LINKÖPINGS UNIVERSITET
Matematiska institutionen
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Tentamen i Diskret Matematik, TATA52, TEN1, 2016–06–03, kl 14–19.

Inga hjälpmedel. Ej räknedosa. Fullständiga motiveringar krävs.

För betyg N behövs 3N-1 poäng.

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1. Visa med induktionsprincipen att $13^n - 6^n$ är delbart med 7 för alla heltal $n \geq 0$.

2. (a) En sammanhängande planär graf har endast hörn med gradtal 4. Ange antal regioner i grafen som en funktion av antal kanter.
   (b) Visa att varje graf med 4 noder är planär
   (c) Betrakta en graf med 5 noder, en av dem med gradtal 2. Visa att grafen är planär

3. (a) 100 studenter ska sudda tavlan vid 12 tillfällen i grupper av två. På hur många sätt kan studenterna sudda tavlan om varje student suddar högst en gång?
   (b) I en statistisk undersökning om mutationer i tre gener har man kommit på att gen A har muterat i 30%, gen B har muterat i 25% och gen C har muterat i 20% av befolkningen. Gener A och B har muterat i 10% av befolkningen, gener A och C i 10%, gener B och C i 10% och slutligen 5% av befolkningen har mutationer i alla tre gener. Hur stor andel av befolkningen ska tas för att vara säker att det förekommer (minst) en individ med någon muterad gen om man arbetar med populationsintervall om 5%?

4. Lös följande system av differensekvationer $\begin{array}{rcrrr} a_{n+1} & = & 2a_n+ & b_n & +n \\ b_{n+1} & = & a_n+ & 2b_n & +3n+1 \end{array}$, där $a_0 = 1 = b_0$.

5. En lärare ger 6 studenter tillbaka deras uppsatser. På hur många sätt kan läraren ge tillbaka uppsatserna till studenterna om
   (a) Exakt en student får tillbaka den egna uppstatsen?
   (b) Minst en student får tillbaka den egna uppsatsen?

6. Betrakta Fibonaccital som definieras av $f_n = f_{n-1} + f_{n-2}$, $n \geq 3$, $f_1 = 1$, $f_2 = 1$. Visa att $sgd(f_n, f_{n-1}) = 1$ för alla $n \geq 2$

Written Examination in Discrete Mathematics TATA52, TEN1, 2016–06-03, kl 14–19.
No calculator.
For grade N are required 3N-1 points.
Complete motivations required.

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1. Show with Mathematical Induction that $13^n - 6^n$ is divisible by 7 for all integers $n \geq 0$.

2. (a) A connected plane graph has vertices of degree exactly 4. Give the number of regions as a function of the number of edges.
   (b) Show that every graph with 4 vertices is a plane graph.
   (c) Consider a graph with 5 vertices, one of them of degree 2. Show that the graph is a plane graph.

3. (a) 100 students shall erase the black board 12 times in groups of two. In how many ways can the students erase the black board if each student erases the black board at most once?
   (b) In one statistical study on mutations of three genes has appeared that gene A has mutated in 30% of the population, gene B has mutated in 25%, gene C in 20%, genes A and B in 10% of population, genes A och C in 10%, genes B och C in 10% and finally 5% of the population has mutations in all three genes. What percentage of the population should be chosen to be sure that it will appear some individual with some mutated gene if the population is divided in intervals of 5%?

4. Solve the following system of difference equations $\begin{array}{rcrrr} a_{n+1} & = & 2a_n+ & b_n & +n \\ b_{n+1} & = & a_n+ & 2b_n & +3n+1 \end{array}$, where $a_0 = 1 = b_0$.

5. A teacher gives 6 students back their essays. In how many ways can the teacher give back the essays to the students if
   (a) just one student gets back his/her own essay?
   (b) at least one student gets back his/her own essay?

6. Consider Fibonacci numbers defined by $f_n = f_{n-1} + f_{n-2}$, $n \geq 3$, $f_1 = 1$, $f_2 = 1$. Show that $gcd(f_n, f_{n-1}) = 1$ for all $n \geq 2$

Answers to TATA52 Discrete Maths 3/6 2016

1) Show that $13^n - 6^n$ is a multiple of 7, $\forall n \geq 0$, using mathematic induction

i) It is true for $n=0$: $13^0 - 6^0 = 1-1 = 0 = (0)(7)$

ii) Assuming $13^p - 6^p = 7h$ for $p \geq 0$, we shall see that $13^{p+1} - 6^{p+1}$ also divisible by 7. But

$13^{p+1} - 6^{p+1} \overset{HI}{=} 13(7h + 6^p) - 6^{p+1} = (13)(7h) + (6)^p(13-6) =$

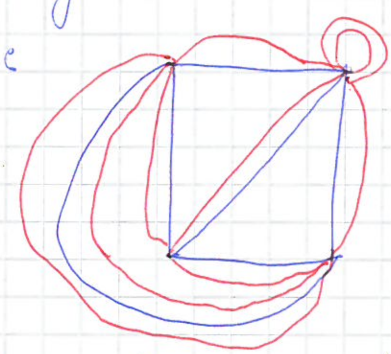
$= 7(\underbrace{13h + 6^p}_{})$, a multiple of 7 as required

We have shown that $13^n - 6^n = 7h$ for all $n \geq 0$

2a) By Euler's formula and handshaking lemma $\begin{cases} 4v = 2e \\ v - e + f = 2 \end{cases}$;

$-\frac{e}{2} + f = 2$; $f = \frac{4+e}{2}$

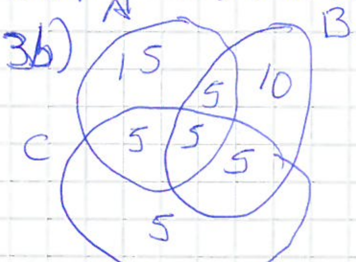
2b) It is enough to see that k_4 and multigraphs of it are planar:



c) The graph in c) is a edge-division of a graph with 4 vertices, and by b) it is planar

3a) $\binom{100}{2}\binom{98}{2}\binom{96}{2}\binom{94}{2}\binom{92}{2}\binom{90}{2}\binom{88}{2}\binom{86}{2}\binom{84}{2}\binom{82}{2}\binom{80}{2}\binom{78}{2} = \frac{100!}{76!\, 2^{12}}$

3b)



$|A \cup B \cup C| = 50\%$. Using Pigeons hole Principle, to be sure ~~to~~ find individuals with mutation one should pick 55% of the population

6) $\gcd(f_n, f_{n-1}) = 1$, where $f_n = f_{n-1} + f_{n-2}$, $f_2 = f_1 = 1$

Consider a divisor d of f_n and f_{n-1}, by division algorithm d divides also $f_n - f_{n-1} = f_{n-2}$, so d divides also $f_{n-1} - f_{n-2} = f_{n-3}$

using division algorithm $n-2$ times we get that d divides

4) Solve $\begin{cases} (1)\ a_{n+1} = 2a_n + b_n + n \\ (2)\ b_{n+1} = a_n + 2b_n + 3n + 1 \end{cases}$ $\quad a_0 = b_0 = 1$

First of all we see that $\begin{cases} b_{n+1} = 2a_{n+1} - 3a_n + (n+1) \text{ and} \\ b_n = 2a_n - 3a_{n-1} + n \end{cases}$ similarly. Setting them in (2), we have $2a_{n+1} - 8a_n + 6a_{n-1} = 4n$, or

$(*)\ a_{n+1} - 4a_n + 3a_{n-1} = 2n$. Characteristic eq is $r^2 - 4r + 3 = 0$

so $a_n^{(h)} = A_1 + A_2(3)^n$, and $a_n^{(p)} = (B_1 n + B_2)n$. Setting it in $(*)$ we get

$$n^2(B_1 - 4B_1 + 3B_1) + n(B_2 - 4B_2 + 3B_2 - 4B_1) + (4B_1 - 2B_2) = 2n$$

So $B_1 = -2/4 = -1/2$, $B_2 = 2B_1 = -1$; $a_n^{(p)} = -\frac{n^2}{2} - n$

$a_n = A_1 + A_2(3)^n - \frac{n^2}{2} - n$. The initial conditions

$$\begin{cases} a_0 = 1 = A_1 + A_2 \\ a_1 = 2+1 = A_1 + 3A_2 - \frac{1}{2} - 1 \end{cases} \Longleftrightarrow \begin{cases} 1 = A_1 + A_2 \\ 9/2 = A_1 + 3A_2 \end{cases} \Rightarrow \begin{matrix} A_2 = 7/4 \\ A_1 = -3/4 \end{matrix}$$

$$\boxed{a_n = \frac{7}{4}(3)^n - \frac{n^2}{2} - n - \frac{3}{4},\quad b_n = \frac{7}{4}(3)^n + \frac{n^2}{2} - n - 3/4}$$

5) We can use Principle for Inclusion and Exclusion directly, but here I'm using derangements

a) If just <u>one</u> student get back the right essay, then there is derangement of the other <u>five</u> students, so:

$$6d_5 = (6)\,5!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) = 264 \text{ ways}$$

b) At least <u>one</u> student getting back the right essay is the complement of derangement in the permutation set of the <u>six</u> students so:

$$6! - d_6 = 6!\left(1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!}\right) =$$

$$= 720\left(1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720}\right) = 455 \text{ ways}$$