Lösningar, TATA77, 2016-10-28

1. $y''(t) - 3y'(t) + 2y(t) = 2e^t \cos 2t$, $t \geq 0$, $y(0) = 1$, $y'(0) = 2$.

Enkelsidig laplacetransform ger ($Y = \mathcal{L}_+ y$):

$$s^2 Y(s) - 1s - 2 - 3(sY(s) - 1) + 2Y(s) = \frac{2(s-1)}{(s-1)^2 + 2^2},$$

/ $s^2 - 3s + 2 = (s-1)(s-2)$ /
$$(s-1)(s-2)Y(s) = \frac{2(s-1)}{(s-1)^2 + 2^2} + s - 1,$$

$$Y(s) = \frac{2}{(s-2)(s^2-2s+5)} + \frac{1}{s-2} = \frac{2/5}{s-2} + \frac{-\frac{2}{5}s + 0}{s^2 - 2s + 5} + \frac{1}{s-2} =$$

$$= \frac{7}{5}\frac{1}{s-2} - \frac{1}{5}\frac{2(s-1)+2}{(s-1)^2+4}, \quad \text{Re } s > 2. \quad \text{Tabell ger:}$$

<u>Svar:</u> $y(t) = \frac{7}{5}e^{2t} - \frac{2}{5}e^t \cos 2t - \frac{1}{5}e^t \sin 2t$, $t \geq 0$.

2. $u(t) = t$, $0 \leq t < 2\pi$, u 2π-per. $T = 2\pi \implies \Omega = \frac{2\pi}{T} = 1$.

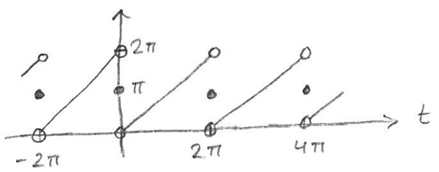
a) $$\hat{u}(n) = \frac{1}{2\pi}\int_0^{2\pi} t e^{-int}\,dt \overset{\substack{n \neq 0 \\ \downarrow}}{=} \frac{1}{2\pi}\left[t\frac{e^{-int}}{-in} - 1\frac{e^{-int}}{(-in)^2}\right]_0^{2\pi} =$$

$$= \frac{1}{2\pi}\left(2\pi\frac{1}{-in} + \frac{1}{n^2} - 0 - \frac{1}{n^2}\right) = \frac{i}{n}, \quad n \neq 0.$$

$$\hat{u}(0) = \frac{1}{2\pi}\int_0^{2\pi} t\,dt = \frac{1}{2\pi}\left[\frac{t^2}{2}\right]_0^{2\pi} = \pi, \quad \text{så:}$$

<u>Svar:</u> $\pi + \sum_{n \neq 0} \frac{i}{n} e^{int}$.

b) Satsen om punktvis konvergens ger (ty u har gen. höger- och vänsterderivator i varje punkt):



c) Parsevals formel ger: $|\pi|^2 + \sum_{n \neq 0} \left|\frac{i}{n}\right|^2 = \frac{1}{2\pi}\int_0^{2\pi} |t|^2\,dt$,

$$\pi^2 + 2\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{2\pi}\left[\frac{t^3}{3}\right]_0^{2\pi} = \frac{4\pi^2}{3}, \quad \text{så:}$$

<u>Svar:</u> $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

3. a) $u \in L^1_T$, $a \in \mathbb{R}$.

$$(u(t-a))^\wedge(n) = \frac{1}{T}\int_T u(t-a)e^{-in\Omega t}\,dt = \Big/ \begin{matrix} r = t-a \\ dr = dt \end{matrix} \Big/ =$$

$$= \frac{1}{T}\int_T u(r)e^{-in\Omega(r+a)}\,dr = e^{-in\Omega a}\,\hat{u}(n), \quad n \in \mathbb{Z}.$$

b) $u, v \in L^1(\mathbb{R})$, u begränsad. $(\Rightarrow u*v \in L^1(\mathbb{R}))$

$$(u*v)^\wedge(\omega) = \int_{-\infty}^{\infty}(u*v)(t)e^{-i\omega t}\,dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u(t-r)v(r)\,dr\, e^{-i\omega t}\,dt =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u(t-r)e^{-i\omega(t-r)}v(r)e^{-i\omega r}\,dr\,dt = \Big/ \text{byt int. ordning} \Big/$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u(t-r)e^{-i\omega(t-r)}\,dt\; v(r)e^{-i\omega r}\,dr = \hat{u}(\omega)\hat{v}(\omega), \quad \omega \in \mathbb{R}.$$

c) $u \in \mathcal{D}'(\mathbb{R})$, $a > 0$.

$$\langle (u(at))^\wedge(s), \varphi(s)\rangle = \langle u(at), \hat{\varphi}(t)\rangle = \frac{1}{a}\langle u(t), \hat{\varphi}(\tfrac{t}{a})\rangle = \ldots$$

$$\left[\ \hat{\varphi}(\tfrac{t}{a}) = \int_{-\infty}^{\infty}\varphi(i\omega)e^{-i\omega\frac{t}{a}}\,d\omega = \Big/ w = \tfrac{\omega}{a} \Big/ = \int_{-\infty}^{\infty}\varphi(iaw)e^{-iwt}\,a\,dw = \widehat{a\varphi(as)}(t)\ \right]$$

$$\ldots = \langle u(t), \widehat{\varphi(as)}(t)\rangle = \langle \hat{u}(s), \varphi(as)\rangle = \langle \tfrac{1}{a}\hat{u}(\tfrac{s}{a}), \varphi(s)\rangle, \quad \varphi \in \mathcal{H},$$

så $\quad (u(at))^\wedge(s) = \frac{1}{a}\hat{u}(\frac{s}{a})$.

4. $(t^2-2t)u = 6\delta + 2$, $\quad u \in \mathcal{D}'(\mathbb{R})$.

$$6\delta = (t-2)(-3\delta) = (t-2)(t(3\delta')) = (t^2-2t)(3\delta') \quad (\text{ty } t\delta' = -\delta),$$

och $\dfrac{2}{t^2-2t} = \dfrac{2}{t(t-2)} = \dfrac{-1}{t} + \dfrac{1}{t-2}$, så en partikulärlösning

är $u_p = 3\delta' - \underline{t}^{-1} + \underline{(t-2)}^{-1}$. Den allmänna lösningen

till $t(t-2)u = 0$ är $u_h = C\delta + D\delta_2$, $C, D \in \mathbb{C}$, så:

<u>Svar:</u> $u = 3\delta' - \underline{t}^{-1} + \underline{(t-2)}^{-1} + C\delta + D\delta_2$, $C, D \in \mathbb{C}$.

5. $u(n) + \sum_{k=n}^{\infty} 2^{n-k} u(k) = \chi(n), \quad n \in \mathbb{Z}.$

Vi har $\chi(n) \xrightarrow{\mathcal{Z}} \frac{z}{z-1}$, $|z|>1$. $\sum_{k=n}^{\infty} 2^{n-k} u(k) = \sum_{k=-\infty}^{\infty} 2^{n-k} \chi(k-n) u(k) =$

$= (f * u)(n)$, där $f(n) = 2^n \chi(-n)$. Vi har $2^{-n}\chi(n) \xrightarrow{\mathcal{Z}} \frac{z}{z-1/2}$, $|z|>\frac{1}{2}$,

så $\hat{f}(z) = \frac{1/z}{1/z - 1/2} = \frac{2}{2-z}$, $0 < |z| < 2$.

$\mathcal{Z}(VL) = \mathcal{Z}(HL)$: $\hat{u}(z) + \frac{2}{2-z}\hat{u}(z) = \frac{z}{z-1}$, $z \in R_u \cap \,]1,2[$, så

$$\hat{u}(z) = \frac{z(z-2)}{(z-1)(z-4)} = z\left(\frac{1/3}{z-1} + \frac{2/3}{z-4}\right) = \frac{1}{3}\frac{z}{z-1} + \frac{2}{3}\frac{z}{z-4}, \quad |z| \in R_u,$$

och vi måste ta $R_u = \,]1,4[$ för att få $R_u \cap \,]1,2[\,\neq \emptyset$.

Vi har:

$$4^{-n}\chi(n) \xrightarrow{\mathcal{Z}} \frac{z}{z-1/4}, \quad |z| > 1/4$$

$$4^n \chi(-n) \qquad \frac{1/z}{1/z - 1/4} = \frac{4}{4-z} = -\frac{4}{z}\frac{z}{z-4}, \quad 0<|z|<4$$

$$4^{n+1}\chi(-(n+1)) \qquad -4\frac{z}{z-4}, \quad 0<|z|<4, \text{ så:}$$

<u>Svar:</u> $u(n) = \frac{1}{3}\chi(n) - \frac{2}{3}4^n\chi(-n-1)$, $n \in \mathbb{Z}$.

6. Sätt $u(t) = \sum_{n=-\infty}^{\infty} \frac{1}{4n^2+4n+1} e^{int}$. $(e^{int})' = ine^{int}$ och $(e^{int})'' = -n^2 e^{int}$,

så $-4u'' - 4iu' + u = \sum_{n=-\infty}^{\infty} e^{int} = \sum_{n=-\infty}^{\infty} 2\pi\delta(t-2\pi n)$.

På $]-2\pi, 2\pi[$: $-4u'' - 4iu' + u = 2\pi\delta$, dvs $(D+\frac{i}{2})^2 u = -\frac{\pi}{2}\delta$.

Part. lösn.: $u_p = \mathcal{L}^{-1}\left(-\frac{\pi}{2(s+i/2)^2}, \operatorname{Re} s > 0\right) = -\frac{\pi t}{2} e^{-it/2}\chi(t)$, $|t| < 2\pi$,

så $u = -\frac{\pi t}{2} e^{-it/2}\chi(t) + (At+B)e^{-it/2}$ på $]-2\pi, 2\pi[$.

u har period 2π, så för $0<t<2\pi$ gäller

$$-\frac{\pi t}{2} e^{-it/2} + (At+B)e^{-it/2} = (A(t-2\pi)+B)e^{-i(t-2\pi)/2},$$

$$-\frac{\pi t}{2} + At + B = -At + 2\pi A - B, \text{ så } A = \frac{\pi}{4}, \; B = \frac{\pi^2}{4}.$$

Alltså: $u(t) = \left(-\frac{\pi t}{2}\chi(t) + \frac{\pi t}{4} + \frac{\pi^2}{4}\right)e^{-it/2}$ på $]-2\pi, 2\pi[$,

och $t=0$ ger:

<u>Svar:</u> $\frac{\pi^2}{4}$.

7. $u(t) = \ln|t|, \quad t \neq 0.$

Sätt $v(t) = (\ln t)\chi(t)$. Då är $u(t) = v(t) + v(-t)$.

Tabell: $(\mathcal{L}v)(s) = -\dfrac{\gamma + \operatorname{Log} s}{s}$, $\operatorname{Re} s > 0$, så för $\varphi \in \mathcal{H}$:

$$\langle (\mathcal{L}v)(s), \varphi(s) \rangle = \int_L -\frac{\gamma + \operatorname{Log} s}{s}\varphi(s)\,\frac{ds}{i} \qquad (L:\ s = 1 + i\omega,\ \omega: -\infty \to \infty)$$

$$= \int_L \left(\gamma \operatorname{Log} s + \tfrac{1}{2}\operatorname{Log}^2 s\right)\varphi'(s)\,\frac{ds}{i} \qquad (\text{Genom part. int.})$$

/ Enbart logaritmisk singularitet, så L kan ersättas med $s = i\omega$, $\omega: -\infty \to \infty$ (Cauchys integralsats). /

$$= \int_{-\infty}^{\infty} \left(\gamma\left(\ln|\omega| + \tfrac{i\pi}{2}\operatorname{sgn}\omega\right) + \tfrac{1}{2}\left(\ln|\omega| + \tfrac{i\pi}{2}\operatorname{sgn}\omega\right)^2\right)\varphi'(i\omega)\,d\omega$$

/ $\varphi_0(\omega) = \varphi(i\omega)$ ger $\varphi_0'(\omega) = i\varphi'(i\omega)$ /

$$= \int_{-\infty}^{\infty} \left(\gamma\ln|\omega| + \frac{i\pi\gamma}{2}\operatorname{sgn}\omega + \frac{1}{2}\ln^2|\omega| + \frac{i\pi}{2}(\operatorname{sgn}\omega)\ln|\omega| - \frac{\pi^2}{8}\right)(-i)\varphi_0'(\omega)\,d\omega.$$

Derivering ger nu att

$$(\mathcal{F}v)(\omega) = i\gamma\,\underline{\omega^{-1}} - \pi\gamma\,\delta(\omega) + \frac{i}{2}\left(\ln^2|\omega|\right)' - \frac{\pi}{2}\left((\operatorname{sgn}\omega)\ln|\omega|\right)'.$$

Detta ger (udda, jämna):

<u>Svar:</u> $\hat{u}(\omega) = -2\pi\gamma\,\delta(\omega) - \pi\left((\operatorname{sgn}\omega)\ln|\omega|\right)'.$