Lösningar, TATA77, 2018-08-30

1. $u(t) = t$ då $0 \le t < \pi$, $u(t) = 0$ då $\pi \le t < 2\pi$, $T = 2\pi \Rightarrow \Omega = 1$.

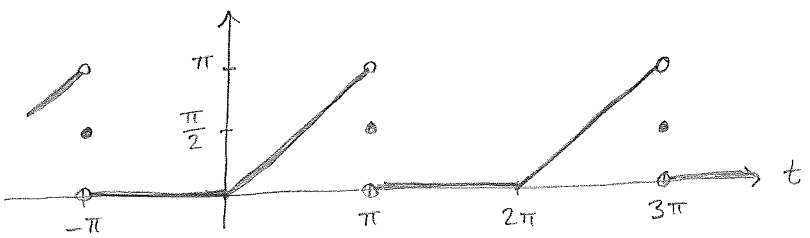
$$\hat{u}(n) = \frac{1}{2\pi}\int_0^{2\pi} u(t)e^{-int}\,dt = \frac{1}{2\pi}\int_0^{\pi} t e^{-int}\,dt \underset{n\neq 0}{=} \frac{1}{2\pi}\left[t\,\frac{e^{-int}}{-in} - 1\,\frac{e^{-int}}{i^2n^2}\right]_0^{\pi} =$$

$$= \frac{1}{2\pi}\left(\pi\frac{(-1)^n}{-in} + \frac{(-1)^n}{n^2} - 0 - \frac{1}{n^2}\right) = \frac{(-1)^n i}{2n} - \frac{1-(-1)^n}{2\pi n^2}, \quad n \neq 0.$$

$$\hat{u}(0) = \frac{1}{2\pi}\int_0^{\pi} t\,dt = \frac{1}{2\pi}\left[\frac{t^2}{2}\right]_0^{\pi} = \frac{\pi}{4}, \text{ så:}$$

<u>Delsvar:</u> u:s f.s. är $\frac{\pi}{4} + \sum_{n\neq 0}\left(\frac{(-1)^n i}{2n} - \frac{1-(-1)^n}{2\pi n^2}\right)e^{int}$.

Fourierseriens summa, enl. satsen om punktvis konvergens:



2. Sätt $u(t) = \begin{cases} 1, & |t| < 1 \\ 0, & |t| \geq 1 \end{cases}$ och $v(t) = e^{-|t|}$.

Tabell ger: $\hat{u}(\omega) = \frac{2\sin\omega}{\omega}$ och $\hat{v}(\omega) = \frac{2}{1+\omega^2}$, så

$$\int_{-\infty}^{\infty} \frac{\sin\omega}{\omega(1+\omega^2)}\,d\omega = 2\pi\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\hat{u}(\omega)}{2}\cdot\frac{\hat{v}(\omega)}{2}\,d\omega =$$

$$= \frac{\pi}{2}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{u}(\omega)\overline{\hat{v}(\omega)}\,d\omega \underset{\text{Parseval}}{=} \frac{\pi}{2}\int_{-\infty}^{\infty} u(t)\overline{v(t)}\,dt =$$

$$= \frac{\pi}{2}\int_{-1}^{1} e^{-|t|}\,dt = \pi\int_0^1 e^{-t}\,dt = \pi\left[-e^{-t}\right]_0^1 = \pi(-e^{-1}+1).$$

<u>Svar:</u> $\pi(1-e^{-1})$.

3. $\sum_{k=0}^{n} u(k) = 2u(n) + n, \quad n \in \mathbb{N}.$

Summan är den "enkelsidiga faltningen" av u och konstanten 1, så enkelsidig z-transform ger $(U = \mathcal{Z}_+ u)$:

$$\frac{z}{z-1} U(z) = 2U(z) + \frac{z}{(z-1)^2}, \qquad z\,U(z) = (2z-2)\,U(z) + \frac{z}{z-1},$$

$$U(z) = -\frac{z}{(z-1)(z-2)} = \frac{z}{z-1} - \frac{z}{z-2}, \qquad |z| > 2.$$

Inverstransform ger: <u>Svar:</u> $u(n) = 1 - 2^n, \quad n \in \mathbb{N}.$

4. $y''(t) - 5y'(t) + 6y(t) = e^t \chi(1-t).$

Laplacetransform ger:

$$s^2\hat{y}(s) - 5s\hat{y}(s) + 6\hat{y}(s) = \left(\frac{e^{-(s-1)}}{-(s-1)}, \operatorname{Re} s < 1\right)_{\mathcal{H}'}$$

$$\begin{array}{lcl} \chi(t) & \xrightarrow{\mathcal{L}} & \frac{1}{s}, \ \operatorname{Re} s > 0 \\ \chi(-t) & & \frac{1}{-s}, \ \operatorname{Re} s < 0 \\ \chi(-(t-1)) & & \frac{e^{-s}}{-s}, \ \operatorname{Re} s < 0 \\ e^t\chi(1-t) & & \frac{e^{-(s-1)}}{-(s-1)}, \ \operatorname{Re} s < 1 \\ e^{ct}\chi(-t) & & \frac{1}{-(s-c)}, \ \operatorname{Re} s < \operatorname{Re} c \end{array}$$

$s^2 - 5s + 6 = (s-2)(s-3)$, så

vi får:

$$\hat{y}(s) = \left(-\frac{e \cdot e^{-s}}{(s-1)(s-2)(s-3)}, \operatorname{Re} s < 1\right)_{\mathcal{H}'} + 2\pi C\delta_2 + 2\pi D\delta_3, \quad C, D \in \mathbb{C}.$$

$$-\frac{1}{(s-1)(s-2)(s-3)} = \frac{-1/2}{s-1} + \frac{1}{s-2} + \frac{-1/2}{s-3}, \quad \text{så:}$$

$$y(t) = \left[e\left(\tfrac{1}{2}e^t - e^{2t} + \tfrac{1}{2}e^{3t}\right)\chi(-t)\right]_{t \mapsto t-1} + Ce^{2t} + De^{3t}.$$

<u>Svar:</u> $y(t) = \frac{1}{2}\left(e^t - 2e^{2t-1} + e^{3t-2}\right)\chi(1-t) + Ce^{2t} + De^{3t},$

$C, D \in \mathbb{C}.$

5. $tu' + 2u = 3t+2, \quad u \in \mathcal{D}'(\mathbb{R}).$

$\Rightarrow \quad t^2u' + 2tu = 3t^2 + 2t$ Ej ekvivalens, så lösningar måste kontrolleras.

$\Leftrightarrow \quad (t^2u)' = 3t^2 + 2t, \quad t^2 u = t^3 + t^2 + C, \quad C \in \mathbb{C},$

$$u = t + 1 + C\underline{t}^{-2} + D\delta + E\delta', \quad D, E \in \mathbb{C}.$$

Kontroll: $u' = 1 + 0 - 2C\underline{t}^{-3} + D\delta' + E\delta''$, så

$$tu' + 2u = t + 0 - \cancel{2C\underline{t}^{-2}} - D\delta - 2E\delta' +$$
$$+ 2t + 2 + \cancel{2C\underline{t}^{-2}} + 2D\delta + 2E\delta' =$$
$$= 3t + 2 + D\delta,$$ så en lösning måste ha $D = 0$.

<u>Svar:</u> $u = t + 1 + C\underline{t}^{-2} + E\delta', \quad C, E \in \mathbb{C}.$

6. $u(t) = \dfrac{\sin t}{t}\,\mathrm{sgn}\, t, \quad t \neq 0.$

$tu(t) = (\sin t)\,\mathrm{sgn}\, t = \frac{1}{2i}(e^{it} - e^{-it})\,\mathrm{sgn}\, t$, så $\mathcal{F}$ ger:

$$i\hat{u}'(\omega) = \frac{1}{2i}\left(\widehat{\mathrm{sgn}}(\omega - 1) - \widehat{\mathrm{sgn}}(\omega + 1)\right).$$

Vi har $\quad 1 \xrightarrow{\mathcal{F}} 2\pi\delta(\omega)$

och $\quad \chi(t) \qquad -i\underline{\omega}^{-1} + \pi\delta(\omega),$

så $\quad \mathrm{sgn}\, t \qquad 2(-i\underline{\omega}^{-1} + \pi\delta(\omega)) - 2\pi\delta(\omega) = -2i\underline{\omega}^{-1}.$

Alltså: $\quad i\hat{u}'(\omega) = -(\underline{\omega - 1})^{-1} + (\underline{\omega + 1})^{-1} =$
$$= \left(-\ln|\omega - 1| + \ln|\omega + 1|\right)',$$ vilket ger att

$$i\hat{u}(\omega) = -\ln|\omega - 1| + \ln|\omega + 1| + C = \ln\left|\frac{\omega+1}{\omega-1}\right| + C,$$ något $C \in \mathbb{C}$.

u är udda, så $\hat{u}$ är udda, och $\ln\left|\frac{-\omega+1}{-\omega-1}\right| = \ln\left|\frac{\omega-1}{\omega+1}\right| =$
$= -\ln\left|\frac{\omega+1}{\omega-1}\right|$ (också udda), så $C = 0$.

<u>Svar:</u> $\hat{u}(\omega) = \frac{1}{i}\ln\left|\frac{\omega+1}{\omega-1}\right|.$

7. $b_n \geq 0$, $n = 1, 2, \ldots$, $\sum_1^\infty b_n < \infty$, och $u(t) = \sum_1^\infty b_n \sin nt$

uppfyller $|u(t+h) - u(t)| \leq C|h|$ för något $C \geq 0$, alla $t, h \in \mathbb{R}$.

Med $t = 0$ har vi för alla $h \geq 0$:

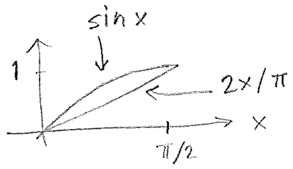
$$Ch \geq |u(0+h) - u(0)| = \left| \sum_1^\infty b_n \sin nh - 0 \right| \geq \sum_1^\infty b_n \sin nh.$$

Låt $a > 0$. Vi får:

$$\frac{Ca^2}{2} = \int_0^a Ch\,dh \geq \int_0^a \left(\sum_1^\infty b_n \sin nh \right) dh = \Big/ \begin{matrix}\text{Likformig konv.}\\ \text{enl. Weierstrass}\end{matrix} \Big/$$

$$= \sum_1^\infty b_n \left[-\frac{\cos nh}{n} \right]_0^a = \sum_1^\infty \frac{b_n}{n}(1 - \cos na) =$$

$$= \sum_1^\infty \frac{b_n}{n} \cdot 2\sin^2 \frac{na}{2} \geq \sum_{1 \leq n \leq \pi/a} \frac{2b_n}{n} \sin^2 \frac{na}{2} \geq$$

$\Big/$ $1 \leq n \leq \frac{\pi}{a} \Rightarrow 0 < \frac{na}{2} \leq \frac{\pi}{2}$, och  $\Big/$

$$\geq \sum_{1 \leq n \leq \pi/a} \frac{2b_n}{n} \left(\frac{2 \cdot \frac{na}{2}}{\pi} \right)^2 = \sum_{1 \leq n \leq \pi/a} \frac{2}{\pi^2} n b_n a^2.$$

För varje $a > 0$ har vi alltså $\sum_{1 \leq n \leq \pi/a} n b_n \leq \frac{C\pi^2}{4}$,

vilket speciellt ger att $\sum_{n=1}^\infty n b_n < \infty$.