Lösningar, TATA77, 2017-08-24

1. $u(t) + \int_{-\infty}^{0} e^{2r} u(t-r)\,dr = 2e^t \chi(-t)$, $t \in \mathbb{R}$.

Integralen $= (f * u)(t)$, där $f(t) = e^{2t}\chi(-t)$, $t \in \mathbb{R}$, så

fouriertransform ger: $\hat{u}(\omega) + \frac{1}{2-i\omega}\hat{u}(\omega) = \frac{2}{1-i\omega}$,

$\frac{3-i\omega}{2-i\omega}\hat{u}(\omega) = \frac{2}{1-i\omega}$, $\hat{u}(\omega) = \frac{2(2-i\omega)}{(1-i\omega)(3-i\omega)} = \frac{1}{1-i\omega} + \frac{1}{3-i\omega}$.

Tabell ger: <u>Svar:</u> $u(t) = (e^t + e^{3t})\chi(-t)$, $t \in \mathbb{R}$.

2. $u(t) = 1 - t^2$, $-1 \le t < 1$, $T = 2 \Rightarrow \Omega = \pi$.

$$\hat{u}(n) = \frac{1}{2}\int_{-1}^{1}(1-t^2)e^{-in\pi t}\,dt = /\text{ om } n \neq 0/$$

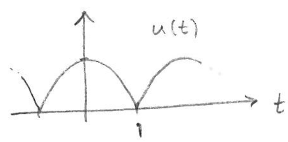
$$= \frac{1}{2}\left[(1-t^2)\frac{e^{-in\pi t}}{-in\pi} - (-2t)\frac{e^{-in\pi t}}{(-in\pi)^2} + (-2)\frac{e^{-in\pi t}}{(-in\pi)^3}\right]_{-1}^{1}$$

$$= \ldots = -\frac{2(-1)^n}{n^2\pi^2}, \quad n \neq 0.$$

$$\hat{u}(0) = \frac{1}{2}\int_{-1}^{1}(1-t^2)\,dt = \frac{2}{3}.$$

<u>Delsvar:</u> Fourierserien är $\frac{2}{3} - \frac{2}{\pi^2}\sum_{n \neq 0}\frac{(-1)^n}{n^2}e^{in\pi t}$.

u har gen. höger- och vänsterderivator överallt. Satsen om punktvis konvergens ger för $t = 1$:



$$\frac{2}{3} - \frac{2}{\pi^2}\sum_{n \neq 0}\frac{(-1)^n}{n^2}e^{in\pi \cdot 1} = \frac{u(1+) + u(1-)}{2} = 0,$$

$$\sum_{n \neq 0}\frac{1}{n^2} = \frac{\pi^2}{3}, \text{ så } \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

3. a) Vi har $(e^t \chi)'' = (e^t\chi + e^t\delta)' = (e^t\chi + \delta)' = e^t\chi + e^t\delta + \delta' =$

$= e^t\chi + \delta + \delta'$. <u>Svar:</u> $e^t\chi + \delta + \delta'$.

b) Vi har $((-\sin t)\chi)'' = ((-\cos t)\chi - (\sin t)\delta)' = ((-\cos t)\chi)' =$

$= (\sin t)\chi - (\cos t)\delta = (\sin t)\chi - \delta$,

och $(t\chi)'' = (\chi + t\delta)' = \chi' = \delta$, så $((t-\sin t)\chi)'' = (\sin t)\chi$.

$u'' = 0 \Leftrightarrow u = Ct + D$, vilket ger:

<u>Svar:</u> $u = (t - \sin t)\chi + Ct + D$, $C, D \in \mathbb{C}$.

4. $y(n+2) - 3y(n+1) + 2y(n) = \chi(-n)$, $n \in \mathbb{Z}$.

$\chi(n) \xrightarrow{\mathscr{Z}} \dfrac{z}{z-1}$, $|z| > 1$, så $\chi(-n) \xrightarrow{\mathscr{Z}} \dfrac{1/z}{1/z - 1}$, $|1/z| > 1$. dvs $0 < |z| < 1$

Z-transformering ger:

$$z^2\hat{y}(z) - 3z\hat{y}(z) + 2\hat{y}(z) = \frac{1}{1-z}, \quad |z| \in R_y \cap]0,1[, \text{ så}$$

$$\hat{y}(z) = \frac{-1}{(z-2)(z-1)^2} = \frac{-1}{z-2} + \frac{1}{(z-1)^2} + \frac{1}{z-1}, \quad |z| \in R_y, \text{ där}$$

$R_y =]0,1[$ eller $]1,2[$ eller $]2,\infty[$. Enbart $R_y =]0,1[$ ger en lösning.

$$2^n\chi(-n) \xrightarrow{\mathscr{Z}} \frac{1}{1-\frac{z}{2}}, \quad 0 < |\tfrac{z}{2}| < 1, \text{ dvs } \frac{2}{2-z}, \quad 0 < |z| < 2.$$

$$n\chi(-n) \xrightarrow{\mathscr{Z}} -z\frac{-1}{(1-z)^2}(-1), \quad 0 < |z| < 1, \text{ dvs } \frac{-z}{(1-z)^2}, \quad 0 < |z| < 1,$$

så $(n-1)\chi(-(n-1)) \xrightarrow{\mathscr{Z}} \dfrac{-1}{(1-z)^2}$, $0 < |z| < 1$. Detta ger:

$$y(n) = \frac{1}{2}2^n\chi(-n) - (n-1)\chi(-n+1) - \chi(-n) = /\, n-1 = 0 \text{ då } n = 1 \,/$$

$$= (2^{n-1} - (n-1) - 1)\chi(-n).$$

<u>Svar:</u> $y(n) = (2^{n-1} - n)\chi(-n)$, $n \in \mathbb{Z}$.

5. $u(t) = t, \quad 0 \le t < 2\pi, \quad T = 2\pi \Rightarrow \Omega = 1.$

$$\hat{u}(n) = \frac{1}{2\pi}\int_0^{2\pi} t e^{-int}\,dt \overset{n\neq 0}{=} \frac{1}{2\pi}\left[t\frac{e^{-int}}{-in} - 1\,\frac{e^{-int}}{(-in)^2}\right]_0^{2\pi} = \frac{i}{n}, \; n \neq 0.$$

För $N \ge 1$ fås nu $\frac{1}{2\pi}\int_0^{2\pi} |u(t) - (S_N u)(t)|^2\,dt = $ / Parseval /

$$= \sum_{n=-\infty}^{\infty} |(u - S_N u)^\wedge(n)|^2 = \sum_{|n|>N} |\hat{u}(n)|^2 = \sum_{|n|>N} \frac{1}{n^2} =$$

$$= 2\sum_{n=N+1}^{\infty} \frac{1}{n^2} \le 2\int_N^\infty \frac{1}{x^2}\,dx = \frac{2}{N},$$ vilket skulle visas.

6. $u(t) = n2^{-|n|}, \quad 2n-1 \le t < 2n+1, \quad n \in \mathbb{Z}.$

$$\int_{2n-1}^{2n+1} n2^{-|n|} e^{-st}\,dt = n2^{-|n|}\left[\frac{e^{-st}}{-s}\right]_{2n-1}^{2n+1} = n2^{-|n|}\,\frac{e^{-s(2n-1)} - e^{-s(2n+1)}}{s} =$$

$$= n2^{-|n|}e^{-2ns}\,\frac{e^{s} - e^{-s}}{s}.$$ (Gäller för $s \in \mathbb{C}$ med $\frac{e^s - e^{-s}}{s} = 2$ då $s = 0$.)

$$\sum_{n=-\infty}^{\infty} n2^{-|n|}e^{-2ns} = \sum_{n=1}^{\infty} n2^{-n}e^{-2ns} - \sum_{n=1}^{\infty} n2^{-n}e^{2ns} =$$

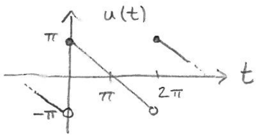
$$= \sum_{n=1}^{\infty} n(2e^{2s})^{-n} - \sum_{n=1}^{\infty} n(2e^{-2s})^{-n} = \text{/ Z-tabell /}$$

$$= \frac{2e^{2s}}{(2e^{2s}-1)^2} - \frac{2e^{-2s}}{(2e^{-2s}-1)^2},$$ om $|2e^{2s}| > 1$ och $|2e^{-2s}| > 1$.

Multiplikation med $\frac{e^s - e^{-s}}{s}$ och förenkling ger:

<u>Svar:</u> $$\hat{u}(s) = \frac{-6(e^{2s} - e^{-2s})(e^s - e^{-s})}{(5 - 2e^{2s} - 2e^{-2s})^2 s}, \quad |\mathrm{Re}\, s| < \tfrac{1}{2}\ln 2.$$

7. $u(t) = \pi - t, \quad 0 \le t < 2\pi, \quad T = 2\pi \Rightarrow \Omega = 1.$



$$\hat{u}(n) = \frac{1}{2\pi}\int_0^{2\pi}(\pi - t)e^{-int}\,dt \overset{n\neq 0}{=} \frac{1}{2\pi}\left[(\pi - t)\frac{e^{-int}}{-in} - (-1)\frac{e^{-int}}{(-in)^2}\right]_0^{2\pi} = \frac{1}{in}, \quad n \neq 0,$$

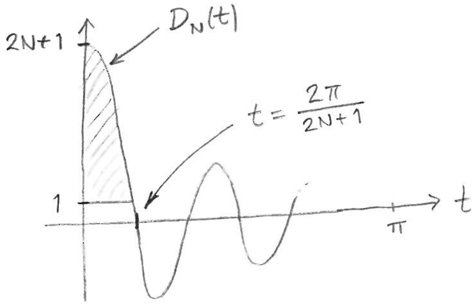
och $\hat{u}(0) = 0$, så $(S_N u)(t) = \sum_{0 \neq |n| \le N} \frac{1}{in} e^{int}$.

Geo. summa

Detta ger att $(S_N u)'(t) = \sum_{0 \neq |n| \le N} e^{int} = \left(\sum_{n=-N}^{N} e^{int}\right) - 1 =$

$$= e^{-iNt}\,\frac{e^{i(2N+1)t} - 1}{e^{it} - 1} - 1 = \underbrace{\frac{\sin\left((2N+1)\frac{t}{2}\right)}{\sin\frac{t}{2}}}_{D_N(t)} - 1 \quad (\text{då } t \neq 2\pi m).$$

D_N svänger med avtagande amplitud på $[0, \pi]$, och



$$(S_N u)(a) = (S_N u)(0) + \int_0^a (S_N u)'(t)\,dt =$$

$$= \int_0^a (D_N(t) - 1)\,dt,$$

så $\max_{t \in [0,\pi]} (S_N u)(t)$ ges av det skuggade området i figuren.

$\frac{2\pi}{2N+1} \to 0$ då $N \to \infty$, så $\max_{t\in[0,\pi]} (S_N u)(t)$ har samma gränsvärde som $\int_0^{2\pi/(2N+1)} D_N(t)\,dt$ då $N \to \infty$, och

$$\int_0^{\frac{2\pi}{2N+1}} \frac{\sin\left((2N+1)\frac{t}{2}\right)}{\sin\frac{t}{2}}\,dt = \int_0^{\pi} \frac{\sin t}{\sin\frac{t}{2N+1}} \cdot \frac{2}{2N+1}\,dt =$$

$$= 2\int_0^{\pi} \frac{\sin t}{t} \cdot \frac{t/(2N+1)}{\sin(t/(2N+1))}\,dt \longrightarrow 2\int_0^{\pi} \frac{\sin t}{t}\,dt \quad \text{då } N \to \infty.$$