Lösningar, TATA77, 2015-01-10

1. $u(t) = \sin t,\quad 0 \le t < \pi/2,\quad T = \pi/2 \;\Rightarrow\; \Omega = 4.$

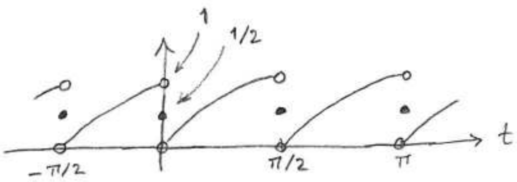
$$\hat{u}(n) = \frac{1}{\pi/2}\int_0^{\pi/2} \sin t \, e^{-in4t}\,dt = \frac{2}{\pi}\int_0^{\pi/2} \frac{1}{2i}\left(e^{it} - e^{-it}\right) e^{-in4t}\,dt =$$

$$= \frac{1}{i\pi}\int_0^{\pi/2}\left(e^{i(1-4n)t} - e^{-i(1+4n)t}\right)dt = \frac{1}{i\pi}\left[\frac{e^{i(1-4n)t}}{i(1-4n)} - \frac{e^{-i(1+4n)t}}{-i(1+4n)}\right]_0^{\pi/2} =$$

$$= \frac{1}{i\pi}\left(\frac{i}{i(1-4n)} - \frac{-i}{-i(1+4n)} - \frac{1}{i(1-4n)} + \frac{1}{-i(1+4n)}\right) = \frac{2-8in}{\pi(1-16n^2)},\quad n \in \mathbb{Z}.$$

<u>Delsvar</u>: Fourierserien är $\displaystyle\sum_{n=-\infty}^{\infty} \frac{2-8in}{\pi(1-16n^2)}\, e^{in4t}$.

Fourierseriens summa, enligt satsen om punktvis konvergens:



2. $y = Sx$ ger att $y'' + 4y' + 3y = x''' + x'$, så impulssvaret h uppfyller

$h'' + 4h' + 3h = \delta''' + \delta'$. Laplacetransform ger att

$(s^2 + 4s + 3)\hat{h} = s^3 + s$, och $s^2 + 4s + 3 = (s+1)(s+3)$, så

$$\hat{h} = \left(\frac{s^3+s}{(s+1)(s+3)},\ \operatorname{Re} s > -1\right)_{\mathcal{H}'} + C\delta_{-1} + D\delta_{-3}$$

$$= \left(s - 4 - \frac{1}{s+1} + \frac{15}{s+3},\ \operatorname{Re} s > -1\right)_{\mathcal{H}'} + C\delta_{-1} + D\delta_{-3}$$

Inverstransform ger: $h = \delta' - 4\delta - e^{-t}\chi + 15e^{-3t}\chi + \frac{C}{2\pi}e^{-t} + \frac{D}{2\pi}e^{-3t}$

<u>Svar</u>: $h = \delta' - 4\delta - e^{-t}\chi + 15e^{-3t}\chi + Ee^{-t} + Fe^{-3t},\quad E, F \in \mathbb{C}.$

3. a) $u(t) = \begin{Bmatrix} 1 - t^2, & 0 \le t < 1 \\ 0, & t < 0 \text{ el. } t \ge 1 \end{Bmatrix} = (1-t^2)(\chi(t) - \chi(t-1))$

Så $u'(t) = (1-t^2)(\delta(t) - \delta(t-1)) - 2t(\chi(t) - \chi(t-1)) =$

$= \delta(t) - 2t(\chi(t) - \chi(t-1))$, och

$u''(t) = \delta'(t) - 2t(\delta(t) - \delta(t-1)) - 2(\chi(t) - \chi(t-1))$, så:

<u>Svar</u>: $u''(t) = \delta'(t) + 2\delta(t-1) - 2(\chi(t) - \chi(t-1))$.

3. b) $tu' + u = 1$, $(tu)' = 1$, $tu = t + C$, $u = 1 + C\underline{t}^{-1} + D\delta$.

<u>Svar</u>: $u = 1 + C\underline{t}^{-1} + D\delta$, $C, D \in \mathbb{C}$.

c) $\langle \delta'(3t-6), \varphi(t)\rangle = \frac{1}{3}\langle \delta'(t), \varphi(\frac{t+6}{3})\rangle = -\frac{1}{3}\langle \delta(t), \varphi'(\frac{t+6}{3})\cdot\frac{1}{3}\rangle =$

$= -\frac{1}{9}\varphi'(2) = \frac{1}{9}\langle \delta_2', \varphi\rangle$, $\varphi \in \mathcal{D}(\mathbb{R})$

<u>Svar</u>: $\frac{1}{9}\delta_2'$.

4. $u(t) = (1-|t|)^2$ då $|t| \le 1$, $u(t) = 0$ då $|t| > 1$.

cos jämn, sin udda

$$\hat{u}(\omega) = \int_{-1}^{1}(1-|t|)^2 e^{-i\omega t}\,dt = \int_{-1}^{1}(1-|t|)^2(\cos\omega t - i\sin\omega t)\,dt =$$

$$= 2\int_0^1 (1-t)^2\cos\omega t\,dt \overset{\omega\neq 0}{=} 2\Big[(1-t)^2\frac{\sin\omega t}{\omega} - 2(1-t)(-1)\frac{-\cos\omega t}{\omega^2} +$$

$$+ 2\,\frac{-\sin\omega t}{\omega^3}\Big]_0^1 = 2\Big(-\frac{2\sin\omega}{\omega^3} + \frac{2}{\omega^2}\Big) = \frac{4(\omega - \sin\omega)}{\omega^3},\quad \omega \neq 0.$$

$\Big(\hat{u}(0) = \int_{-1}^{1}(1-|t|)^2\,dt = \frac{2}{3}.\Big)$

<u>Delsvar</u>: $\hat{u}(\omega) = \dfrac{4(\omega - \sin\omega)}{\omega^3}$ $\quad(\hat{u}(0) = \frac{2}{3})$.

Parsevals formel ger att

$$\int_{-\infty}^{\infty}\frac{4^2(\omega - \sin\omega)^2}{\omega^6}\,d\omega = 2\pi\int_{-1}^{1}(1-|t|)^4\,dt = 2\pi\cdot 2\cdot\frac{1}{5}.$$

<u>Delsvar</u>: $\displaystyle\int_{-\infty}^{\infty}\frac{(\omega-\sin\omega)^2}{\omega^6}\,d\omega = \frac{\pi}{20}$.

5. $$\hat{u}(z) = \frac{z^3}{(z-1)^2(z-2)} = z\,\frac{z^2}{(z-1)^2(z-2)} = z\Big(\frac{-3}{z-1} + \frac{-1}{(z-1)^2} + \frac{4}{z-2}\Big) =$$

$$= -\frac{3z}{z-1} - \frac{z}{(z-1)^2} + \frac{4z}{z-2},\quad 1 < |z| < 2.$$

Så $u(n) = -3\chi(n) - n\chi(n) + v(n)$, där $\hat{v}(z) = \dfrac{4z}{z-2}$, $0 < |z| < 2$.

$v(n) \xrightarrow{\mathcal{Z}} \dfrac{4z}{z-2}$, $0 < |z| < 2$

$v(-n)$ $\quad \dfrac{4/z}{1/z - 2} = \dfrac{4}{1-2z} = -\dfrac{2}{z - 1/2}$, $0 < |1/z| < 2$

$v(-(n+1))$ $\quad -\dfrac{2z}{z-1/2}$, $|z| > 1/2$

forts...

5. forts: Så $v(-n-1) = -2(1/2)^n \chi(n)$, $v(n) = -2(1/2)^{-n-1}\chi(-n-1)$.

<u>Svar:</u> $u(n) = -(n+3)\chi(n) - 4\cdot 2^n \chi(-n-1)$, $n \in \mathbb{Z}$.

6. $u(t) = \sin n$ då $n \le t < n+1$, $n = 0, 1, 2, \dots$, och $u(t) = 0$ då $t < 0$.

$$\hat{u}(s) = \int_{-\infty}^{\infty} u(t) e^{-st}\,dt = \sum_{n=0}^{\infty} \int_n^{n+1} (\sin n)\, e^{-st}\,dt \overset{s \neq 0}{=}$$

$$= \sum_{n=0}^{\infty} \sin n \left[\frac{e^{-st}}{-s}\right]_n^{n+1} = \sum_{n=0}^{\infty} \sin n \,\frac{e^{-sn} - e^{-s(n+1)}}{s} =$$

$$= \frac{1-e^{-s}}{s} \sum_{n=0}^{\infty} (\sin n)(e^s)^{-n} = /\text{ z-transformtabell }/$$

$$= \frac{1-e^{-s}}{s}\,\frac{(\sin 1)e^s}{e^{2s} - 2(\cos 1)e^s + 1} \quad \text{då } |e^s| > 1, \text{ dvs då } \operatorname{Re} s > 0.$$

<u>Svar:</u> $\hat{u}(s) = \dfrac{(\sin 1)(e^s - 1)}{s(e^{2s} - 2(\cos 1)e^s + 1)}$, $\operatorname{Re} s > 0$.

7. $u : \mathbb{R} \to \mathbb{C}$ kont. och $u(t) = \frac{1}{t} + \mathcal{O}(\frac{1}{t^2})$ då $t \to \pm\infty$.

Sätt $v(t) = u(t) - \dfrac{t}{t^2+1} = \dfrac{1}{t} + \mathcal{O}(\frac{1}{t^2}) - \dfrac{t}{t^2+1} =$

$$= \frac{t^2+1-t^2}{t(t^2+1)} + \mathcal{O}\left(\frac{1}{t^2}\right) = \frac{1}{t(t^2+1)} + \mathcal{O}\left(\frac{1}{t^2}\right) \quad \text{då } t \to \pm\infty.$$

Så $v \in \mathcal{L}^1(\mathbb{R})$, så $\hat{v}$ är en kontinuerlig funktion.

$$\hat{u}(\omega) = \left(\frac{t}{t^2+1}\right)^{\wedge}(\omega) + \hat{v}(\omega) = -\pi i e^{-|\omega|}\operatorname{sgn}\omega + \hat{v}(\omega),$$

så $\hat{u}$ ges av en funktion som är kontinuerlig utom i origo,

och $\hat{u}(0+) - \hat{u}(0-) = -\pi i + \hat{v}(0) - (-\pi i(-1) + \hat{v}(0)) = \underline{\underline{-2\pi i}}$.