Tentamen i Diskret Matematik, TATA52, TEN1, 2013–08-22, kl 8–13.
No calculator.
For grade N are required 3N-1 points.
Complete motivations required.

1. Show that $1^2 - 2^2 + 3^2 + \ldots + (-1)^{n+1}n^2 = \dfrac{(-1)^{n+1}n(n+1)}{2}$ for any integer $n \geq 1$.

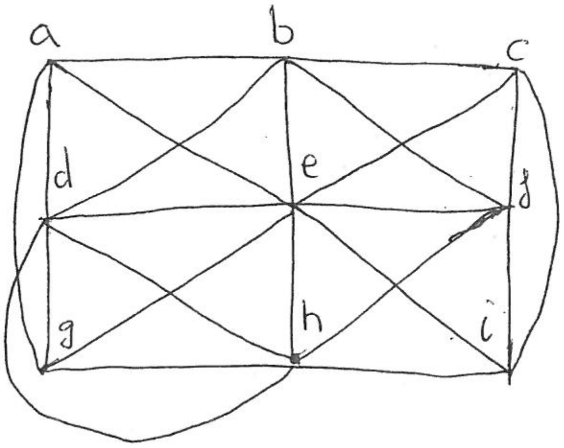
2. (a) Is the graph A below Eulerian?
   (b) Is the graph A below Hamiltonian?
   (c) The weighted graph B below shows the costs to make roads between the new areas in a town. Find the cheapest road system that connects all the new areas.

3. (a) How many positive divisors does 30! have? (2p)
   (b) How many positive divisors which are the square of an integer has 30!? (1p)

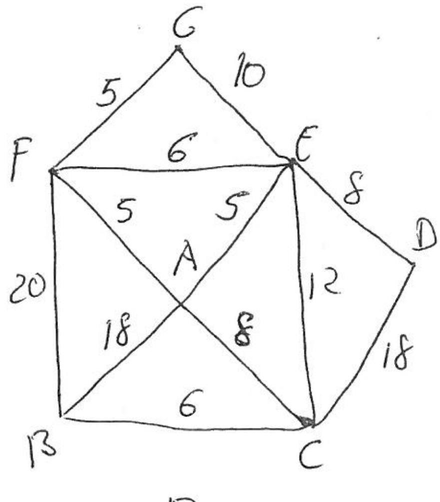
4. How many integer solutions does $x_1+x_2+x_3+x_4 \leq 24$ have if $0 \leq x_i \leq 8$ for $1 \leq i \leq 4$?

5. (a) Solve the linear recurrence equation $b_n = b_{n-1} + 2b_{n-2}$, $n \geq 2$, $b_0 = b_1 = 1$.
   (b) Use (a) to solve the recurrence equation $\sqrt{a_n} = \sqrt{a_{n-1}} + 2\sqrt{a_{n-2}}$, $n \geq 2$, $a_0 = a_1 = 1$.

6. (a) Find strictly positive integers a, b, c such that 31 divides $5a + 7b + 11c$. (1p)
   (b) Show that if 31 divides $5a + 7b + 11c$ for some integers a, b, c then 31 divides $11a + 3b + 18c$ as well. (2p).



A



B

LINKÖPINGS UNIVERSITET
Matematiska institutionen
Milagros Izquierdo

Tentamen i Diskret Matematik, TATA52, TEN1, 2013–08–22, kl 8–13.

Inga hjälpmedel. Ej räknedosa. Fullständiga motiveringar krävs.

För betyg N behövs 3N-1 poäng.

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1. Visa att för alla heltal $n \geq 1$ gäller att $1^2 - 2^2 + 3^2 + \ldots + (-1)^{n+1}n^2 = \dfrac{(-1)^{n+1}n(n+1)}{2}$.

2. (a) Är grafen A nedan eulersk?

   (b) Är grafen A nedan hamiltonsk?

   (c) Viktade grafen B nedan visar kostnaderna för att göra olika vägförbindelser mellan nya områden i en stad. Hitta det billigaste vägnätet som föbinder **alla** de nya områdena med varandra.

3. (a) Hur många positiva delare har 30! ? (2p)

   (b) Hur många positiva delare som är kvadrater av ett heltal har 30! ? (1p)

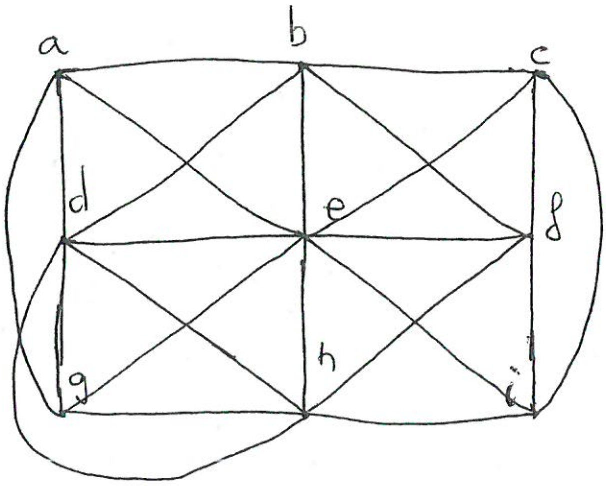
4. Hur många heltalslösningar har $x_1 + x_2 + x_3 + x_4 \leq 24$ om $0 \leq x_i \leq 8$ för $1 \leq i \leq 4$?

5. (a) Lös den linjära rekursiva ekvationen $b_n = b_{n-1} + 2b_{n-2}$, $n \geq 2$, $b_0 = b_1 = 1$.

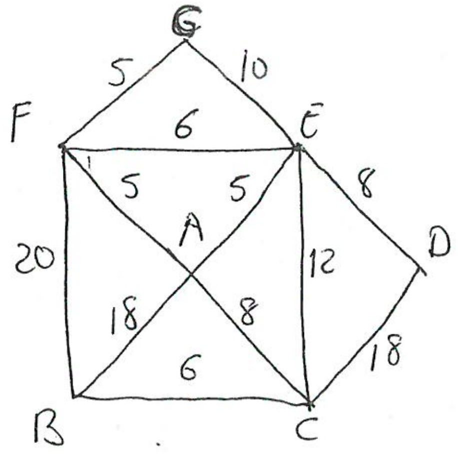
   (b) Använd (a) för att lösa den rekursiva ekvationen $\sqrt{a_n} = \sqrt{a_{n-1}} + 2\sqrt{a_{n-2}}$, $n \geq 2$, $a_0 = a_1 = 1$.

6. (a) Hitta strikt positiva heltal a, b, c så att 31 delar $5a + 7b + 11c$. (1p)

   (b) Visa att om 31 delar $5a + 7b + 11c$ för några heltal a, b, c så delar 31 också $11a + 3b + 18c$ (2p).



A



B

Answers TATA 52 22/8 2013

1) Show that $\sum_{k=1}^{n}(-1)^{k+1}k^2 = \frac{(-1)^{n+1}n(n+1)}{2}$ for $n \geq 1$

With Math. Ind. we show

i) for $n = 1$ $(-1)^2 1^2 = \frac{(-1)^2 2 \times 1}{2}$, as wanted

ii) We assume that for $p \geq 1$ $\sum_{1}^{p}(-1)^{k+1}k^2 = \frac{(-1)^{p+1}p(p+1)}{2}$

and we show that:

$$\sum_{1}^{p+1}(-1)^{k+1}k^p = \sum_{1}^{p}(-1)^{k+1}k^2 + (-1)^{p+2}(p+1)^2 \overset{\text{Ind. Hyp.}}{=}$$

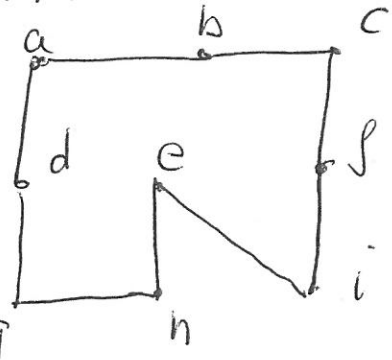
$$= \frac{(-1)^{p+1}(p+1)p}{2} + (-1)^{p+2}(p+1)^2 = (-1)^{p+1}(p+1)\left[\frac{p}{2} - p+1\right]$$

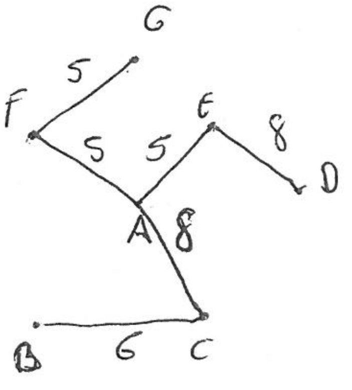
$$= (-1)^{p+1}(p+1)\left(\frac{-p-2}{2}\right) = (-1)^{p+2}\frac{(p+1)(p+2)}{2},$$ as desired.

2) a) Graph A is <u>not</u> Eulerian since $\deg(b) = \deg(f) = 5$

However it contains an open Euler path.

b) Grapha A is Hamiltonian: the following is a Hamiltonian circuit



c) 

with total cost <u>37</u> units.

4) Integer solutions to $x_1+x_2+x_3+x_4 \leq 24$, $0 \leq x_i \leq 8$.
Equivalently integer solutions $x_1+x_2+x_3+x_4+x_5=24$
$0 \leq x_1, x_2, x_3, x_4 \leq 8$ and $x_5 \geq 0$
We calculate first number of solutions when $x_i \geq 0$
: $\binom{28}{4}$ and take away the number of solutions
when x_1, x_2, x_3 or $x_4 \geq 9$. These are:
$4\binom{19}{4}+\binom{4}{2}\binom{10}{4}$. Totally $\binom{28}{4}-4\binom{19}{4}+6\binom{10}{4}$

3) $30! = 2^{15+7+3+1} \times 3^{10+3+1} \times 5^{6+1} \times 7^4 \times 11^2 \times 13^2 \times 17 \times 19 \times 23 \times 29$
a) So positive divisors (integer) are:
$27 \times 15 \times 8 \times 5 \times 3 \times 3 \times 2 \times 2 \times 2 \times 2 \ldots$
b) Here are the exponents of the prime factors even so:
$14 \times 8 \times 4 \times 3 \times 2 \times 2 \times 1 \times 1 \times 1 \times 1 \ldots$

5a) Solve $b_n = b_{n-1} + 2b_{n-2}$, $b_0 = b_1 = 1$
C.E $r^2 - r - 2 = 0 \Rightarrow r = 2, -1$ Then $b_n = A_1(-1)^n + A_2(2)^n$
Int. Cond. $\begin{cases} b_0 = 1 = A_1 + A_2 \\ b_1 = 1 = -A_1 + 2A_2 \end{cases} \Rightarrow \begin{cases} A_2 = 2/3 \\ A_1 = 1/3 \end{cases}$
$$b_n = \frac{2^{n+1} + (-1)^n}{3}$$
5b) We may do the change of variable $b_n = \sqrt{a_n}$
$b_0 = \sqrt{a_0} = \sqrt{1} = 1$, $b_1 = \sqrt{a_1} = \sqrt{1} = 1$. Then as
we know b_n, we know that $a_n = \frac{1}{9}\left(2^{n+1}+(-1)^n\right)^2$

6a) $a=1, b=5, c=2$ is such a set $(5+35+22=62=2\times 31)$
$a=5, b=5, c=3$ another possible set $(93 = 3 \times 31)$
b) If $31 \mid (5a+7b+11c)$, as $31 \mid (31a+31b+62c)$
Then 31 divides $31a+31b+62c-4(5a+7b+11c)$
$= (31-20)a + (31-28)b + (62-44)c$
$= 11a + 3b + 18c$ as wanted