LINKÖPINGS UNIVERSITET
Matematiska institutionen
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Tentamen i Diskret Matematik, TATA52, TEN1, 2013–06-01, kl 08–13.
Inga hjälpmedel. Ej räknedosa.

För betyg N behövs 3N-1 poäng.

Fullständiga motiveringar krävs.

1. (a) Är grafen G_1 hamiltonsk? (1p)
   (b) Visa att grafen G_2 är planär. Ange en planär inbäddning av grafen (1p)
   (c) Visa att man kan rita grafen G_3 utan att lyfta pennan från pappret och utan att gå samma kant 2 gånger. Ange en sådan väg (1p)

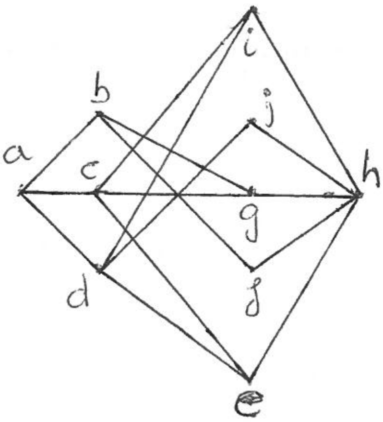
2. (a) Till en konferens handlar matematiska institutionen bakelser av två typer: dyrare och billigare. Varje bakelse kostar 42 eller 30 SEK. Hur många bakelser köpte man av varje typ om man betalade totalt 2706 SEK? (1p)
   (b) Om ett "hashtal" vet man att det ligger mellan 1150 och 1550 och att det är kongruent med talet $69^{480819}\,(mod\,385)$. Hitta talet (2p)

3. (a) Hur många reflexiva relationer på $A = \{1, 2, \ldots, 16\}$ finns det? (1p)
   (b) Sverige har en lista med 100 idrottare som kan skickas till OS 2014. Hur många olika delegationer kan man skicka om Ada, Blaise, Carl, David, Emmy, Fredrik, Gertrude, Jane, Isaac och Linda redan är kvalificerade? (1p)
   (c) Hur många binära listor av längd ~~högst~~ 19 finns det om de första 5 symbolerna i varje lista är ettor? (1p)

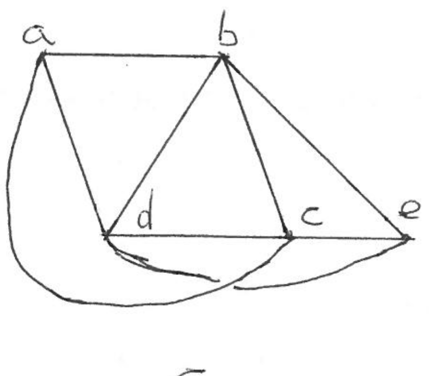
4. (a) Hitta rekursiva ekvationen för antalet binära listor av längd n, $n \geq 1$, som uppfyller villkoret att de innehåller ett udda antal 0:or. (1p)
   (b) Lös den rekursiva ekvationen $a_n + a_{n-1} = 2^{n-1}$, $n \geq 2$, $a_1 = 1$ (2p)

5. På hur många olika sätt kan man omordna bokstäverna i SAMTAL så att inga av S, M, T, L hamnar på sina ursprungliga platser?

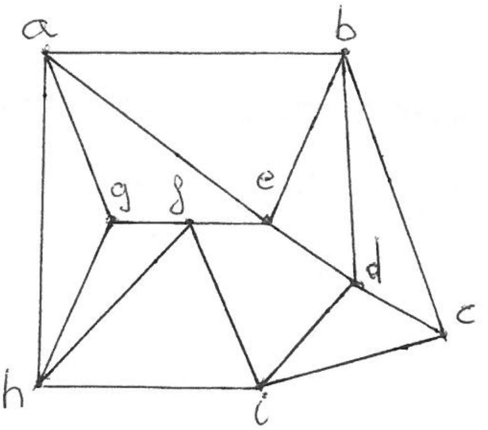
6. Visa med induktionsprincipen att $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$ för $n \geq 1$.



G_1



G_2



G_3

Written Examination in Discrete Mathematics TATA52, TEN1, 2013–06-01, kl 08–13.

No calculator.

For grade N are required 3N-1 points.

Complete motivations required.

1. (a) Is the graph G_1 Hamiltonian? (1p)
   (b) Show that the graph G_2 is planar. Give a planar embedding of the graph (1p)
   (c) Show that one can draw the graph G_3 without lifting the pen from the paper and without drawing the same edge twice. Give such a way of drawing. (1p)

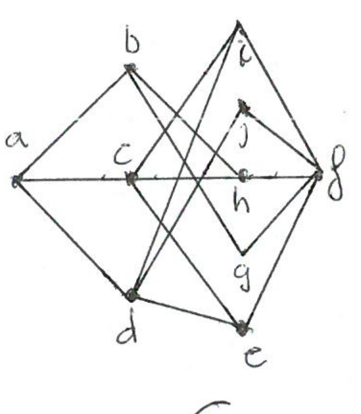
2. (a) MAI buys to a conference two kinds of cakes: cheaper and more expensive. Each cake costs 42 or 30 SEK. How many cakes of each type does MAI buy if it pays 2706 SEK in total? (1p)
   (b) We know that a "hash number"is between 1150 and 1550. We also know that it is congruent with the number $69^{480819}\,(mod\,385)$. Find the hash number. (2p)

3. (a) How many reflexive relations on $A = \{1, 2, \ldots, 16\}$ are there? (1p)
   (b) Sweden has a list with 100 athletes that can be sent to the Olympic Games 2014. How many different delegations can Sweden send if Ada, Blaise, Carl, David, Emmy, Fredrik, Gertrude, Jane, Isaac and Linda are already qualified? (1p)
   (c) How many binary lists of length ~~at most~~ 19 are there if the first 5 symbols in each list are ones? (1p)

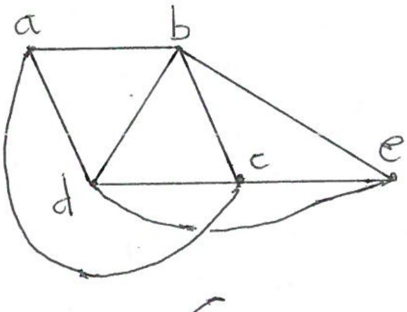
4. (a) Give the recurrence equation for the number of binary lists of length n, $n \geq 1$, that contain an odd number of zeros. (1p)
   (b) Solve the recurrence equation $a_n + a_{n-1} = 2^{n-1}$, $n \geq 2$, $a_1 = 1$ (2p)

5. In how many ways can one rearrange the letters in SAMTAL such that none ofS, M, T, L never come back to its original position?

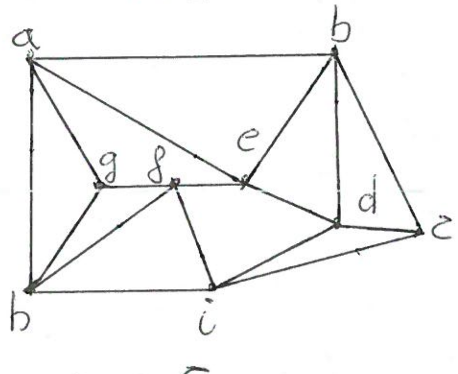
6. Show with mathematical induction that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$ för $n \geq 1$.



G_1



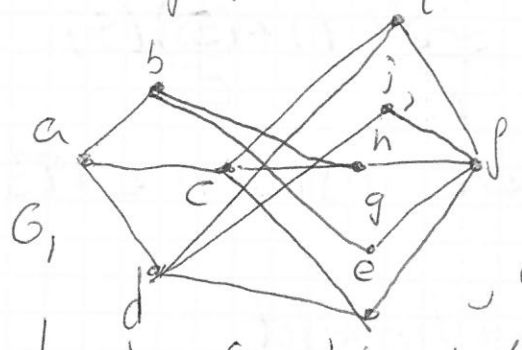
G_2



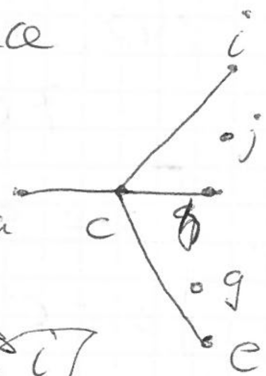
G_3

TATA52 Discrete Maths. 1/6 2013

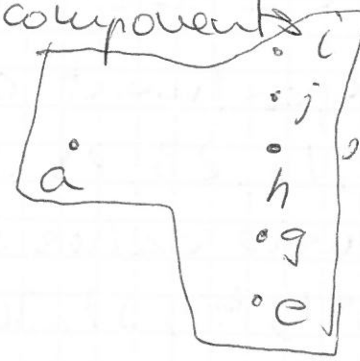
1a) Is the graph G_1 Hamiltonian? No, since



, $G_1 - \{b, d, f\}$ is

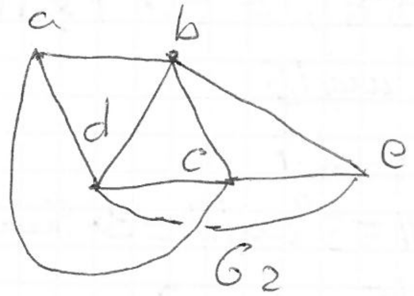


, it has 3 components, but $G_1 - \{b, d, f, g, e\}$ is

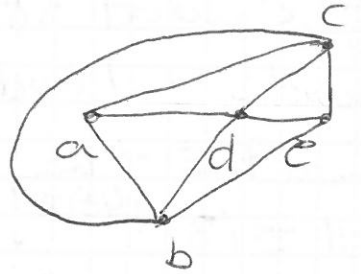


, with 6 components, and $6 > 5$.

b) Show that G_2 is planar

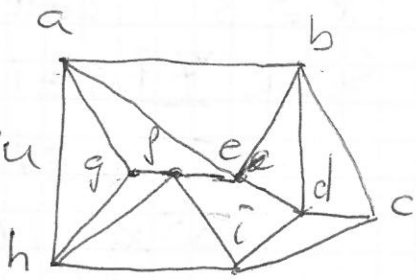


$\cong$



planar embedding

c) There is an open Eulerian path in



since $d(a) = d(b) = d(h) = d(f) = d(e)$
$= d(i) = d(d) = 4$ and $d(g) = d(c) = 3$

The open Eulerian path starts in g and finishes in c or vice-versa.

3a) Any reflexive relation contains the pair (i,i) $1 \leq i \leq 16$
So there are $2^{16 \times 16 - 16} = 2^{240}$

b) Any of the remaining 90 athletes can compete or not in OG 2014 so $2^{100-10} = 2^{90} = \sum_{i=10}^{100} \binom{100}{i}$

c) Again $2^{19-5} = 2^{14}$ (lists of length 19)

2a) $42E + 30C = 2706$ (E=expensive, C=cheap)

$(42, 30) = 6$, and $6 \mid 2706$ ($2706 = 451 \times 6$)

$7E + 5C = 451$ $\quad 1 = \underbrace{(-2)}_{E_0}(7) + \underbrace{(3)}_{C_0}(5)$

$$\begin{cases} 0 \le E = -902 + 5n \\ 0 \le C = 1353 - 7n \end{cases}, \text{ where } 181 \le n \le 193$$

So the department could have bought 3, 8, 13, 18, 23, 28, 33, 38, 43, 48, 53, 58 or 63 more expensive cakes and respectively 86, 79, 72, 65, 58, 51, 44, 37, 30, 23, 16, 9 or 2 cheaper cakes.

2b) $69^{480819} \equiv x \bmod 385$, $385 = 5 \times 7 \times 11$

$x \equiv (-1)^{480819} \bmod 5 \equiv -1 \bmod 5 \equiv \underline{4 \bmod 5}$

$x \equiv (-1)^{480819} \bmod 7 \equiv -1 \bmod 7 \equiv \underline{6 \bmod 7}$

$x \equiv 3^{480819} \bmod 11 \equiv (3^{10})^{48081} \cdot 3^9 \bmod 11 \equiv 3^9 \bmod 11 \equiv 3 \times 5 \bmod 11 \equiv \underline{4 \bmod 11}$

Solve $\underline{77} x_1 \equiv 1 \bmod 5 \Leftrightarrow 2x_1 \equiv 1 \bmod 5$, $x_1 \equiv 3$

$55 x_2 \equiv 1 \bmod 7 \Leftrightarrow -x_2 \equiv 1 \bmod 7$; $x_2 \equiv 6$ $(\equiv -1)$

$35 x_3 \equiv 1 \bmod 11 \Leftrightarrow 2x_3 \equiv 1 \bmod 11$; $x_3 \equiv 6$

$x \equiv 4 \times 77 \times 3 + 6 \times 55 \times 6 + 4 \times 35 \times 6 \bmod 385$

$x \equiv 924 + 1980 + 840 \bmod 385 \equiv 279 \bmod 385$

So $x = 279 + 3 \times 385 = 279 + 1155 =$

$$\boxed{x = 1434}$$

5) We may count with PIE how many ~~arrangements~~ arrangements if some of S, M, T or L stays in the original place.

First of all there are $\frac{6!}{2!}$ arrangements of the 6 letters

For each of S, M, T and L, there are $\frac{5!}{2!}$ arrangements where the letter is fixed (more letters could be fixed)

There are $\frac{4!}{2!}$ arrangements where two of S, M, T and L are fixed (more letters could be fixed)

There are $\frac{3!}{2!}$ arrangements where three of the four letters are fixed

Finally there is $\frac{2!}{2!}$ arrangement where all the four letters (and so the A's) are fixed

So totally $\frac{6!}{2!} - \left(4 \times \frac{5!}{2!} - \binom{4}{2}\frac{4!}{2!} + \binom{4}{3}\frac{3!}{2!} - \binom{4}{4}\frac{2!}{2!}\right)$

$= 360 - (240 - 72 + 12 - 1) = 360 - 59 = \underline{\underline{301 \text{ ways}}}$

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4c) a_n = # binary list with odd number of zeros

$a_1 = 1$; $a_2 = 2$ In general $a_n = f(a_{n-1})$

Two cases last digit is a 0: $\underbrace{\_\ \_\ \_\ \_\ \_}_{\text{even number of 0's with length } n-1}\ 0$

binary lists with odd number of 0's = # binary lists with even number of 0's so case 1 we have a_{n-1}

Case 2: last digit is a 1: $\underbrace{\_\ \_\ \_\ \_}_{\text{length } n-1 \text{ and odd number of 0's}}\ 1$

So in case 2 we have also a_{n-1}. Thus

$$\boxed{a_n = a_{n-1} + a_{n-1} = 2a_{n-1} \ ; \ a_1 = 1}$$

4b) Solve $a_n + a_{n-1} = 2^{n-1}$, $n \geq 2$, $a_1 = 1$

Homogeneous part a_n^H solves $a_n^H + a_{n-1}^H = 0$ $(r+1=0)$

$$a_n^H = A(-1)^n$$

Particular solution $a_n^P = B(2)^n$, where

$$2B(2)^{n-1} + B(2)^{n-1} = (2)^{n-1};\ 3B = 1;\ B = 1/3$$

$$a_n = \frac{2^n}{3} + A(-1)^n$$

Initial cond.: $a_1 = 1 = \frac{2}{3} - A$; $A = -\frac{1}{3}$

$$\boxed{a_n = \frac{(2)^n + (-1)^{n+1}}{3}}$$

6) Show using Math. Ind. that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$, $n \geq 1$

i) $n=1$ $\quad 1\binom{1}{1} = 1 = 1 \times 2^0$. Ok

ii) Assume that $\sum_{k=1}^{p} k\binom{p}{k} = p2^{p-1}$ and show that

$$\sum_{k=1}^{p+1} k\binom{p+1}{k} = (p+1)2^p.$$

$$\sum_{k=1}^{p+1} k\binom{p+1}{k} = \sum_{k=1}^{p} k\binom{p+1}{k} + (p+1)\binom{p+1}{p+1} = \sum_{k=1}^{p}\left[k\binom{p}{k} + k\binom{p}{k-1}\right] + (p+1)$$

$$= \sum_{k=1}^{p} k\binom{p}{k} + \sum_{k-1=0}^{p-1}(k-1)\binom{p}{k-1} + \sum_{k-1=0}^{p-1} 1\binom{p}{k-1} + (p+1)$$

We use math ind twice $\sum_{k=1}^{p} k\binom{p}{k} = p2^{p-1}$

Math Ind

$$\sum_{k-1=0}^{p-1}(k-1)\binom{p}{k-1} = p2^{p-1} - p\binom{p}{p} = p2^{p-1} - p$$

We also know that $\sum_{k-1=0}^{p-1}\binom{p}{k-1} = 2^p - \binom{p}{p} = 2^p - 1$

Altogether $p2^{p-1} + p2^{p-1} - p + 2^p - 1 + p + 1 = 2p2^{p-1} + 2^p$

$$= 2^p(p+1) \text{ as required}$$