Written Examination in Discrete Mathematics TATA82, TEN1, 2018–11–02, kl 8–13.
No calculator.
For grade 3 are required 9 points, 12 points for grade 4 and 16 for grade 5.
Complete motivations required.

1. Show with Mathematical Induction that $(1-\frac{1}{4})(1-\frac{1}{9})\dots(1-\frac{1}{n^2}) = \frac{n+1}{2n}$ for all $n \geq 2$.

2. (a) Is the graph below Hamiltonian? Eulerian? In affirmative case, provide a Hamiltonian cycle and a closed eulerian path. (2p)
   (b) Determine the graph's chromatic number. Is the graph bipartite ? (1p)

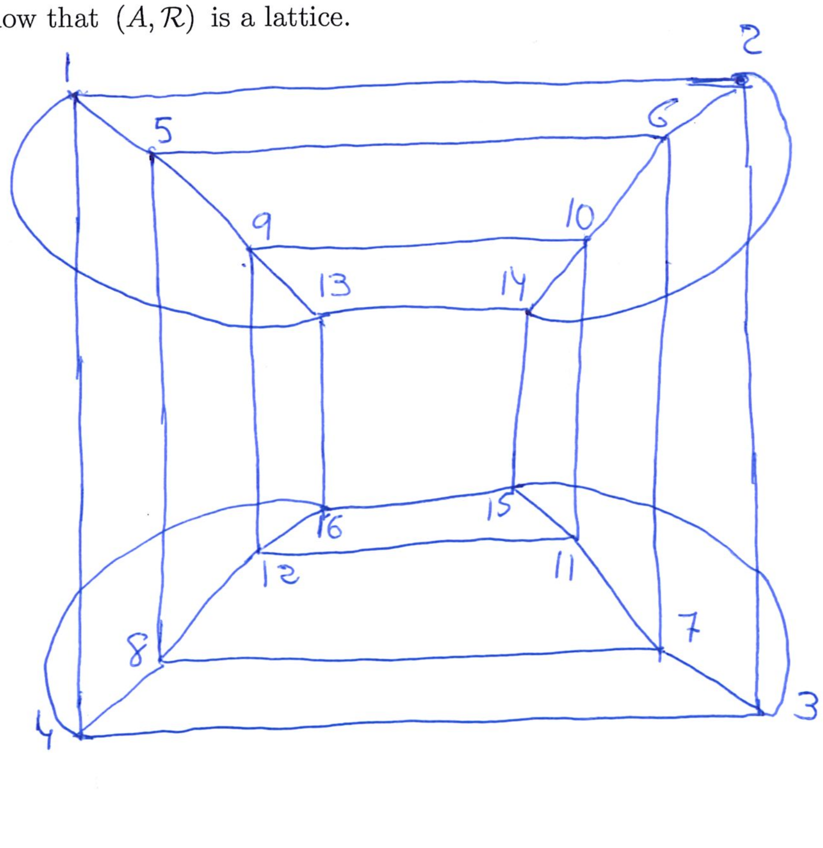
3. Consider 5-digits numbers such that they contain any digit at most once:
   (a) How many such numbers begin with 1 and finish with 23 or 32?
   (b) How many such numbers beginning with 1 are multiples of four? (a number is a multiple of four if its two last digits form a number which is a multiple of four itself).
   (c) How many such numbers do begin with 1 but do not end with 23 or 32?

4. (a) Show that $1^{12}+2^{12}+3^{12}+4^{12}+5^{12}+6^{12} \equiv 6\,(mod 7)$ (2p)
   (b) Two primes p and q are called *twins* if $q = p+2$; e.g. p = 11 and q=13. Show that the average of two twin primes is never a prime. (1p)

5. (a) Solve $b_n - 5b_{n-1} + 6b_{n-2} = n+1,\ n \geq 3,\ b_1 = 1, b_2 = 4$. (2p)
   (b) Use Part (a) to solve $na_n - 5(n-1)a_{n-1} + 6(n-2)a_{n-2} = n+1,\ n \geq 3,\ a_1 = 1, a_2 = 2$.

6. Determine the number of permutations of 1, 2, 3, 4, 5, 6, 7, 8, 9 and X that satisfy that none of 2, 4, 6 and 8 is fixed.

7. Consider the set A of all people resident in Sweden, and as such with Swedish identification number. We define a relation $\mathcal{R}$ on A by $P_1 \mathcal{R} P_2$ if the identification number for P_1 is smaller or equal to the identification number for P_2
   (a) Show that $(A, \mathcal{R})$ is a poset.
   (b) Given two people P_1, P_2, determine $\mathbf{inf}(P_1, P_2)$ och $\mathbf{sup}(P_1, P_2)$.
   (c) Show that $(A, \mathcal{R})$ is a lattice.



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Tentamen i Diskret Matematik, TATA82, TEN1, 2018–11–02, kl 08–13.

Inga hjälpmedel. Ej räknedosa. Fullständiga motiveringar krävs.

För betyg 3 behövs 9 poäng, för betyg 4, 12 poäng och 16 poäng för betyg 5.

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1. Visa med induktionsprincipen att $(1-\frac{1}{4})(1-\frac{1}{9})\ldots(1-\frac{1}{n^2}) = \frac{n+1}{2n}$ för alla $n \geq 2$.

2. (a) Är grafen nedan hamiltonsk? Eulersk? I så fall ange en hamiltonsk cykel och en sluten eulersk väg. (2p)

   (b) Bestäm grafens kromatiska tal. Är grafen bipartit? (1p)

3. Betrakta femsiffriga tal, sådana där ingen siffra förekommer mer än en gång

   (a) Hur många sådana tal har som första siffra 1:a och slutar med 23 eller 32?

   (b) Hur många sådana tal har som första siffra en 1:a och är en multipel av fyra? (ett tal t är en multipel av fyra om och endast om dess två sista siffror bildar en multipel av fyra).

   (c) Hur många sådana tal har som första siffra 1:a och slutar inte med 23 eller 32?

4. (a) Visa att $1^{12}+2^{12}+3^{12}+4^{12}+5^{12}+6^{12} \equiv 6\,(mod 7)$ (2p)

   (b) Två primtal p och q kallas *tvillingar* om $q = p+2$; t.ex. p = 11 och q=13. Visa att medelvärdet av två tvilling-primtal aldrig är ett primtal (1p)

5. (a) Lös den rekursiva ekvationen $b_n - 5b_{n-1} + 6b_{n-2} = n+1$, $n \geq 3$, $b_1 = 1, b_2 = 4$. (2p)

   (b) Använd Del (a) för att lösa den rekursiva ekvationen $na_n - 5(n-1)a_{n-1} + 6(n-2)a_{n-2} = n+1$, $n \geq 3$, $a_1 = 1, a_2 = 2$. (1p)

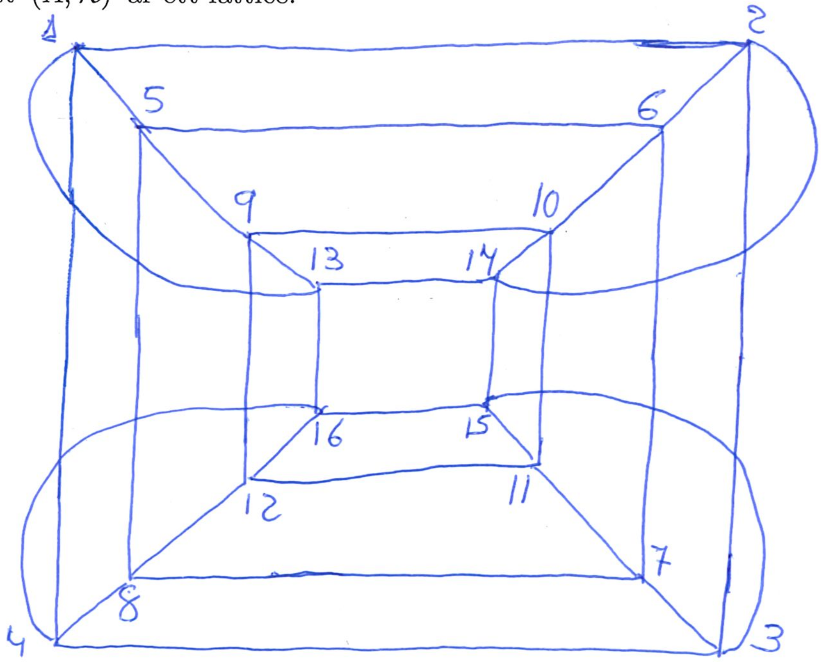
6. Beräkna antalet permutationer av 1, 2, 3, 4, 5, 6, 7, 8, 9 och X som uppfyller att inget av 2, 4, 6 eller 8 är fixerat.

7. Betrakta mängden A av alla personer bosatta i Sverige med personnummer. Vi definierar en relation $\mathcal{R}$ på A genom $P_1\mathcal{R}P_2$ om personnummer för P_1 är mindre eller lika med personnummer för P_2

   (a) Visa att $\mathcal{R}$ är en partialordning.

   (b) Givet två personer P_1, P_2, bestäm **sub**(P_1, P_2) och **möb**(P_1, P_2).

   (c) Visa att $(A, \mathcal{R})$ är ett lattice.



Answers TATA82 Discrete Mathematics 2/11 2018

1) Show with Math Ind $\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right) = \frac{n+1}{2n}, \forall n \geq 2$

i) Control for $n=2$: $VL_2 = 1-\frac{1}{4} = \frac{3}{4} = \frac{2+1}{2(2)} = HL_2$ TRUE

ii) Assume that $\prod_{k=2}^{p}\left(1-\frac{1}{k^2}\right) = \frac{p+1}{2p}$ for some $p \geq 2$

and prove the corresponding claim for $p+1$

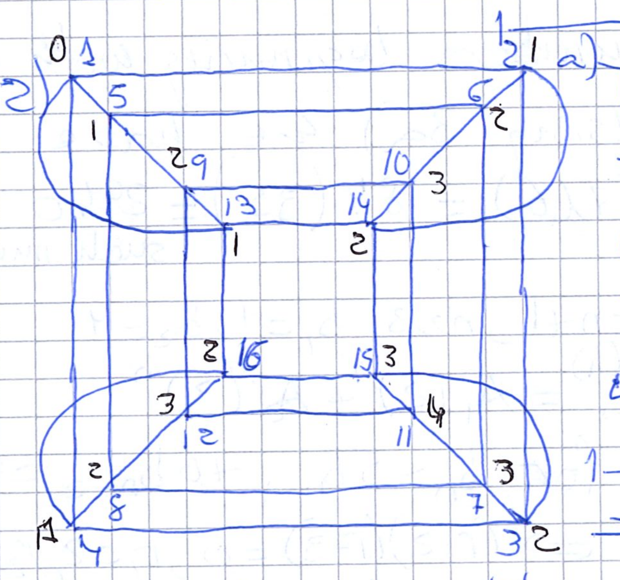
$$VL_{p+1} = \prod_{k=2}^{p+1}\left(1-\frac{1}{k^2}\right) = \left(1-\frac{1}{(p+1)^2}\right)\prod_{2}^{p}\left(1-\frac{1}{k^2}\right) \overset{\text{Assumption}}{=}$$

$$= \left(1-\frac{1}{(p+1)^2}\right)\frac{(p+1)}{2p} = \frac{((p+1)^2-1)(p+1)}{(p+1)^2(2)p} = \frac{(p+1)(p+2)\cancel{p}}{(p+1)^2(2)\cancel{p}}$$

$$= \frac{p+2}{2(p+1)} = \frac{(p+1)+1}{2(p+1)} = HL_{p+1} \text{ as required.}$$

Math Ind. tells us that $\prod_{2}^{n}\left(1-\frac{1}{k^2}\right) = \frac{n+1}{2n} \quad \forall n \geq 2$

2)



a) The graph is Eulerian since for any node i, $1 \leq i \leq 16$ $\deg i = 4$; even.

The graph is Hamiltonian, where a Hamilton-cycle is

$1 \to 2 \to 3 \to 4 \to 8 \to 7 \to 6 \to 5 \to 9 \to 10 \to 11 \to 12 \to 16 \to 15 \to 14 \to 13 \to 1$

b) The graph is bipartite with bipartition $A = \{1, 3, 6, 8, 9, 11, 14, 16\}$ and $B = \{2, 4, 5, 7, 10, 12, 13, 15\}$. Then $\chi(G) = 2$

4a) By Fermat's Small Th. $a^{\phi(7)} \equiv 1 \mod 7$ for a s.t $\gcd(7,a)=1$ where $\phi(7)=6$. Then $1^{12} \equiv (1^6)^2 \equiv 1$, $2^{12} \equiv (2^6)^2 \equiv 1$; $3^{12} \equiv (3^6)^2 \equiv 1$; $4^{12} \equiv (4^6)^2 \equiv 1$; $5^{12} \equiv (5^6)^2 \equiv 1$ and $6^{12} \equiv (6^6)^2 \equiv 1$ and

$$1^{12}+2^{12}+3^{12}+4^{12}+5^{12}+6^{12} \equiv 1+1+1+1+1+1 \equiv 6 \mod 7$$

4b) If p and q are twin primes ($q=p+2$ prime) then p, q are odd numbers since if $p=2$ $q=2+2=4$ is not prime. Thus the average $\frac{p+q}{2}=\frac{2p+2}{2}=p+1$ is an even number distinct from 2 and never prime

3a) $\underline{1}\ \underline{x}\ \underline{x}\ \underline{2}\ \underline{3}$ They are (7)(6) such numbers equally for numbers finishing with 32

Totally $2(7)(6)=84$ such numbers

b) If the numbers are multiple of 4 the two last digits are: 04, 08, 12, 16, 20, 24, 28, 32, 36, 40, 48, 52, 56, 60, 64, 68, 72, 76, 80, 84, 92, 96. For any of the 22 cases we have again (7)(6) possibilities

Totally (2)(11)(7)(6) = 924 such numbers

c) They are (9)(8)(7)(6) numbers beginning with 1 and no more conditions. Using 3a) we have in total $(9)(8)(7)(6)-(2)(7)(6)=84(35)=2940$ such numbers

5a) Solve $b_n-5b_{n-1}+6b_{n-2}=n+1$, $n\geq 3$, $b_1=1$, $b_2=4$

$b_n=b_n^{(h)}+b_n^{(p)}$ where $b_n^{(h)}=A_1(2)^n+A_2(3)^n$

since is the general solution for $b_n-5b_{n-1}+6b_{n-2}=0$

with charac. eq. $r^2-5r+6=0 \Leftrightarrow (r-2)(r-3)=0$, $r_1=2$, $r_2=3$

$b_n^{(p)}=1\_(1)^n(Bn+C)$ where

$Bn+C-5B(n-1)-5C+6B(n-2)+6C=n+1$ $\begin{cases}2B=1\\2C-7B=1\end{cases}$

$B=\frac{1}{2}$, $C=\frac{9}{4}$ $b_n^{(p)}=\frac{2n+9}{4}$

$a_n=A_1(2)^n+A_2(3)^n+\frac{2n+9}{4}$. BV: $\begin{cases}b_1=1=2A_1+3A_2+\frac{11}{4}\\b_2=4=4A_1+9A_2+\frac{13}{4}\end{cases}$

$A_1=-3$, $A_2=\frac{17}{12}$

$$b_n=\frac{17}{4}(3)^{n-1}-3(2)^n+\frac{2n+9}{4}$$

5b) To solve $na_n - 5(n-1)a_{n-1} + 6(n-2)a_{n-2} = n+1$
$a_1 = 1$, $a_2 = 2$. We do the change of variable
$b_n = n a_n$. Now the equ. get
$b_n - 5b_{n-1} + 6b_{n-2} = n+1$, $b_1 = 1(1) = 1$, $b_2 = 2(2) = 4$
By 5a) $b_n = \frac{17}{4}(3)^{n-1} - 3(2)^n + \frac{2n+9}{4}$ and
$a_n = \frac{b_n}{n}$, $n \geq 1$ so $\boxed{a_n = \frac{17}{4n}(3)^{n-1} - \frac{3}{n}(2)^n + \frac{1}{2} + \frac{9}{4n}}$

6) Let $\mathcal{U} = \{$perm. of $1, 2, \dots, X\}$. We look for the number of permutations in the complement of $A_1 \cup A_2 \cup A_3 \cup A_4$ where
$A_1 = \{$perm in $\mathcal{U}$, fixing $2\}$, $A_2 = \{$permut in $\mathcal{U}$ fixing $4\}$
$A_3 = \{$perm in $\mathcal{U}$ fixing $6\}$ and $A_4 = \{$permut in $\mathcal{U}$ fixing $8\}$
By the Principle of Inclusion and Exclusion

$$|\overline{A_1 \cup A_2 \cup A_3 \cup A_4}| = |\mathcal{U}| - \left(\sum_{1}^{4} |A_i| - \sum_{1 \leq i_1 < i_2 \leq 4} |A_{i_1} \cap A_{i_2}| + \sum_{1 \leq i_1 < i_2 < i_3 \leq 4} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - |A_1 \cap A_2 \cap A_3 \cap A_4| \right)$$

Now $|A_1| = |A_2| = |A_3| = |A_4| = 9!$ (one of 10 symbols fixed)
For any pair i_1, i_2 $|A_{i_1} \cap A_{i_2}| = 8!$
For any triple i_1, i_2, i_3 $|A_{i_1} \cap A_{i_2} \cap A_{i_3}| = 7!$,
and $|A_1 \cap A_2 \cap A_3 \cap A_4| = 6!$
$|\mathcal{U}| = 10!$

So, in total $|\overline{A_1 \cup A_2 \cup A_3 \cup A_4}| = 10! - \binom{4}{1}9! + \binom{4}{2}8! - \binom{4}{3}7! + 6!$
permutations <u>not</u> fixing <u>any</u> of 2, 4, 6 and 8

7) $A = \{$ persons in Sweden with ident number $\}$
$P_1 \, R \, P_2$ if ident num. $P_1 \leq$ ident numb P_2

(a) R is a partial ordering
i) R reflexiv: for every person P in A
ident numb $P \leq$ ident numb of P
ii) R antisymmetric:
if $P_1 R P_2$ and $P_2 R P_1$ then P_1 and P_2 have the same ident number, which is unique, so $P_1 = P_2$
iii) R transitive: if $P_1 R P_2$ and $P_2 R P_3$
then ident $P_1 \leq$ ident no. $P_2 \leq$ ident no. P_3; so $P_1 R P_3$

b) Given P_1, with ident no. t_1 and P_2 with ident no. t_2
Say that $t_1 < t_2$ so $\min\{t_1, t_2\} = t_1$
$\max\{t_1, t_2\} = t_2$. Then

$$\inf(P_1, P_2) = \text{sub}(P_1, P_2) = P_1$$
$$\sup(P_1, P_2) = \text{mob}(P_1, P_2) = P_2$$

c) (A, R) is a lattice, this is proved in b)