)|=$

$=|A\cap D|-|(A\cap D)\cap(B\cup C)|=|A\cap D|-|(A\cap D\cap C)\cup(A\cap D\cap B)|$

Where $|(A\cap D\cap B)\cup(A\cap D\cap C)|=|A\cap D\cap B|+|A\cap D\cap C|-$

$-|A\cap B\cap D\cap C|=8+10-4=14$

$|A\cap D|-|(A\cap D)\cap(B\cup C)|=15-14=1$, Answer 1%

c) $\dfrac{|(A\cap D)\setminus(B\cup C)|}{|A|}=\boxed{\dfrac{1}{30}}$

7) $A=\{$simple graphs with 10 nodes$\}$. $G\,\mathcal{R}\,G'$ if G and G' isomorphic, i.e there is a function $f:V(G)\to V(G')$ f invertible, such that every edge $e=\{u,v\}$ in G corresponds to the edge $e'=\{fu,fv\}$ in G'

a) $\mathcal{R}$ is an equivalence relation

i) $\mathcal{R}$ reflexive: for every simple graph G the identity $I:V(G)\to V(G)$ $I(v)=v$ is an isomorphism $G\,\mathcal{R}\,G$

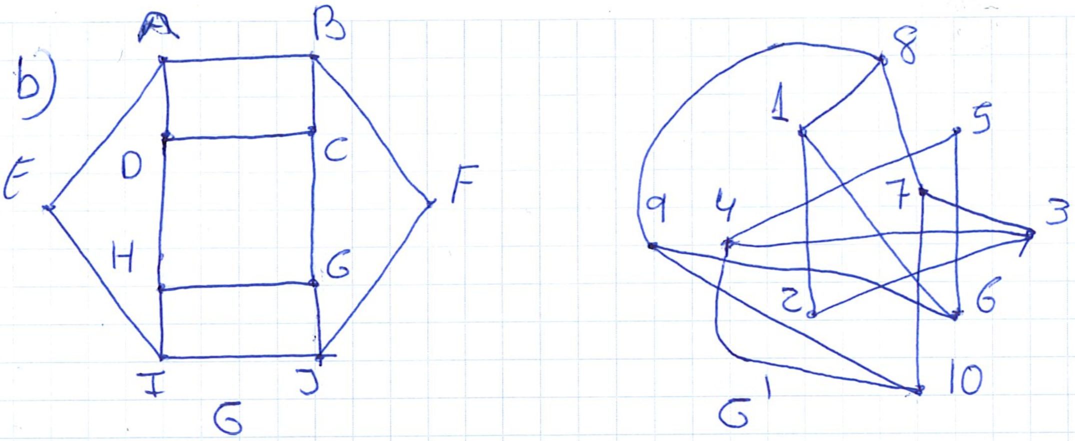
ii) $\mathcal{R}$ symmetric: if $G\,\mathcal{R}\,G'$ we have ~~an~~ isomorphism $f:V(G)\to V(G')$, then $f^{-1}:V(G')\to V(G)$

is an isomorphism making $G' \mathcal{R} G$, as wanted

iii) $\mathcal{R}$ transitive; if $G_1 \mathcal{R} G_2$ and $G_2 \mathcal{R} G_3$ we have the isomorphisms $f: V(G_1) \to V(G_2)$ and $g: V(G_2) \to V(G_3)$, then the composition $g \circ f: V(G_1) \to V(G_3)$ is an isomorphism between G_1 and G_3, and $G_1 \mathcal{R} G_3$. We see it

Any edge $e = \{u, v\}$ in G_1 corresponds to the edge $e' = \{fu, fv\}$ in G_2, which corresponds to the edge $e'' = \{g \circ f(u), g \circ f(v)\}$ in G_3

b)



An isomorphism is, for instance

$f: V(G) \to V(G')$

$f(A) = 1$, $f(E) = 2$, $f(I) = 3$, $f(J) = 4$, $f(F) = 5$

$f(B) = 6$, $f(D) = 8$, $f(C) = 9$, $f(G) = 10$, $f(H) = 7$