Lösningar, TATA77, 2017-10-19

1. $y(n+2) - 5y(n+1) + 6y(n) = 6\cdot 2^n,\ n \in \mathbb{N},\quad y(0)=3,\ y(1)=4.$

Enkelsidig z-transform ger ($Y = \mathcal{Z}_+ y$):

$$z^2 Y(z) - 3z^2 - 4z - 5(zY(z) - 3z) + 6Y(z) = \frac{6z}{z-2}, \text{ så}$$

$$(z^2 - 5z + 6)Y(z) = \frac{6z}{z-2} + 3z^2 - 11z = \frac{3z^3 - 17z^2 + 28z}{z-2}, \text{ så}$$

$$Y(z) = z\,\frac{3z^2 - 17z + 28}{(z-3)(z-2)^2} = z\left(\frac{4}{z-3} + \frac{-6}{(z-2)^2} + \frac{-1}{z-2}\right) =$$

$$= \frac{4z}{z-3} - \frac{6z}{(z-2)^2} - \frac{z}{z-2}, \quad |z| > 3. \quad \text{Ur tabell fås:}$$

$$y(n) = 4\cdot 3^n \chi(n) - 3n\cdot 2^n \chi(n) - 2^n \chi(n), \quad n \in \mathbb{N}.$$

<u>Svar:</u> $y(n) = 4\cdot 3^n - (3n+1)2^n,\ n \in \mathbb{N}.$

2. a) $\langle e^t \delta'(t/2), \varphi(t)\rangle = \langle \delta'(t/2), e^t\varphi(t)\rangle =$

$$= \frac{1}{1/2}\langle \delta'(t), e^{2t}\varphi(2t)\rangle = -2\langle \delta(t), e^{2t}\varphi'(2t)\cdot 2 + 2e^{2t}\varphi(2t)\rangle =$$

$$= -4\varphi'(0) - 4\varphi(0) = \langle 4\delta' - 4\delta, \varphi\rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}).$$

<u>Svar:</u> $4\delta' - 4\delta.$

b) $(t^2 - t)u = t - 2, \quad u \in \mathcal{D}'(\mathbb{R}).$

PBU: $\dfrac{t-2}{t^2-t} = \dfrac{t-2}{t(t-1)} = \dfrac{2}{t} + \dfrac{-1}{t-1}$, så en partikulärlösning

är $u_p = 2\underline{t}^{-1} - (\underline{t-1})^{-1}$. Alla lösningar till den

homogena ekvationen $t(t-1)u = 0$ ges av $u_h = C\delta + D\delta_1$,

så:

<u>Svar:</u> $u = 2\underline{t}^{-1} - (\underline{t-1})^{-1} + C\delta + D\delta_1,\quad C, D \in \mathbb{C}.$

3. $a \in \mathbb{R} \setminus \mathbb{Z}$, $u(t) = e^{iat}$ då $0 \le t < 2\pi$, $T = 2\pi \Rightarrow \Omega = 1$.

$$\hat{u}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{iat} e^{-int}\,dt = \frac{1}{2\pi}\int_0^{2\pi} e^{i(a-n)t}\,dt = \frac{1}{2\pi}\left[\frac{e^{i(a-n)t}}{i(a-n)}\right]_0^{2\pi} =$$

$$= \frac{e^{i2\pi a} - 1}{2\pi i(a-n)} = \frac{e^{i\pi a}(e^{i\pi a} - e^{-i\pi a})}{2\pi i(a-n)} = \frac{e^{i\pi a}\sin \pi a}{\pi(a-n)}, \quad n \in \mathbb{Z}.$$

Delsvar: u:s f.s. är $\displaystyle\sum_{n=-\infty}^{\infty} \frac{e^{i\pi a}\sin \pi a}{\pi(a-n)} e^{int}$.

Parsevals formel ger: $\displaystyle\sum_{n=-\infty}^{\infty} \left|\frac{e^{i\pi a}\sin \pi a}{\pi(a-n)}\right|^2 = \frac{1}{2\pi}\int_0^{2\pi} |e^{iat}|^2\,dt$,

dvs $\displaystyle\sum_{n=-\infty}^{\infty} \frac{\sin^2 \pi a}{\pi^2(a-n)^2} = 1$, så: $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(a-n)^2} = \frac{\pi^2}{\sin^2 \pi a}$, VSV.

4. $\displaystyle u(t) + \int_{-\infty}^{t} e^{-(t-r)} u(r)\,dr = 4e^{-2|t|}$, $t \in \mathbb{R}$.

Integralen $= (f * u)(t)$, där $f(t) = e^{-t}\chi(t)$, så fouriertransform

ger: $\displaystyle\hat{u}(\omega) + \frac{1}{1+i\omega}\hat{u}(\omega) = \frac{16}{4+\omega^2}$, $\displaystyle\frac{2+i\omega}{1+i\omega}\hat{u}(\omega) = \frac{16}{(2+i\omega)(2-i\omega)}$,

$$\hat{u}(\omega) = \frac{16(1+i\omega)}{(2+i\omega)^2(2-i\omega)} = \frac{-4}{(2+i\omega)^2} + \frac{3}{2+i\omega} + \frac{3}{2-i\omega}, \quad \omega \in \mathbb{R}.$$

$$e^{-2t}\chi(t) \xrightarrow{\mathcal{F}} \frac{1}{2+i\omega}, \quad \text{så} \quad te^{-2t}\chi(t) \xrightarrow{\mathcal{F}} i\frac{d}{d\omega}\left(\frac{1}{2+i\omega}\right) = \frac{1}{(2+i\omega)^2}.$$

Så $-4te^{-2t}\chi(t) + 3e^{-2t}\chi(t) + 3e^{2t}\chi(-t)$ har rätt

transform (men är inte kontinuerlig).

Svar: $u(t) = \begin{cases} (3-4t)e^{-2t}, & t \geq 0, \\ 3e^{2t}, & t \leq 0. \end{cases}$

5. $y'(t) - 2y(t) = \delta' + e^{|t|}$, $y \in \mathcal{D}'(\mathbb{R})$.

$e^{|t|} = e^{t}\chi(t) + e^{-t}\chi(-t)$ (för $t \neq 0$), och

$\chi(t) \xrightarrow{\mathcal{L}} \frac{1}{s}$, $\text{Re } s > 0$

$\chi(-t)$ $\quad \frac{1}{-s}$, $\text{Re}(-s) > 0$, dvs $-\frac{1}{s}$, $\text{Re } s < 0$, så:

$e^{ct}\chi(-t)$ $\quad -\frac{1}{s-c}$, $\text{Re}(s-c) < 0$, dvs $-\frac{1}{s-c}$, $\text{Re } s < \text{Re } c$.

Laplacetransform av diffekvationen ger:

$$s\hat{y}(s) - 2\hat{y}(s) = s \cdot 1 + \left(\frac{1}{s-1}, \text{Re } s > 1\right)_{\mathcal{H}'} + \left(-\frac{1}{s+1}, \text{Re } s < -1\right)_{\mathcal{H}'},$$

$$\hat{y}(s) = \left(\frac{s}{s-2}, \text{Re } s > 2\right)_{\mathcal{H}'} + \left(\frac{1}{(s-2)(s-1)}, \text{Re } s > 2\right)_{\mathcal{H}'} + \left(\frac{-1}{(s-2)(s+1)}, \text{Re } s < -1\right)_{\mathcal{H}'} + 2\pi C \delta_2(s).$$

$$\frac{s}{s-2} = 1 + \frac{2}{s-2}, \quad \frac{1}{(s-2)(s-1)} = \frac{1}{s-2} + \frac{-1}{s-1} \quad \text{och} \quad \frac{-1}{(s-2)(s+1)} = \frac{-1/3}{s-2} + \frac{1/3}{s+1},$$

så $y(t) = \delta + 2e^{2t}\chi(t) + (e^{2t} - e^{t})\chi(t) + \left(\frac{1}{3}e^{2t} - \frac{1}{3}e^{-t}\right)\chi(-t) + Ce^{2t}$

Svar: $y(t) = \delta + (3e^{2t} - e^{t})\chi(t) + \frac{1}{3}(e^{2t} - e^{-t})\chi(-t) + Ce^{2t}$, $C \in \mathbb{C}$.

6. $\displaystyle\int_{t-1}^{t} (1 - t + r)u(r)\,dr = t\chi(t)$, $t \in \mathbb{R}$.

Integralen $= (f * u)(t)$, där $f(t) = \begin{cases} 1-t, & 0 \le t \le 1, \\ 0, & \text{annars}. \end{cases}$

$$\hat{f}(s) = \int_0^1 (1-t)e^{-st} \underset{s \neq 0}{=} \left[(1-t)\frac{e^{-st}}{-s} - (-1)\frac{e^{-st}}{s^2}\right]_0^1 = 0 + \frac{e^{-s}}{s^2} + \frac{1}{s} - \frac{1}{s^2} =$$

$$= \frac{s - 1 + e^{-s}}{s^2}, \quad s \in \mathbb{C} \quad (\text{analytisk funktion, hävbar sing. i } s = 0).$$

Laplacetransform ger: $\frac{s-1+e^{-s}}{s^2}\hat{u}(s) = \frac{1}{s^2}$, $\text{Re } s \in \Sigma_u \cap]0, \infty[$.

Om $|e^{-s}| < |s-1|$, tex om $\text{Re } s > 2$, så är $s - 1 + e^{-s} \neq 0$, så en lösning är $\hat{u}(s) = \frac{1}{s-1+e^{-s}} = \frac{1}{(s-1)(1 + e^{-s}/(s-1))} =$

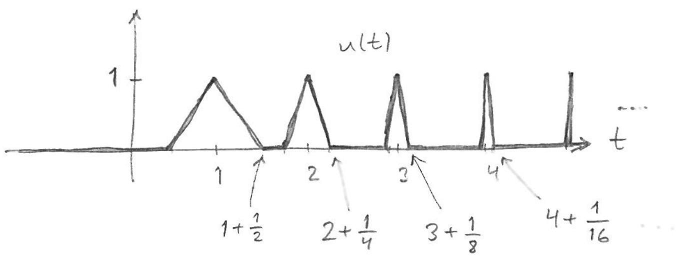
$$= \frac{1}{s-1}\left(1 - \frac{e^{-s}}{s-1} + \frac{e^{-2s}}{(s-1)^2} - \dots\right), \quad \text{Re } s > 2$$ (2 ej minsta möjliga.)

Inverstransform (tabell + räkneregler) ger:

Svar: $u(t) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (t-n)^n e^{t-n}}{n!}\chi(t-n)$, $t \in \mathbb{R}$.

(fouriertransform)

7. Antag att $u \in \mathcal{L}^1(\mathbb{R})$ är kontinuerlig. Om $\hat{u} \in \mathcal{L}^1(\mathbb{R})$ så gäller att $\hat{\hat{u}} = 2\pi \check{u}$ (inversionsformeln), vilket ger att $u(t) \to 0$ då $t \to \pm\infty$ enligt Riemann-Lebesgues lemma. Så om $u(t) \not\to 0$ då $t \to \infty$ fås att $\hat{u} \notin \mathcal{L}^1(\mathbb{R})$. Ett konkret exempel:



u är kontinuerlig, $u \in \mathcal{L}^1(\mathbb{R})$ ty $\int_{-\infty}^{\infty} |u(t)|\,dt = \frac{1}{2} + \frac{1}{4} + \dots = 1$, och $u(t) \not\to 0$ då $t \to \infty$.