Lösningar, TATA77, 2016-08-25

1. $y'''(t) - 8y(t) = 8,\ t \geq 0, \qquad y(0) = 1,\ y'(0) = 1,\ y''(0) = 2.$

Enkelsidig laplacetransform ger $(Y = \mathcal{L}_+ y)$:

$$s^3 Y(s) - 1s^2 - 1s - 2 - 8Y(s) = \frac{8}{s}, \quad \text{så} \quad (s^3 - 8)Y(s) = s^2 + s + 2 + \frac{8}{s},$$

$$Y(s) = \frac{s^3 + s^2 + 2s + 8}{s(s-2)(s^2+2s+4)} = \frac{-1}{s} + \frac{1}{s-2} + \frac{s+1}{s^2+2s+4}, \quad \operatorname{Re} s > 2.$$

Eftersom $s^2 + 2s + 4 = (s+1)^2 + 3$ fås ur tabell:

Svar: $y(t) = -1 + e^{2t} + e^{-t}\cos\sqrt{3}t, \quad t \geq 0.$

2. a) $\langle t\delta(3t-6), \varphi(t)\rangle = \langle \delta(3t-6), t\varphi(t)\rangle =$

$$= \frac{1}{3}\langle \delta(t), \tfrac{t+6}{3}\varphi\left(\tfrac{t+6}{3}\right)\rangle = \frac{1}{3}\cdot 2\varphi(2) = \langle \tfrac{2}{3}\delta_2, \varphi\rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}).$$

Svar: $\frac{2}{3}\delta_2$.

b) Svar: $u = A\delta + B\delta_1 + C\delta_1' + D\delta_1''$, $A, B, C, D \in \mathbb{C}$.

c) $u(t) = |t| = t \operatorname{sgn} t$, så $u' = t \cdot 2\delta + 1 \cdot \operatorname{sgn} t = \operatorname{sgn} t$,

och $u'' = 2\delta$. Svar: 2δ.

3. $u(t) = \alpha e^{-\beta t^2}, \quad t \in \mathbb{R}, \quad \alpha, \beta > 0.$

Då $(u * u)(t) = e^{-t^2}$ ger fouriertransformering att

$$\hat{u}(\omega)^2 = \sqrt{\pi}\, e^{-\omega^2/4}, \quad \left(\alpha\sqrt{\tfrac{\pi}{\beta}}\, e^{-\omega^2/4\beta}\right)^2 = \sqrt{\pi}\, e^{-\omega^2/4},$$

$$\frac{\alpha^2 \pi}{\beta} e^{-\omega^2/2\beta} = \sqrt{\pi}\, e^{-\omega^2/4}, \quad \omega \in \mathbb{R}. \quad \text{Så } 2\beta = 4,$$

dvs $\beta = 2$, och $\frac{\alpha^2 \pi}{2} = \sqrt{\pi}$.

Svar: $\alpha = \frac{\sqrt{2}}{\sqrt[4]{\pi}}, \quad \beta = 2.$

4. u π-per., $\quad \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} u(t-r)\cos r \, dr = \sin^2 t, \quad t \in \mathbb{R}.$

$T = \pi \Rightarrow \Omega = 2.$ Sätt $f(t) = |\cos t|, \quad t \in \mathbb{R}.$

Då har vi att $(u *_T f)(t) = \sin^2 t = \left(\frac{e^{it} - e^{-it}}{2i}\right)^2 =$

$= -\frac{1}{4}e^{i2t} + \frac{1}{2} - \frac{1}{4}e^{-i2t},$ så $\hat{u}(n)\hat{f}(n) = \begin{cases} -1/4, & n = \pm 1, \\ 1/2, & n = 0, \\ 0, & \text{annars.} \end{cases}$

$$\hat{f}(n) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos t \cdot e^{-in2t} \, dt = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \left(e^{i(1-2n)t} + e^{-i(1+2n)t}\right) dt =$$

$$= \frac{1}{2\pi}\left[\frac{e^{i(1-2n)t}}{i(1-2n)} + \frac{e^{-i(1+2n)t}}{-i(1+2n)}\right]_{-\pi/2}^{\pi/2} = \ldots = \frac{2(-1)^n}{\pi(1-4n^2)}, \quad n \in \mathbb{Z}.$$

Detta ger $\hat{u}(n) = \begin{cases} -3\pi/8, & n = \pm 1, \\ \pi/4, & n = 0, \\ 0, & \text{annars.} \end{cases}$ <u>Svar</u>: $u(t) = \frac{\pi}{4} - \frac{3\pi}{4}\cos 2t, \quad t \in \mathbb{R}.$

5. $y = Sx$ ger att $y'' + y' - 2y = x'' + x'$, så impulssvaret h uppfyller $h'' + h' - 2h = \delta'' + \delta'$. Laplacetransform ger att $(s^2 + s - 2)\hat{h} = s^2 + s$, och $s^2 + s - 2 = (s-1)(s+2)$, så

$$\hat{h} = \left(\frac{s^2+s}{s^2+s-2}, \operatorname{Re} s > 1\right)_{\mathcal{H}'} + C\delta_1 + D\delta_{-2} = /\text{partialbråksuppdelning}/$$

$$= \left(1 + \frac{2/3}{s-1} - \frac{2/3}{s+2}, \operatorname{Re} s > 1\right)_{\mathcal{H}'} + C\delta_1 + D\delta_{-2}.$$

Inverstransform ger $h = \delta + \frac{2}{3}e^t \chi(t) - \frac{2}{3}e^{-2t}\chi(t) + \frac{C}{2\pi}e^t + \frac{D}{2\pi}e^{-2t}.$

$h = 0$ på $]0, \infty[$, så $\frac{C}{2\pi} = -\frac{2}{3}, \quad \frac{D}{2\pi} = \frac{2}{3}$, vilket ger:

<u>Delsvar</u>: $h = \delta - \frac{2}{3}e^t\chi(-t) + \frac{2}{3}e^{-2t}\chi(-t).$

Formel för y: $y(t) = (h * x)(t) = /\ \delta * x = x\ / =$

$$= x(t) + \int_{-\infty}^{\infty}\left(-\frac{2}{3}e^{t-r} + \frac{2}{3}e^{-2(t-r)}\right)\chi(-(t-r))\, x(r)\, dr =$$

$$= x(t) - \frac{2}{3}\int_t^{\infty}\left(e^{t-r} - e^{-2(t-r)}\right) x(r)\, dr, \quad t \in \mathbb{R}.$$

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6. $\quad nu(n) - 2\sum_{k=0}^{n} u(k) = n, \quad n \in \mathbb{N}, \quad u(2) = 1.$

Enkelsidig z-transform ger ($U = \mathcal{Z}_+ u$):

$$-z\,U'(z) - 2\,\frac{z}{z-1}\,U(z) = \frac{z}{(z-1)^2},$$

$$U'(z) + \frac{2}{z-1}\,U(z) = -\frac{1}{(z-1)^2}, \quad \Big/ \text{Integrerande faktor: } (z-1)^2 \Big/$$

$$\left((z-1)^2 U(z)\right)' = -1, \quad (z-1)^2 U(z) = -z + C,$$

$$U(z) = -\frac{z}{(z-1)^2} + \frac{C}{(z-1)^2} = -\frac{z}{(z-1)^2} + \frac{C}{z}\,\frac{z}{(z-1)^2}, \quad |z| > 1.$$

Inverstransform ger $\quad u(n) = -n + C(n-1)\chi(n-1), \quad n \in \mathbb{N}$

$u(2) = 1$ ger nu $\quad 1 = -2 + C$, dvs $\quad C = 3.$

<u>Svar:</u> $\quad u(n) = -n + 3(n-1)\chi(n-1), \quad n \in \mathbb{N}.$

7. $\quad u \in C^\infty(\mathbb{R}), \quad u(t) = \sqrt{t}$ då $t > 1$, $\quad u(t) = 0$ då $t < 0$.

Låt $k \in \mathbb{N}$ och sätt $\quad v(t) = \dfrac{d^{k+2}}{dt^{k+2}}\left(t^k u(t)\right), \quad t \in \mathbb{R}.$

$v \in C^\infty(\mathbb{R})$, $\quad v(t) = \text{konstant} \cdot t^{-3/2}$ då $t > 1$, och

$v(t) = 0$ då $t < 0$, så $v \in L^1(\mathbb{R})$, vilket medför

att $\hat{v}$ är en kontinuerlig funktion.

Eftersom $\quad \hat{v}(\omega) = (i\omega)^{k+2}\, i^k\, \hat{u}^{(k)}(\omega)$ följer att

$\hat{u}^{(k)}$:s restriktion till $]0, \infty[$ ges av en kont. fkn,

så $\hat{u}$:s restriktion till $]0, \infty[$ ges av en C^∞-fkn.