Lösningar, TATA77, 2015-10-30

1. $$2u(n) = 1 + \sum_{k=0}^{n} (-1)^{n-k} u(k), \quad n \in \mathbb{N}.$$

Enkelsidig z-transform ger ($U = \mathcal{Z}_+ u$): $2U(z) = \frac{z}{z-1} + \frac{z}{z-(-1)} U(z)$,

så $\left(2 - \frac{z}{z+1}\right) U(z) = \frac{z}{z-1}$, $\quad \frac{z+2}{z+1} U(z) = \frac{z}{z-1}$,

$$U(z) = \frac{z(z+1)}{(z+2)(z-1)} = z\left(\frac{1/3}{z+2} + \frac{2/3}{z-1}\right) = \frac{1}{3}\frac{z}{z+2} + \frac{2}{3}\frac{z}{z-1}, \quad |z| > 2.$$

Tabell ger: $\quad$ <u>Svar:</u> $u(n) = \frac{(-2)^n + 2}{3}, \quad n \in \mathbb{N}.$

2. $u(t) = e^t, \quad 0 \le t < \pi, \quad T = \pi \Rightarrow \Omega = 2\pi/T = 2.$

a) $$\hat{u}(n) = \frac{1}{\pi}\int_0^{\pi} e^t e^{-in2t}\,dt = \frac{1}{\pi}\left[\frac{e^{(1-i2n)t}}{1-i2n}\right]_0^{\pi} = \frac{1}{\pi}\frac{e^{\pi}-1}{1-i2n}, \quad n \in \mathbb{Z}.$$

<u>Svar:</u> $$\sum_{n=-\infty}^{\infty} \frac{1}{\pi}\frac{e^{\pi}-1}{1-i2n} e^{i2nt}.$$

b) u har generaliserad höger- och vänsterderivata i $t = \pi$, så satsen om punktvis konvergens ger att fourierseriens summa i $t = \pi$ är $\frac{u(\pi+) + u(\pi-)}{2} = \frac{u(0+) + e^{\pi}}{2} = \underline{\underline{\frac{1+e^{\pi}}{2}}}.$

c) Parsevals formel ger: $$\frac{1}{\pi}\int_0^{\pi} |e^t|^2\,dt = \sum_{n=-\infty}^{\infty} \left|\frac{1}{\pi}\frac{e^{\pi}-1}{1-i2n}\right|^2.$$

$$VL = \frac{1}{\pi}\int_0^{\pi} e^{2t}\,dt = \frac{1}{\pi}\left[\frac{e^{2t}}{2}\right]_0^{\pi} = \frac{1}{\pi}\frac{e^{2\pi}-1}{2} = \frac{(e^{\pi}-1)(e^{\pi}+1)}{2\pi},$$

$$HL = \frac{(e^{\pi}-1)^2}{\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{1+4n^2}, \quad \text{så:}$$

<u>Svar:</u> $$\sum_{n=-\infty}^{\infty}\frac{1}{4n^2+1} = \frac{\pi}{2}\frac{e^{\pi}+1}{e^{\pi}-1}.$$

3. a) $\langle \delta'(2t), \varphi(t) \rangle = \frac{1}{2} \langle \delta'(t), \varphi(\frac{t}{2}) \rangle = -\frac{1}{2} \langle \delta(t), \varphi'(\frac{t}{2}) \cdot \frac{1}{2} \rangle =$

$= -\frac{1}{4}\varphi'(0) = \frac{1}{4}\langle \delta', \varphi \rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}).$ <u>Svar:</u> $\frac{1}{4}\delta'$.

b) $t^2 u = \delta, \quad u \in \mathcal{D}'(\mathbb{R})$.

Laplacetransform ger: $(-1)^2 \hat{u}''(s) = 1$, så $\hat{u}'(s) = s + C$,

$\hat{u}(s) = \frac{s^2}{2} + Cs + D, \quad C, D \in \mathbb{C}$. $\delta \xrightarrow{\mathcal{L}} 1$, så $\delta' \xrightarrow{\mathcal{L}} s$

och $\delta'' \xrightarrow{\mathcal{L}} s^2$, så: <u>Svar:</u> $u = \frac{1}{2}\delta'' + C\delta' + D\delta, \quad C, D \in \mathbb{C}$.

c) $t|t| = t \cdot t \operatorname{sgn} t = t^2(2\chi(t) - 1), \quad t \in \mathbb{R}$.

$1 \xrightarrow{\mathcal{F}} 2\pi\delta(\omega)$

$\chi(t) \quad -i\omega^{-1} + \pi\delta(\omega)$

$2\chi(t) - 1 \quad -2i\omega^{-1}$

Så $\widehat{t|t|}(\omega) = i^2 \frac{d^2}{d\omega^2}(-2i\omega^{-1}) = 2i(-1)(-2)\omega^{-3} = \underline{\underline{4i\omega^{-3}}}$.

4. $u(t) + \int_t^\infty e^{t-r} u(r)\, dr = 3e^{-|t|}, \quad t \in \mathbb{R}$.

$\int_t^\infty e^{t-r} u(r)\, dr = \int_{-\infty}^\infty e^{t-r} \chi(-(t-r)) u(r)\, dr$, så

fouriertransform av ekvationen ger:

$\hat{u}(\omega) + \frac{1}{1-i\omega}\hat{u}(\omega) = \frac{3 \cdot 2}{1+\omega^2}, \quad \frac{2-i\omega}{1-i\omega}\hat{u}(\omega) = \frac{6}{(1-i\omega)(1+i\omega)}$,

$\hat{u}(\omega) = \frac{6}{(1+i\omega)(2-i\omega)} = \frac{2}{1+i\omega} + \frac{2}{2-i\omega}, \quad \omega \in \mathbb{R}$.

Funktionen $2e^{-t}\chi(t) + 2e^{2t}\chi(-t)$ har enl. tabell rätt transform, men är diskontinuerlig i $t = 0$.

<u>Svar:</u> $u(t) = \begin{cases} 2e^{-t}, & t \geq 0, \\ 2e^{2t}, & t \leq 0. \end{cases}$

5. $y''(t) - 3y'(t) + 2y(t) = \delta(t+3) + \delta(t-3)$, $y(t) = 0$ då $t > 3$.

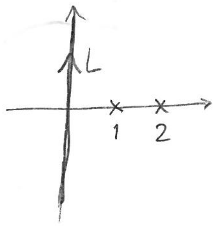
Laplacetransform ger: $s^2\hat{y}(s) - 3s\hat{y}(s) + 2\hat{y}(s) = e^{3s} + e^{-3s}$.

$s^2 - 3s + 2 = (s-1)(s-2)$, så

$$(s-1)(s-2)\hat{y}(s) = (e^{3s} + e^{-3s})(s-1)(s-2)\left(\frac{1}{(s-1)(s-2)}, \operatorname{Re} s < 1\right)_{\mathcal{H}'},$$

så $\hat{y}(s) = (e^{3s} + e^{-3s})\left(\frac{1}{(s-1)(s-2)}, \operatorname{Re} s < 1\right)_{\mathcal{H}'} + C\delta_1(s) + D\delta_2(s)$

Residysats + uppskattningar ger:



$$\frac{1}{2\pi i}\int_L \frac{e^{st}}{(s-1)(s-2)}\,ds = \left\{\begin{array}{l} 0, \quad t \geq 0 \\ -\left(\operatorname*{Res}_{s=1} + \operatorname*{Res}_{s=2}\right)\frac{e^{st}}{(s-1)(s-2)}, \quad t \leq 0 \end{array}\right\} =$$

$$= \left\{\begin{array}{l} 0, \quad t \geq 0 \\ -(-e^t + e^{2t}), \quad t \leq 0 \end{array}\right\} = \underbrace{(e^t - e^{2t})\chi(-t)}_{f(t)}, \quad t \in \mathbb{R}.$$

Så $y(t) = f(t+3) + f(t-3) + \frac{C}{2\pi}e^t + \frac{D}{2\pi}e^{2t}$. $C = D = 0$ ger $y(t) = 0$ då $t > 3$, eftersom $f(t) = 0$ då $t > 0$.

<u>Svar:</u> $y(t) = (e^{t+3} - e^{2(t+3)})\chi(-t-3) + (e^{t-3} - e^{2(t-3)})\chi(-t+3)$, $t \in \mathbb{R}$.

6. Sätt $u(t) = \sum_{n=-\infty}^{\infty} \frac{e^{int}}{1 - 16n^2}$. $(e^{int})'' = -n^2 e^{int}$, så

$$16u'' + u = \sum_{n=-\infty}^{\infty} e^{int} = \sum_{n=-\infty}^{\infty} 2\pi\delta(t - 2\pi n).$$

På $]-2\pi, 2\pi[$ har vi: $16u'' + u = 2\pi\delta$.

Kar.ekv.: $16r^2 + 1 = 0$, $r = \pm\frac{i}{4}$, så $u_h = A\cos\frac{t}{4} + B\sin\frac{t}{4}$.

$u_f(0) = 0$ och $u_f'(0) = \frac{1}{16}$ $\Rightarrow$ $u_f = \frac{1}{4}\sin\frac{t}{4}$

$\Rightarrow f(t) = \left(\frac{1}{4}\sin\frac{t}{4}\right)\chi(t)$ $\Rightarrow$ $u_p = 2\pi f = \left(\frac{\pi}{2}\sin\frac{t}{4}\right)\chi(t)$.

Så $u = u_p + u_h$ på $]-2\pi, 2\pi[$, och u är 2π-periodisk, vilket för $0 < t < 2\pi$ ger att

$$\frac{\pi}{2}\sin\frac{t}{4} + A\cos\frac{t}{4} + B\sin\frac{t}{4} = A\cos\frac{t-2\pi}{4} + B\sin\frac{t-2\pi}{4} =$$

$$= A\sin\frac{t}{4} - B\cos\frac{t}{4}.$$

Forts. →

6. forts. Så $\frac{\pi}{2} + B = A$ och $A = -B$, dvs $A = \frac{\pi}{4}$, $B = -\frac{\pi}{4}$.

Svar: Serien har period 2π och ges av $\frac{\pi}{4}\sin\frac{t}{4} + \frac{\pi}{4}\cos\frac{t}{4}$ då $0 \le t < 2\pi$.

7. $u(t) = t$ då $0 \le t < 1$ och u 1-periodisk.

$$(\mathcal{L}_+u)(s) = \int_0^\infty u(t)e^{-st}\,dt = \sum_{n=0}^{\infty}\int_n^{n+1} u(t)e^{-st}\,dt$$

$$= \sum_{n=0}^{\infty}\int_0^1 u(t+n)e^{-s(t+n)}\,dt = \sum_{n=0}^{\infty} e^{-sn}\int_0^1 u(t)e^{-st}\,dt$$

$$= \frac{1}{1-e^{-s}}\int_0^1 te^{-st}\,dt, \quad \operatorname{Re} s > 0.$$

Så $$(\mathcal{L}_+u)(1+2\pi in) = \frac{1}{1-e^{-(1+2\pi in)}}\int_0^1 te^{-(1+2\pi in)t}\,dt$$

$$= \frac{1}{1-e^{-1}}\int_0^1 te^{-t}e^{-in2\pi t}\,dt = \frac{e}{e-1}(\mathcal{F}_T v)(n), \quad n \in \mathbb{Z},$$

där $v(t) = te^{-t}$ då $0 \le t < 1$ och v är 1-periodisk.

Satsen om punktvis konvergens ger att

$$\lim_{N\to\infty}\sum_{n=-N}^{N}(\mathcal{L}_+u)(1+2\pi in) = \lim_{N\to\infty}\sum_{n=-N}^{N}\frac{e}{e-1}(\mathcal{F}_T v)(n)$$

$$= \frac{e}{e-1}\,\frac{v(0+)+v(0-)}{2} = \frac{e}{e-1}\,\frac{0+e^{-1}}{2} = \underline{\underline{\frac{1}{2(e-1)}}}.$$