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Matematiska institutionen
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Tentamen i Diskret Matematik, TATA52, TEN1, 2014–08–21, kl 8–13.

Inga hjälpmedel. Ej räknedosa. Fullständiga motiveringar krävs.

För betyg N behövs 3N-1 poäng.

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1. Betrakta Fibonaccital som definieras $F_0 = 0, F_1 = 1, F_{n+1} = F_n + F_{n-1}, n \geq 1$. Visa att för alla heltal $n \geq 10$, $F_n - F_{n-5} = 10F_{n-5} + F_{n-10}$.

2. Ägaren till en ö med femhörningsform vill fördela ön mellan sina barn så att varje barn får en del som är en femhörning. För att göra kartan med fördelningen behöver ägaren 11 milstolpar (varav fem finns på randen/stranden) som bas för att rita gränsdragningarna mellan de olika egendomarna. Hur många egendomar finns det? Hur många gränsdragningar? Rita en sådan karta.

3. På hur många sätt kan man välja fem (5) tal ur $\{1, 2, \ldots, 25\}$ om summan av de fem talen ska vara ett udda tal?

4. För att hitta ett hemligt tal mellan 1 och 200 löser vi följande system av kongruensekvationer:
$$\begin{cases} 3x \equiv 1 \; mod(11) \\ 4x \equiv 3 \; mod(13) \end{cases}$$
Vilket är det hemliga talet? Har vi hittat det egentligen?

5. Lös den rekursiva ekvationen $a_{n+2} - 4a_{n+1} + 3a_n = 2(3)^n + 3n$, $a_0 = 0$, $a_1 = 1$

6. En automata behöver lista alla funktioner från $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ till $B = \{1, 2, 3, 4\}$ med villkoren att $f(1) \neq 1$, $f(2) \neq 2$, $f(3) \neq 3$ och $f(4) \neq 4$. Hur många sådana funktioner finns det?

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No calculator.
For grade N are required 3N-1 points.
Complete motivations required.

1. Consider Fibonacci numbers defined by $F_0 = 0, F_1 = 1, F_{n+1} = F_n + F_{n-1}, n \geq 1$. Show that for any integer $n \geq 10$, $F_n - F_{n-5} = 10F_{n-5} + F_{n-10}$.

2. The owner of an island with pentagonal form will divide the island among relatives so that each relative gets a piece of land with pentagonal form. To do a map of the division the owner needs 11 milestones (5 of them on the shore/boundary of the island) as basis to draw the demarcations between the different properties. How many properties and demarcations are there? Draw such a map.

3. In how many ways can one choose five (5) numbers out of $\{1, 2, \ldots, 25\}$ such that the sum of the five is an odd integer?

4. To find **the** mystery number between 1 and 200 we solve the system of congruence equations:
$$\begin{cases} 3x & \equiv & 1 \ \ mod(11) \\ 4x & \equiv & 3 \ \ mod(13) \end{cases}$$
What is the mystery number? Have we really found it?

5. Solve the recurrence equation $a_{n+2} - 4a_{n+1} + 3a_n = 2(3)^n + 3n,\ a_0 = 0,\ a_1 = 1$

6. An automaton needs to list all functions from $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ to $B = \{1, 2, 3, 4\}$ with the conditions that $f(1) \neq 1$, $f(2) \neq 2$, $f(3) \neq 3$ and $f(4) \neq 4$. How many such functions are there?

Lösningar TATA52 21/8 2014

1) With Math Ind. i) $n \geq 10$ $f_{10} = 55$, $f_0 = 0$, $f_5 = 5$

So $10 - 5 = 50 = 10 \times 5 + 0$, true

ii) Assume that for $p \geq 10$ $f_p = 11 f_{p-5} + f_{p-10}$

Is still f_{p+1} eleven times f_{p-4} plus f_{p-9} ??

$$f_{p+1} = f_p + f_{p-1} \underset{\text{Math Indu}}{=} 11 f_{p-5} + f_{p-10} + 11 f_{p-6} + f_{p-11}$$

$$= 11(f_{p-5} + f_{p-6}) + f_{p-10} + f_{p-11} \underset{\text{Fibnrs}}{=} 11 f_{p-4} + f_{p-9}$$

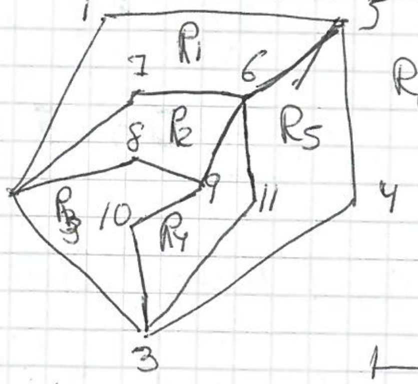
as required.

2) So we have a planar graph with 11 vertices where all the regions (faces) have boundary with 5 edges. We have $11 - e + f = 2$ and $\deg(R_i) = 5$, $1 \leq i \leq f$

$$\sum_{1}^{f} \deg(R_i) = 5f = 2e \;; \; f = \frac{2}{5} e$$

$$11 - e + \frac{2}{5} e = 2 \;; \; 45 = 3e, \; e = 15$$

$f = 6 - 1 = 5$. It will be 5 different ~~demarcations~~ properties and $15 - 5 =$ 10 inland demarcations (15 if one counts the demarcations on the beach) like in



(water).

3) If the sum is odd we can choose 1, 3 or 5 odd numbers between 1 and 25 (there are 13 odd numbers)

$$\binom{13}{1}\binom{12}{4} + \binom{13}{3}\binom{12}{2} + \binom{13}{5}\binom{12}{0} = 26\,598$$

choices

4) We solve the system $\begin{cases} 3x \equiv 1 \bmod 11 \\ 4x \equiv 3 \bmod 13 \end{cases}$,

which is equivalent to $\begin{cases} x \equiv 4 \bmod 11 \\ x \equiv 4 \bmod 13 \end{cases}$

$\begin{pmatrix} 3\times 4 = 12 = 11+1 \\ 4\times 4 = 16 = 13+3 \end{pmatrix}$

using Chinese Remainder th. The parameters are $b_1 = 4$, $N_1 = 13$, $b_2 = 4$, $N_2 = 11$, $N = 11\times 13 = 143$

$13x_1 \equiv 1 \bmod 11$, $2x_1 \equiv 1 \bmod 11$, $x_1 = 6$; $11x_2 \equiv 1 \bmod 13$

$x_2 = 6$

$x \equiv (4\times 13\times 6 + 4\times 11\times 6) \bmod 143 \equiv 24^2 \bmod 143$

$\equiv (-2)^2 \bmod 143 \equiv 4 \bmod 143$

So either $x = 4$ or $x = 147$. We cannot choose!!

5) $a_{n+2} - 4a_{n+1} + 3a_n = 2(3)^n + 3n$

For $a_n^{(h)}$: $r^2 - 4r + 3 = 0$; $r = 1, 3$. $a_n^{(h)} = A_1 + A_2(3)^n$

So $a_n^{(p)} = B(3)^n n + Cn^2 + Dn$. Setting it in the equation we get $B = 1/3$, $C = -3/4$, $D = 0$

$$a_n = (3)^{n-1} n - \frac{3n^2}{4} + A_1 + A_2(3)^n$$

In Cond $\begin{cases} a_0 = 0 = A_1 + A_2 \\ a_1 = 1 = 1 - 3/4 + A_1 + 3A_2 \end{cases} \Longrightarrow A_2 = 3/8, A_1 = -3/8$

$$a_n = (3)^{n-1}\left(n + \frac{9}{8}\right) - \frac{6n^2+3}{8}$$

6) Consider the sets $A_i = \{$function f like $f(i) = i\}$, $1 \le i \le 4$

We look for $|\overline{A_1 \cup A_2 \cup A_3 \cup A_4}|$. There are 4^8 functions

$|A_i| = 4^7$; $|A_{i_1} \cap A_{i_2}| = 4^6$, $|A_{i_1} \cap A_{i_2} \cap A_{i_3}| = 4^5$

$|A_1 \cap A_2 \cap A_3 \cap A_4| = 4^4$

So there are $4^8 - 4\times 4^7 + 6\times 4^6 - 4\times 4^5 + 4^4 = 5\times 4^6 + 4^4$

$= 4^4(16\times 5 + 1) = 256\times 81 = 20736$ good functions