Written Examination in Discrete Mathematics TATA82, TEN1, 2018–08-23, kl 08–13.

No calculator.

For grade 3 are required 9 points, 12 points for grade 4 and 16 for grade 5.

Complete motivations required.

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1. Show using mathematical induction that $(14)^{2n+1} - (6)^n + 2$ is a multiple of 5 for all $n \geq 0$.

2. (a) Show that the weighted graph below is not Hamiltonian.

   (b) Determine a minimal spanning tree for the weighted graph below.

   (c) A complete binary tree of height h has 14 nodes with degree 3. Determine the height, the number of nodes and the number of edges in the tree.

3. (a) To a robot-conference in Mathburgh come 105 robots (all equal). Mathburgh has 5 hotel for robots. In how many ways can the robots be divided between the hotels if each hotel houses at least one robot?

   (b) Traveling through France, Spain, Portugal, Italy and Greece I collect one-euro (1) coins. My collection consists of 100 coins. How many different collections could I have?

   (c) How many sequences of length 100 are there so that 0 occurs 40 times, 1 occurs 30 times, 2 occurs 20 times, and 3 occurs 10 times?

4. (a) Solve the system of congruence equations:

$$\begin{cases} x & +2y & +3z & \equiv & 9 \ mod(19) \\ -x & +3y & +5z & \equiv & 8 \ mod(19) \\ 2x & +3y & +4z & \equiv & 7 \ mod(19) \end{cases} \qquad (2p)$$

   (b) Show that $(8n+3, 3n+1) = 1$ for all $n \geq 0$ (1p)

5. Solve the system of recurrence equations:

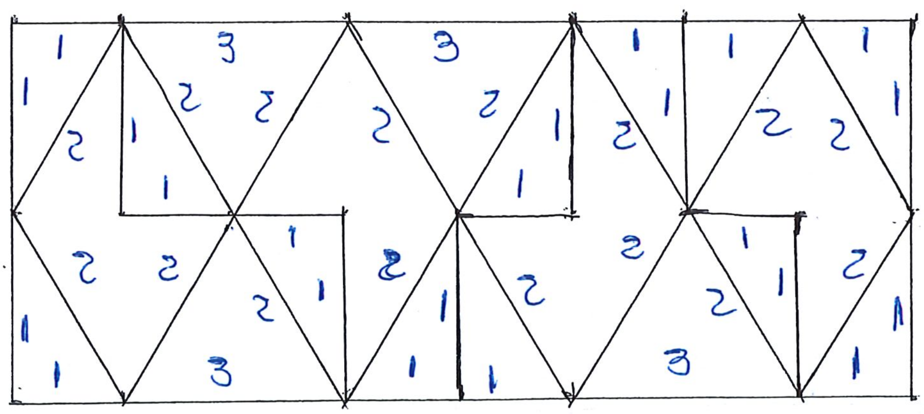
$$\begin{cases} a_{n+1} & = & 7a_n & -4b_n & +4n \\ b_{n+1} & = & 8a_n & -5b_n & +4n \end{cases} \quad with \quad a_0 = 1, \quad b_0 = 1.$$

6. How many integers between 1 and 300 must one pick to be sure that at least one of the numbers is relative prime with 105?

7. Consider the set A of spanning trees in the graph below. We define a relation $\mathcal{R}$ on A by $T_1 \mathcal{R} T_2$ if T_1 and T_2 have the same total cost (or weight). Show that $\mathcal{R}$ is an equivalence relation.

   Provide a tree in the equivalence class determined by the total cost 24.

   Provide a tree in the equivalence class determined by the total cost 25.



LINKÖPINGS UNIVERSITET
Matematiska institutionen
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Tentamen i Diskret Matematik, TATA82, TEN1, 2018–08–23, kl 08–13.

Inga hjälpmedel. Ej räknedosa. Fullständiga motiveringar krävs.

För betyg 3 behövs 9 poäng, för betyg 4 12 poäng och 16 poäng för betyg 5.

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1. Visa med induktionsprincipen att $(14)^{2n+1} - (6)^n + 2$ är ett multipel av 5 för alla $n \geq 0$.

2. (a) Visa att den viktade grafen nedan inte är hamiltonsk.
   (b) Bestäm ett minimalt spännande träd för grafen nedan.
   (c) Ett fullständigt binärt träd av höjden h har 14 noder med gradtal 3. Bestäm höjden, antal noder och antal kanter i trädet.

3. (a) Till en robot-konferens i Mathburgh kommer 105 robotar (alla lika). Mathburgh har 5 hotell för robotar. På hur många sätt kan robotarna delas upp mellan hotellen om varje hotell har minst en robot?
   (b) Under en resa genom France, Spain, Portugal, Italy och Greece samlar jag på en-euro (1) mynt. Min samling består av 100 mynt. Hur många olika samlingar kan jag ha?
   (c) Hur många olika följder av längd 100 finns det med villkoret att 0 förekommer 40 gånger, 1 förekommer 30 gånger, 2 förekommer 20 gånger och 3 förekommer 10 gånger?

4. (a) Lös ekvationsystemet:

$$\begin{cases} x & +2y & +3z & \equiv & 9 & mod(19) \\ -x & +3y & +5z & \equiv & 8 & mod(19) \\ 2x & +3y & +4z & \equiv & 7 & mod(19) \end{cases} \qquad (2p)$$

   (b) Visa att $(8n+3, 3n+1) = 1$ för alla $n \geq 0$ (1p)

5. Lös systemet av rekursiva ekvationer

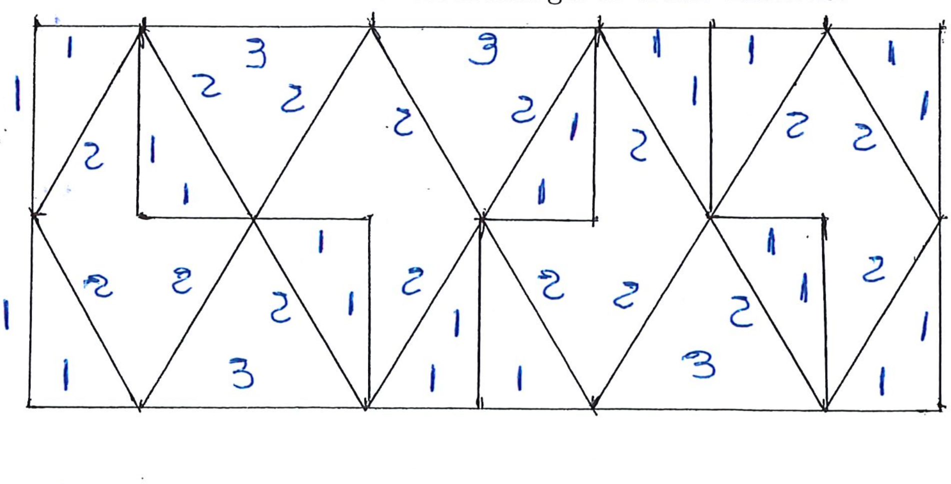
$$\begin{cases} a_{n+1} & = & 7a_n & -4b_n & +4n \\ b_{n+1} & = & 8a_n & -5b_n & +4n \end{cases} \quad med \quad a_0 = 1, \quad b_0 = 1.$$

6. Hur många positiva heltal mellan 1 och 300 ska man välja för att vara säkert på att minst ett är relativt primtal med 105?

7. Betrakta mängden A av alla spännande träd i grafen nedan. Vi definierar en relation $\mathcal{R}$ på A genom $T_1 \mathcal{R} T_2$ om T_1 och T_2 har samma totalt vikt (eller kostnad). Visa att $\mathcal{R}$ är en ekvivalensrelation.

   Ange ett träd som hör till ekvivalensklassen som ges av totala vikten 24.

   Ange ett träd som hör till ekvivalensklassen som ges av totala vikten 25.



Svar TATA82 Discrete Mathematics 23/8 2018

1) Show, using Math Induct, $5 \mid ((14)^{2n+1} - (6)^n + 2)\ \forall n \geq 0$

i) We see that it is true for $n = 0$:

$5 \mid (14 - 1 + 2)$ i.e $5 \mid 15$ so TRUE

ii) We assume that $5 \mid ((14)^{2p+1} - (6)^p + 2)$, for $p \geq 0$,

$(14)^{2p+1} = 5m + (6)^p - 2$ for some $m \in \mathbb{Z}$

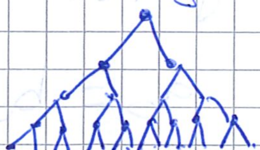
Now, we show that the claim is true for $p+1$

Now $(14)^{2p+3} - (6)^{p+2} + 2 = (196)(5m + (6)^p - 2) - (6)^p + 2 =$

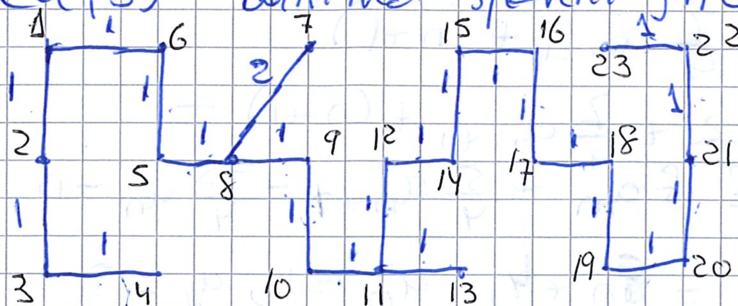
$= (196m)(5) + \underbrace{(196-6)}_{190}(6)^p - (2)\underbrace{(196-1)}_{195}$ which is a multiple of 5

As required

2c) A complete binary tree has vertices of degree 2 (the root), degree 1: the leaves, and degree 3. In each level there are double as many as in the preceeding level so $14 = 2 + 4 + 8$ and the height is $h = 4$, there are in total $1 + 14 + 16 = 31$ vertices

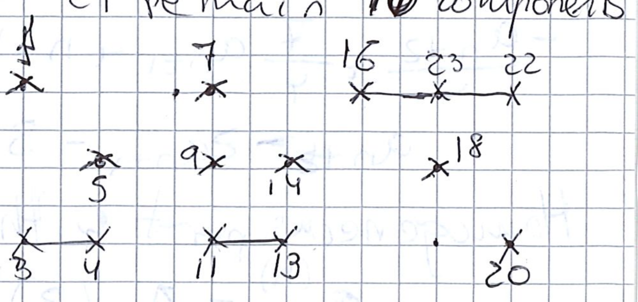


2a, b) A minimal spanning tree



The minimal cost is 23u.

b) After removing vertices 2, 6, 8, 10, 12, 15, 17, 19 and 21 remain 10 components



3a We have $\binom{100+5-1}{5-1}$ ways of dividing 100 robots in 5 hotels (5 other robots were housed one in each hotel) : $\binom{104}{4}$

3b) There are $\binom{100+5-1}{5-1} = \binom{104}{4}$ ways of dividing 100 coins with 5 different (heads) crowns

3c) There are $\frac{100!}{40!\,30!\,20!\,10!}$ sequences

4a) Solve $\begin{cases} x+2y+3z \equiv 9 \mod (19) \\ -x+3y+5z \equiv 8 \mod (19) \\ 2x+3y+4z \equiv 7 \mod (19) \end{cases} \iff \begin{cases} x+2y+3z \equiv 9 \\ 5y+8z \equiv -2 \\ 9y+14z \equiv 4 \end{cases}$

$\iff \begin{cases} x+2y+3z \equiv 9 \\ 5y+8z \equiv -2 \\ 2z \equiv 0 \end{cases}$ $9(r2)-5(r3)$, $\underline{x \equiv 6 \mod (19)}$, $\underline{y \equiv (-2)(4) \equiv 11 \mod 19}$, $\underline{z \equiv 0 \mod (19)}$

4b) Show that $\gcd(8n+3, 3n+1) = 1 \quad \forall n \geq 0$ with Euclides' algorithm

$8n+3 = 2(3n+1) + 2n+1$; $3n+1 = 1(2n+1) + n$

$2n+1 = 2(n) + \underline{1}$ and finally $\gcd(8n+3, 3n+1) = \underline{1}$

5) Solve $\begin{cases} a_{n+1} = 7a_n - 4b_n + 4n \\ b_{n+1} = 8a_n - 5b_n + 4n \end{cases}$ $a_0 = b_0 = 1$

From (1) $b_n = -\frac{a_{n+1}}{4} + \frac{7}{4}a_n + n$

$b_{n+1} = -\frac{a_{n+2}}{4} + \frac{7}{4}a_{n+1} + (n+1)$

So (2) becomes $-\frac{a_{n+2}}{4} + \frac{7}{4}a_{n+1} + (n+1) =$

$-\frac{a_{n+2}}{4} + \frac{7}{4}a_{n+1} + n+1 = 8a_n + \frac{5}{4}a_{n+1} - \frac{35}{4}a_n - n$

$a_{n+2} - 2a_{n+1} - 3a_n = \underline{8n+4}, \quad a_0 = 1, a_1 = 3$

Homogeneous part with C.E $r^2 - 2r - 3 = 0$; $r = 3, -1$

$a_n^{(h)} = A_1(3)^n + A_2(-1)^n$

Particular sequence $a_n^{(p)} = B_1 n + B_2$

Control $B_1(n+2) + B_2 - 2B_1(n+1) - 2B_2 - 3B_1 n - 3B_2 = 8n + 4$

$-4B_1 n = 8n$, $B_1 = -2$

$-4B_2 = 4$, $B_2 = -1$

$a_n = A_1 (3)^n + A_2 (-1)^n - 2n - 1$

$$\begin{cases} a_0 = 1 = A_1 + A_2 - 1 \\ a_1 = 3 = 3A_1 - A_2 - 3 \end{cases} \quad \begin{cases} 2 = A_1 + A_2 \\ 6 = 3A_1 - A_2 \end{cases} \quad \begin{matrix} A_1 = 2 \\ A_2 = 0 \end{matrix}$$

$$a_n = +2 (3)^n - 2n - 1$$

$$b_n = -\frac{1}{2}(3)^{n+1} + \frac{(n+1)}{2} + \frac{1}{4} + \frac{7}{2}(3)^n - \frac{7}{2}n - \frac{7}{4} + n$$

$$b_n = +2 (3)^n - 2n - 1$$

Observe that $\{b_n\}$ satisfies $b_{n+2} - 2b_{n+1} - 3b_n = 8n + 4$, $b_0 = 1$, $b_1 = 3$

6) We calculate how many integers between 1 and 300 are not relative prime with 105, i.e they are a multiple of 3, 5 or 7

To do it consider $T = \{\text{multiples of } 3\}$, $F = \{\text{multiple of } 5\}$, $S = \{\text{multiples of } 7\}$ and we look for $|T \cup F \cup S|$

$|T| = \lfloor \frac{300}{3} \rfloor = 100$; $|F| = \lfloor \frac{300}{5} \rfloor = 60$; $|S| = \lfloor \frac{300}{7} \rfloor = 42$

$|T \cap S| = \lfloor \frac{300}{21} \rfloor = 14$; $|T \cap F| = \lfloor \frac{300}{15} \rfloor = 20$; $|F \cap S| = \lfloor \frac{300}{35} \rfloor$

$|F \cap S| = 8$ and $|T \cap F \cap S| = \lfloor \frac{300}{105} \rfloor = 2$

So $|T \cup S \cup F| = 100 + 60 + 42 - 14 - 20 - 8 + 2 = 162$

So by taking $163 = 162 + 1$ integers we are sure (at least) one is relative prime with 105.

7) Consider the set A of spanning trees in the graph below. Define the relation $\mathcal{R}$ on A by $T_1 \mathcal{R} T_2$ if T_1 and T_2 have the same total cost.

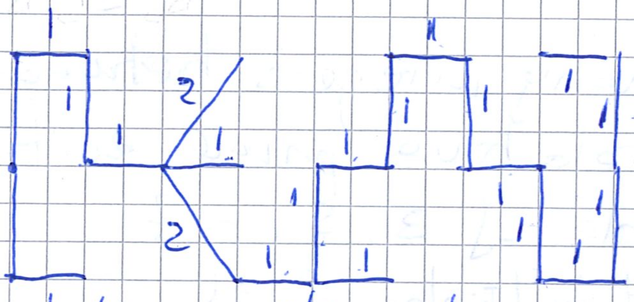
a) Show that $\mathcal{R}$ is an equivalence relation

i) $\mathcal{R}$ reflexive since for every spanning tree T, $T \mathcal{R} T$ since $c(T) = c(T)$, c the cost-functi

ii) $\mathcal{R}$ symmetric: if $T_1 \mathcal{R} T_2$ then $c(T_1) = c(T_2)$ and $c(T_2) = c(T_1)$, then $T_2 \mathcal{R} T_1$

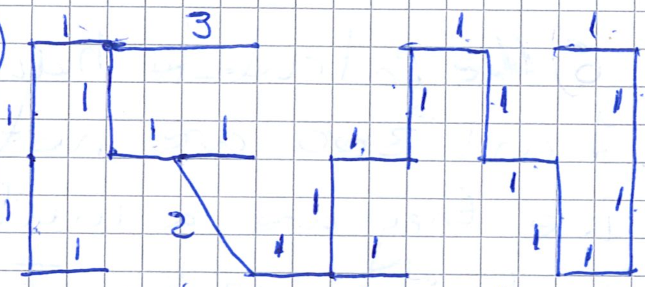
iii) $\mathcal{R}$ transitive since $T_1 \mathcal{R} T_2$ and $T_2 \mathcal{R} T_3$ then $c(T_1) = c(T_2) = c(T_3)$ and $T_1 \mathcal{R} T_3$

b)



is a tree with cost 24

c)



is a tree with cost 25