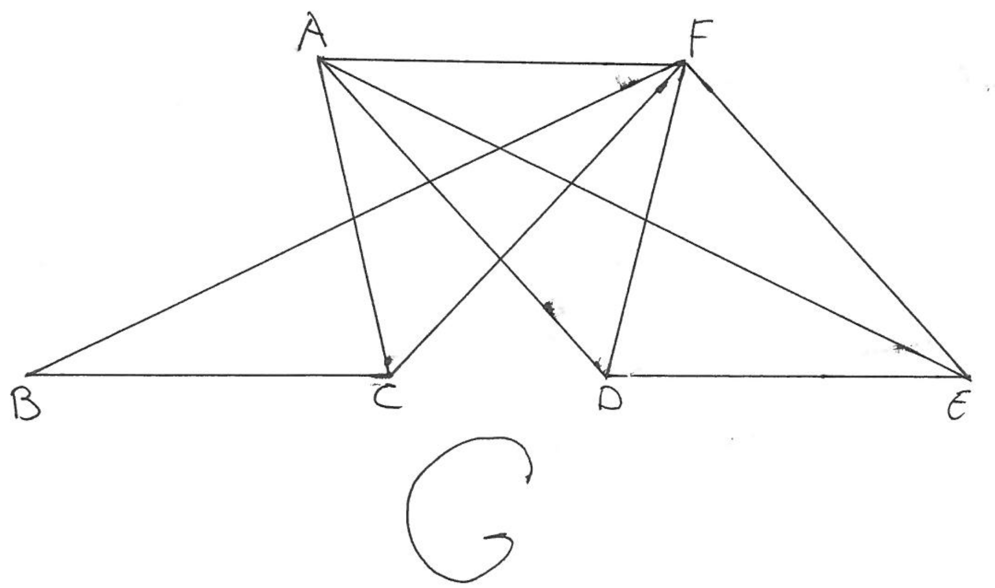
Written Examination in Discrete Mathematics TATA52, TEN1, 2014–06-04, kl 14–19.
No calculator.

For grade N are required 3N-1 points.

Complete motivations required.

1. Show with Mathematical Induction that $3^{2n} - 2^{2n}$ is divisible by 5, for $n \geq 0$.
2. Is the graph G in the figure below Hamiltonian?, Eulerian?, Planar?, Bipartite?
3. (a) Find the integer between 75 and 100 that is congruent with $19^{23} - 2^{23} - 13^{23}$ modulus 23 (1p)
   (b) Solve the Diophantic equation $70x + 98y = 196$, where $x,\ y \geq 0$. (1p)
   (c) Find the positive integer $t \leq 1000$ that is congruent with 5 modulus 11, with 4 modulus 13 and with 3 modulus 7 (1p)
4. Solve the recurrence equation $a_{n+2} - 2a_{n+1} + a_n = 3n + 2$, $n \geq 0$, $a_0 = 0$, $a_1 = -2$
5. A company produces 156 models of dishwashers. each model is identified with a number N and a sequence of four symbols $X_1X_2X_3X_4$, where X_1 can be 0 or 5; X_2 can be 0, 1 or 2; X_3 can be L or R, and X_4 is one of S, T, P, I. How many different values of the number N are needed so that all the models are completely identified?
6. Determine the probability that no group has more than 10 people when 40 people are divided in 5 groups.



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Tentamen i Diskret Matematik, TATA52, TEN1, 2014–06-04, kl 14–19.

Inga hjälpmedel. Ej räknedosa.

För betyg N behövs 3N-1 poäng.

Fullständiga motiveringar krävs.

1. Visa med induktionsprincipen att $3^{2n} - 2^{2n}$ är delbart med 5 för $n \geq 0$.

2. Är grafen G i figuren hamiltonsk?, Eulersk?, Planär?, Bipartit?

3. (a) Hitta heltalet som ligger mellan 75 och 100 och är kongruent med $19^{23} - 2^{23} - 13^{23}$ modulus 23 (1p)

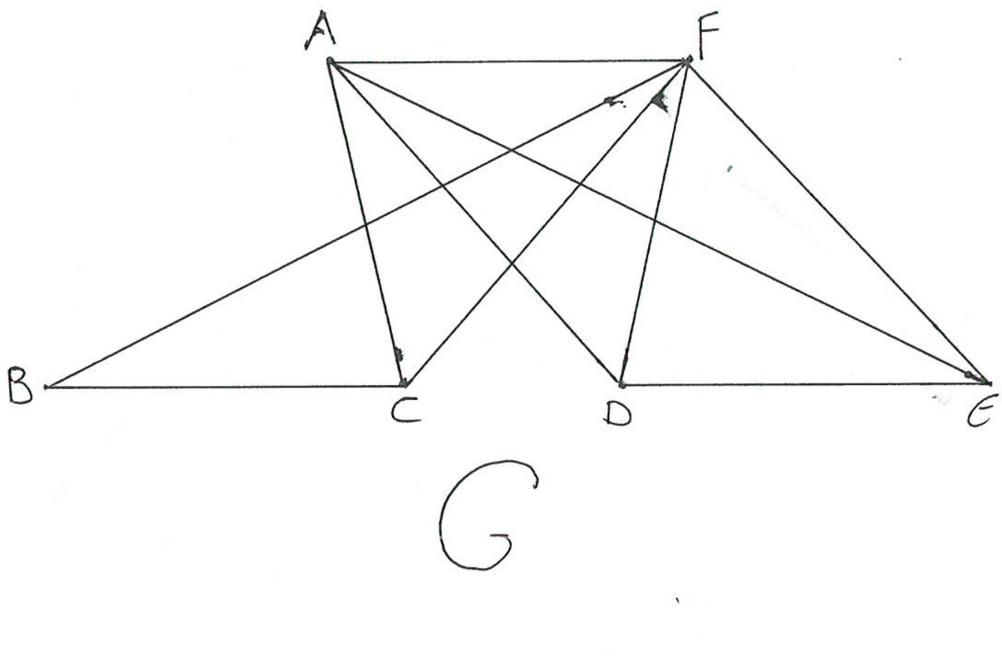
   (b) Lös den diofantiska ekvationen $70x + 98y = 196$, där $x, y \geq 0$. (1p)

   (c) Hitta det positiva heltalet $t \leq 1000$ som är kongruent med 5 modulus 11, med 4 modulus 13 och med 3 modulus 7 (1p)

4. Lös den rekursiva ekvationen $a_{n+2} - 2a_{n+1} + a_n = 3n + 2$, $n \geq 0$, $a_0 = 0$, $a_1 = -2$

5. Ett företag producerar 156 modeller av diskmaskiner. Varje modell är kodad med ett tal N och en följd av fyra symboler $X_1X_2X_3X_4$, där X_1 kan vara 0 eller 5; X_2 kan vara 0, 1 eller 2; X_3 kan vara L eller R och X_4 är en (1) av S, T, P, I. Hur många olika värden på talet N behövs så att alla modeller är fullständigt identifierade?

6. Bestäm sannolikheten att ingen grupp har fler än 10 personer när man delar 40 personer i 5 grupper.



Solutions TATA52 Discrete Mathematics 4/6 2014

1) Show with Math-Ind that $3^{2n} - 2^{2n} = 5k$ for $n \geq 0$

i) For $n = 0$ $\quad 3^0 - 2^0 = 1 - 1 = 0 = 0 \times 5$ true

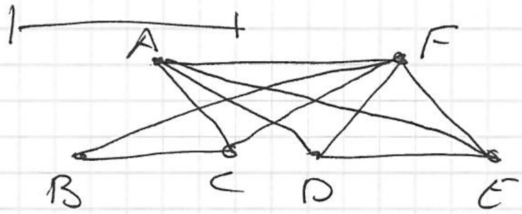
ii) Assume that $3^{2p} - 2^{2p} = 5k_p$ for some $p \geq 0$

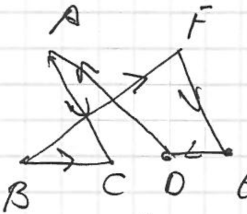
Is still $3^{2(p+1)} - 2^{2(p+1)}$ a multiple of 5?

$$(3)^{2p+2} - (2)^{2p+2} = 9\,(3)^{2p} - 4\,(2)^{2p} \underset{\text{Hyp}}{=} 9\,(5k_p + (2)^{2p}) - 4\,(2)^{2p}$$

$$= 9\,(5k_p) + 9\,(2)^{2p} - 4\,(2)^{2p} = 5\,(9k_p + (2)^{2p}) \text{ as wanted}$$

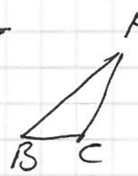
With Math Ind. we have that $3^{2n} - 2^{2n} = 5k \ \forall n \geq 0$.

2) The graph G  is

i) Hamiltonian la Hamiltonian cycle 

ii) Not Eulerian since $\deg F = 5$, $\deg C = \deg D = \deg E = 3$

iii) Planar with plane-embedding 

iv) Not bipartite since it contains the triangle 

3a) Since 23 is a prime number $19^{23} \equiv 19 \bmod 23$, $2^{23} \equiv 2 \bmod 23$ and $13^{23} \equiv 13 \bmod 23$ so

$$19^{23} - 2^{23} - 13^{23} \equiv 19 - 2 - 13 \equiv 4 \bmod 23.$$

The number we ~~search~~ after is $4 + 4 \times 23 = \underline{\underline{96}}$

b) $70x + 98y = 196$, $x, y \geq 0$ $\quad sgd(70, 98) = 14$ we have

$5x + 7y = 14$, $x, y \geq 0$ $\quad sgd(5, 7) = 1$ $\quad 1 = 5(3) + 7(-2)$

$$\begin{cases} x = (3)14 - 7n \geq 0 & 7n \leq 42 \quad n \leq 6 \\ y = (-2)14 + 5n \geq 0 & 28 \leq 5n \quad n \geq 6 \end{cases} \qquad \{\, n = 6 \qquad \begin{matrix} x = 0 \\ y = 2 \end{matrix}$$

c) $\begin{cases} t \equiv 5 \bmod 11 \\ t \equiv 4 \bmod 13 \\ t \equiv 3 \bmod 7 \end{cases}$ $t \le t \le 1000$ Using the Chinese Remainder Theorem we get

$t \equiv 5 \times 91 \times x_1 + 4 \times 77 \times x_2 + 3 \times 143 \times x_3 \bmod 1001$

where $91x_1 \equiv 1 \bmod 11$, $77x_2 \equiv 1 \bmod 13$ and $143x_3 \equiv 1 \bmod 7$

$3x_1 \equiv 1 \bmod 11$, $x_1 = 4$; $-x_2 \equiv 1 \bmod 13$, $x_2 = 12$; $3x_3 \equiv 1 \bmod 7$, $x_3 = 5$

$t \equiv 5 \times 4 \times 91 + 4 \cancel{\times} (-1) \times 77 + 3 \times 143 \times 5 \equiv 3657 \bmod 1001$

$\equiv (3657 - 3003) \bmod 1001 \equiv \underline{654}$

4) Solve $a_{n+2} - 2a_{n+1} + a_n = 3n + 2$; $a_0 = 0$; $a_1 = -2$

i) $a_n^{(h)}$: $a_{n+2} - 2a_{n+1} + a_n = 0$; $r^2 - 2r + 1$; $r = 1$ (double)

$$a_n^{(h)} = A_1 n + A_2$$

ii) $a_n^{(p)} = (B_1 n + B_2) n^2 = B_1 n^3 + B_2 n^2$, we set it in the equ:

$B_1(n^3 + 6n^2 + 12n + 8) + B_2(n^2 + 4n + 4) - 2B_1(n^3 + 3n^2 + 3n + 1) - 2B_2(n + 2n + 1)$
$+ B_1 n^3 + B_2 n^2 = 3n + 2$

$n^3(B_1 - 2B_1 + B_1) + n^2(6B_1 - 6B_1 + B_2 - 2B_2 + B_2) + n(12B_1 - 6B_1 + 4B_2 - 4B_2)$
$+ 8B_1 - 2B_1 + 4B_2 - 2B_2 = 3n + 2$. Comparing

$0 = 0$; $0 = 0$; $6B_1 = 3$, $B_1 = 1/2$; $6B_1 + 2B_2 = 2$, $B_2 = -1/2$

$a_n^{(p)} = \dfrac{n^3 - n^2}{2}$; $a_n = \dfrac{n^3 - n^2}{2} + A_1 n + A_2$

Int. Cond. $\begin{cases} a_0 = 0 = A_2 & A_2 = 0 \\ a_1 = -2 = \frac{1-1}{2} + A_1 + A_2 & A_1 = -2 \end{cases}$

$$a_n = \frac{1}{2}(n^3 - n^2 - 4)$$

5) $N - x_1 x_2 x_3 x_4$. There are 156 models and each value of N identifies ($|X_1| = 2$, $|X_2| = 3$, $|X_3| = 2$, $|X_4| = 4$) $2 \times 3 \times 2 \times 4 = 48$ models so one needs

$\underline{4} = 3 + 1 = \left\lfloor \frac{156}{48} \right\rfloor + 1$ numbers N

6) Calculate the probability of no group having more of 10 people when dividing 40 people in 5 gr:

i) The total number of divisions is $\binom{40+5-1}{4} = \binom{44}{4}$

ii) We consider the complement: that some group contains at least 11 people. Consider the sets
$A_i = \{$divisions where group i contains at least 11 people$\}$

We want $\dfrac{\binom{44}{4} - |A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5|}{\binom{44}{4}}$

iii)

$|A_i| = \binom{29+4}{4} = \binom{33}{4}$

$|A_{i_1} \cap A_{i_2}| = \binom{18+4}{4} = \binom{22}{4}$

$|A_{i_1} \cap A_{i_2} \cap A_{i_3}| = \binom{7+4}{4} = \binom{11}{4}$

$|A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap A_{i_4}| = |A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5| = 0$

iv) $\dfrac{\binom{44}{4} - 5\binom{33}{4} + 10\binom{22}{4} - 10\binom{11}{4}}{\binom{44}{4}} = \dfrac{1104}{37044}$ ca 3%