Lösningar, TATA77, 2019-01-08

1. $y(n+2) - 2y(n+1) - 3y(n) = 12 - 6\cdot 2^n$, $n \in \mathbb{N}$, $y(0) = 6$, $y(1) = -6$.

Enkelsidig z-transform ger ($Y = \mathcal{Z}_+ y$):

$$z^2 Y(z) - 6z^2 + 6z - 2(zY(z) - 6z) - 3Y(z) = \frac{12z}{z-1} - \frac{6z}{z-2},$$

$$(z^2 - 2z - 3)Y(z) = \frac{6z(2z-4-z+1)}{(z-1)(z-2)} + 6z^2 - 18z,$$

$$(z-3)(z+1)Y(z) = \frac{6z(z-3)}{(z-2)(z-1)} + 6z(z-3),$$

$$Y(z) = \frac{6z(z^2-3z+3)}{(z-2)(z-1)(z+1)} = z\left(\frac{2}{z-2} + \frac{-3}{z-1} + \frac{7}{z+1}\right) =$$

$$= \frac{2z}{z-2} - \frac{3z}{z-1} + \frac{7z}{z+1}, \quad |z| > 2.$$

Tabell ger nu $y(n)$.

Svar: a) $\dfrac{6z(z^2-3z+3)}{(z-2)(z-1)(z+1)}$, $|z| > 2$.

b) $2\cdot 2^n - 3 + 7(-1)^n$, $n \in \mathbb{N}$.

2. $\hat{u}(s) = \dfrac{s^3 - 13s}{s^2 - 4s + 3}$, $1 < \operatorname{Re} s < 3$.

$$\hat{u}(s) = s + 4 - \frac{12}{(s-1)(s-3)} =$$

$$= s + 4 + \frac{6}{s-1} + \frac{-6}{s-3}, \quad 1 < \operatorname{Re} s < 3.$$

$\chi(t) \xrightarrow{\mathcal{L}}$	$\frac{1}{s}$, $\operatorname{Re} s > 0$
$\chi(-t)$	$\frac{1}{-s}$, $\operatorname{Re}(-s) > 0$
$e^{3t}\chi(-t)$	$-\frac{1}{s-3}$, $\operatorname{Re} s < 3$

Detta, och tabell, ger:

Svar: $u(t) = \delta'(t) + 4\delta(t) + 6e^t\chi(t) + 6e^{3t}\chi(-t)$.

3. u π-periodisk, $\frac{1}{\pi}\int_0^{\pi} u(t-r)r\,dr = \cos^2 t$, $t\in\mathbb{R}$.

$T=\pi \Rightarrow \Omega = 2$. $VL = (u *_T f)(t)$, där

$f(t) = t$, $0\le t<\pi$, och f är π-periodisk.

$HL = \cos^2 t = \left(\frac{e^{it}+e^{-it}}{2}\right)^2 = \frac{1}{4}e^{i2t} + \frac{1}{2} + \frac{1}{4}e^{-i2t}$,

så $\widehat{VL}(n) = \widehat{HL}(n)$ ger: $\hat{u}(n)\hat{f}(n) = \begin{cases} 1/2, & n=0 \\ 1/4, & n=\pm 1 \\ 0, & \text{annars}\end{cases}$

$$\hat{f}(n) = \frac{1}{\pi}\int_0^{\pi} t e^{-in2t}\,dt \overset{n\neq 0}{=} \frac{1}{\pi}\left[t\frac{e^{-in2t}}{-in2} - 1\cdot\frac{e^{-in2t}}{(-in2)^2}\right]_0^{\pi} = \frac{i}{2n},\ n\neq 0.$$

$\hat{f}(0) = \frac{1}{\pi}\int_0^{\pi} t\,dt = \frac{\pi}{2}$. Så $\hat{u}(n) = \begin{cases} 1/\pi, & n=0, \\ 1/2i, & n=1, \\ -1/2i, & n=-1, \\ 0, & \text{annars},\end{cases}$

dvs $u(t) = \frac{1}{\pi} + \frac{1}{2i}e^{i2t} - \frac{1}{2i}e^{-i2t}$.

<u>Svar:</u> $u(t) = \frac{1}{\pi} + \sin 2t$, $t\in\mathbb{R}$.

4. a) $\sin|t| = \begin{cases}\sin t, & t>0\\ -\sin t, & t<0\end{cases}$, så $(\sin|t|)' = \begin{cases}\cos t, & t>0\\ -\cos t, & t<0\end{cases} + (0-0)\delta_0$

och $(\sin|t|)'' = \begin{cases}-\sin t, & t>0\\ \sin t, & t<0\end{cases} + (1-(-1))\delta_0$.

<u>Svar:</u> $-\sin|t| + 2\delta$.

b) $tu' = \delta$. $t\delta' = -\delta$, så $tu' = t(-\delta')$, så

$u' = -\delta' + C\delta$, så: <u>Svar:</u> $u = -\delta + C\chi + D$, $C, D\in\mathbb{C}$.

c) $u\in D'(\mathbb{R})$. Vi har att $\langle (u'(t))^\wedge(s), \varphi(s)\rangle = \langle u'(t), \hat{\varphi}(t)\rangle =$

$= -\langle u(t), \hat{\varphi}'(t)\rangle = \ldots$

$$\hat{\varphi}'(t) = \frac{d}{dt}\int_{-\infty}^{\infty}\varphi(i\omega)e^{-i\omega t}\,d\omega = \int_{-\infty}^{\infty}\varphi(i\omega)e^{-i\omega t}(-i\omega)\,d\omega =$$

$$= -(s\varphi(s))^\wedge(t),\quad t\in\mathbb{R}$$

$\ldots = -\langle u(t), -(s\varphi(s))^\wedge(t)\rangle = \langle \hat{u}(s), s\varphi(s)\rangle = \langle s\hat{u}(s), \varphi(s)\rangle$,

för $\varphi\in\mathcal{H}$, så $(u'(t))^\wedge(s) = s\hat{u}(s)$.

5. $y'(t) - y(t) = \delta'(t) - \chi(-t)$, $\quad y \in \mathcal{D}'(\mathbb{R})$.

Laplacetransform ger:

$$\chi(t) \xrightarrow{\mathcal{L}} \frac{1}{s},\ \mathrm{Re}\, s > 0$$
$$\chi(-t) \qquad -\frac{1}{s},\ \mathrm{Re}\, s < 0$$
$$e^t \chi(-t) \qquad -\frac{1}{s-1},\ \mathrm{Re}\, s < 1$$

$$s\hat{y}(s) - \hat{y}(s) = s \cdot 1 + \left(\frac{1}{s},\ \mathrm{Re}\, s < 0\right)_{\mathcal{H}'} =$$

$$= \left(\frac{s^2+1}{s},\ \mathrm{Re}\, s < 0\right)_{\mathcal{H}'},\ \text{så}$$

$$\hat{y}(s) = \left(\frac{s^2+1}{(s-1)s},\ \mathrm{Re}\, s < 0\right)_{\mathcal{H}'} + 2\pi C \delta_1(s) =$$

$$= \left(1 + \frac{2}{s-1} + \frac{-1}{s},\ \mathrm{Re}\, s < 0\right)_{\mathcal{H}'} + 2\pi C \delta_1(s), \quad C \in \mathbb{C}.$$

Detta ger $y(t) = \delta(t) - 2e^t \chi(-t) + \chi(-t) + Ce^t$.

<u>Svar:</u> $y(t) = \delta(t) - (2e^t - 1)\chi(-t) + Ce^t, \quad C \in \mathbb{C}$.

6. $u_n(t) = e^{t^2/2}(e^{-t^2})^{(n)}$, $\quad t \in \mathbb{R}$ och $n \in \mathbb{N}$.

Det följer att $u_{n+1}(t) = e^{t^2/2}\left((e^{-t^2})^{(n)}\right)' = e^{t^2/2}\left(e^{-t^2/2} u_n(t)\right)' =$

$$= e^{t^2/2}\left(-te^{-t^2/2} u_n(t) + e^{-t^2/2} u_n'(t)\right) = -tu_n(t) + u_n'(t), \quad n \in \mathbb{N}.$$

Vi har $u_0(t) = e^{t^2/2} e^{-t^2} = e^{-t^2/2}$, så $\hat{u}_0(\omega) = \sqrt{2\pi}\, u_0(\omega)$, och

om det för något $n \geq 0$ gäller att $\hat{u}_n(\omega) = \sqrt{2\pi}(-i)^n u_n(\omega)$ så

följer att $\widehat{u_{n+1}}(\omega) = (-tu_n(t) + u_n'(t))^\wedge(\omega) =$

$$= -i\hat{u}_n'(\omega) + i\omega\hat{u}_n(\omega) = -i\sqrt{2\pi}(-i)^n u_n'(\omega) + i\omega\sqrt{2\pi}(-i)^n u_n(\omega) =$$

$$= \sqrt{2\pi}(-i)^{n+1}\left(u_n'(\omega) - \omega u_n(\omega)\right) = \sqrt{2\pi}(-i)^{n+1} u_{n+1}(\omega).$$

Alltså gäller $\mathcal{F}u_n = \sqrt{2\pi}(-i)^n u_n$ för alla $n \in \mathbb{N}$.

7. Sätt $I_n = \int_{-\infty}^{\infty} \frac{1-e^{-i\omega}}{i\omega} \cdot \frac{1-e^{-i\omega/2}}{i\omega/2} \cdot \ldots \cdot \frac{1-e^{-i\omega/2^n}}{i\omega/2^n} e^{2i\omega}\, d\omega, \ n \geq 1.$

Låt $\rho(t) = \begin{cases} 1, & 0 \leq t \leq 1, \\ 0, & \text{annars.} \end{cases}$ ρ:s fouriertransform är

$$\hat{\rho}(\omega) = \int_0^1 1 e^{-i\omega t}\, dt = \left[\frac{e^{-i\omega t}}{-i\omega} \right]_0^1 = \frac{1-e^{-i\omega}}{i\omega} \qquad (= 1 \text{ då } \omega = 0).$$

Med $\rho_n(t) = 2^n \rho(2^n t)$ fås då $\hat{\rho_n}(\omega) = \frac{1-e^{-i\omega/2^n}}{i\omega/2^n}$, $n \geq 0$.

Så $\frac{1-e^{-i\omega}}{i\omega} \cdot \ldots \cdot \frac{1-e^{-i\omega/2^n}}{i\omega/2^n} = \hat{\rho_0}(\omega) \cdot \ldots \cdot \hat{\rho_n}(\omega) =$

$= (\rho_0 * \ldots * \rho_n)^\wedge(\omega)$ och Fouriers inversionsformel ger

att $I_n = 2\pi (\rho_0 * \ldots * \rho_n)(2)$, $n \geq 1$.

Eftersom $\rho_n(t) \neq 0$ enbart då $t \in [0, 2^{-n}]$ följer att

$(\rho_0 * \ldots * \rho_n)(t) \neq 0$ enbart då $t \in [0, 2^{-0} + 2^{-1} + \ldots + 2^{-n}]$,

dvs då $t \in [0, 2 - 2^{-n}]$, så $\underline{\underline{I_n = 0, \ n \geq 1.}}$