Lösningar, TATA77, 2014-10-31

1. $y(n+2) - y(n+1) + y(n) = 1, \quad n \in \mathbb{N}, \quad y(0) = 4, \quad y(1) = 2.$

Enkelsidig z-transform ger ($Y = \mathcal{Z}_+ y$):

$$z^2 Y(z) - 4z^2 - 2z - (zY(z) - 4z) + Y(z) = \frac{z}{z-1}, \text{ så}$$

$$(z^2 - z + 1)\,Y(z) = \frac{z}{z-1} + 4z^2 - 2z = \frac{z + 4z^3 - 6z^2 + 2z}{z-1}, \text{ så}$$

$$Y(z) = z \cdot \frac{4z^2 - 6z + 3}{(z^2 - z + 1)(z-1)} = z\left(\frac{3z-2}{z^2-z+1} + \frac{1}{z-1}\right) =$$

$$= \frac{3z^2 - 2z}{z^2 - z + 1} + \frac{z}{z-1} = \frac{3(z^2 - (\cos\frac{\pi}{3})z) - \frac{1}{2}\frac{2}{\sqrt{3}}(\sin\frac{\pi}{3})z}{z^2 - 2(\cos\frac{\pi}{3})z + 1} + \frac{z}{z-1}, \quad |z| > 1.$$

Ur tabell fås: <u>Svar</u>: $y(n) = 3\cos\frac{n\pi}{3} - \frac{1}{\sqrt{3}}\sin\frac{n\pi}{3} + 1, \quad n \in \mathbb{N}.$

2. a) $\langle (\cos t)\delta'', \varphi\rangle = \langle \delta'', (\cos t)\varphi\rangle = (-1)^2\langle \delta, ((\cos t)\varphi)''\rangle =$

$= \langle \delta, (\cos t)\varphi'' - 2(\sin t)\varphi' - (\cos t)\varphi\rangle = 1\cdot\varphi''(0) - 0 - 1\varphi(0) =$

$= \langle (-1)^2\delta'' - \delta, \varphi\rangle, \quad \varphi \in \mathcal{D}(\mathbb{R}).$ <u>Svar</u>: $\delta'' - \delta$.

b) $t^2 u' = 1.$ $t^2 u' = t^2 \underline{t}^{-2}$, så $u' = \underline{t}^{-2} + C\delta + D\delta'$, så:

<u>Svar</u>: $u = -\underline{t}^{-1} + C\alpha + D\delta + E, \quad C, D, E \in \mathbb{C}.$

c) $u \in \mathcal{D}'(\mathbb{R})$. Vi har att $\langle (u'(t))\hat{}\,(s), \psi(s)\rangle = \langle u'(t), \hat{\psi}(t)\rangle =$

$= -\langle u(t), \hat{\psi}'(t)\rangle = \ldots$

$$\left[\begin{array}{l} \hat{\psi}'(t) = \frac{d}{dt}\int_{-\infty}^{\infty}\psi(i\omega)e^{-i\omega t}\,d\omega = \int_{-\infty}^{\infty}\psi(i\omega)\,e^{-i\omega t}(-i\omega)\,d\omega = \\ \qquad = -(s\psi(s))\hat{}\,(t), \quad t \in \mathbb{R}. \end{array}\right.$$

$\ldots = -\langle u(t), -(s\psi(s))\hat{}\,(t)\rangle = \langle \hat{u}(s), s\psi(s)\rangle = \langle s\hat{u}(s), \psi(s)\rangle,$

för $\psi \in \mathcal{H}$, så $(u'(t))\hat{}\,(s) = s\hat{u}(s)$.

3. $u(t) = e^{iat}$, $-\pi \le t < \pi$, $T = 2\pi \Rightarrow \Omega = 1$, $a \in \mathbb{R} \setminus \mathbb{Z}$.

$$\hat{u}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iat} e^{-int}\,dt = \frac{1}{2\pi}\left[\frac{e^{i(a-n)t}}{i(a-n)}\right]_{-\pi}^{\pi} =$$

$$= \frac{1}{2\pi}\,\frac{e^{ia\pi}e^{-in\pi} - e^{-ia\pi}e^{in\pi}}{i(a-n)} = \frac{(-1)^n \sin a\pi}{\pi(a-n)}, \quad n \in \mathbb{Z}.$$

<u>Delsvar:</u> Fourierserien är $\displaystyle\sum_{n=-\infty}^{\infty} \frac{(-1)^n \sin a\pi}{\pi(a-n)} e^{int}$.

u har gen. höger- och vänsterderivata i $t = \pi$, så satsen om punktvis konvergens ger att

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \frac{(-1)^n \sin a\pi}{\pi(a-n)} e^{in\pi} = \frac{u(\pi+) + u(\pi-)}{2} = \frac{u((-\pi)+) + e^{ia\pi}}{2} =$$

$$= \frac{e^{ia(-\pi)} + e^{ia\pi}}{2} = \cos a\pi, \quad \text{så} \quad \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{a-n} = \frac{\pi}{\tan a\pi}.$$

4. $y'' + y' - 2y = \delta''$, $y \in S'$. Fouriertransform ger:

$(i\omega)^2 \hat{y} + i\omega\hat{y} - 2\hat{y} = (i\omega)^2$. $(i\omega)^2 + i\omega - 2 = (i\omega - 1)(i\omega + 2)$ har inga reella nollställen, så

$$\hat{y} = \frac{(i\omega)^2}{(i\omega)^2 + i\omega - 2} = 1 + \frac{-i\omega + 2}{(i\omega - 1)(i\omega + 2)} = 1 + \frac{1/3}{i\omega - 1} + \frac{-4/3}{i\omega + 2}.$$

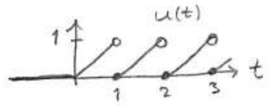
Tabell ger: <u>Svar:</u> $y = \delta - \frac{1}{3}e^{t}\chi(-t) - \frac{4}{3}e^{-2t}\chi(t)$.

5. $$\hat{u}(s) = \frac{1 - (s+1)e^{-s}}{s^2(1 - e^{-s})} = \hat{v}(s)\,(1 + e^{-s} + e^{-2s} + \dots), \quad \operatorname{Re} s > 0,$$

där $\hat{v}(s) = \frac{1}{s^2} - \left(\frac{1}{s} + \frac{1}{s^2}\right)e^{-s}$, $\operatorname{Re} s > 0$. Tabell ger att

$$v(t) = t\chi(t) - (1 + (t-1))\chi(t-1) = \begin{cases} t, & 0 \le t < 1, \\ 0, & \text{annars.} \end{cases}$$

Så $u(t) = v(t) + v(t-1) + v(t-2) + \dots$:



<u>Svar:</u> $u(t) = \begin{cases} 0, & t < 0, \\ t - n, & n \le t < n+1, \ n \in \mathbb{N}. \end{cases}$

6. $\displaystyle\int_{-\infty}^{\infty} e^{-2|t-r|}\, u(r)\,dr = t^2 e^t \alpha(t), \quad t\in\mathbb{R}.$

VL är $(e^{-2|t|} * u(t))(t)$, och $e^{-2|t|} = e^{-2t}\alpha(t) + e^{2t}\alpha(-t)$, $t \neq 0$.

$$e^{-2t}\alpha(t) \xrightarrow{\mathcal{L}} \frac{1}{s+2}, \quad \mathrm{Re}\, s > -2,$$

$$e^{2t}\alpha(-t) \qquad \frac{1}{-s+2}, \quad \mathrm{Re}(-s) > -2.$$

Så $\displaystyle e^{-2|t|} \xrightarrow{\mathcal{L}} \frac{1}{s+2} - \frac{1}{s-2} = -\frac{4}{s^2-4}, \quad -2 < \mathrm{Re}\, s < 2.$

$\mathcal{L}(VL) = \mathcal{L}(HL)$ ger: $\displaystyle -\frac{4}{s^2-4}\hat{u}(s) = \frac{2}{(s-1)^3}, \quad 1 < \mathrm{Re}\, s < 2,$ så

$$\hat{u}(s) = -\frac{s^2-4}{2(s-1)^3} = -\frac{(s-1)^2 + 2(s-1) - 3}{2(s-1)^3}, \quad \mathrm{Re}\, s > 1, \text{ så tabell ger:}$$

$$u(t) = -\frac{1}{2}e^t\alpha(t) - te^t\alpha(t) + \frac{3}{4}t^2e^t\alpha(t).$$

<u>Svar</u>: $\displaystyle u(t) = \frac{3t^2-4t-2}{4}e^t\alpha(t), \quad t\in\mathbb{R}.$

7. Visa att $\displaystyle\int_{-\infty}^{\infty} \frac{\sin\omega}{\omega}\,\frac{\sin(\omega/2)}{\omega/2}\cdots\frac{\sin(\omega/2^n)}{\omega/2^n}\,d\omega = \pi, \quad n\in\mathbb{N}.$

Sätt, för $a>0$, $\displaystyle p_a(t) = \begin{cases} 1/2a, & |t|\le a, \\ 0, & |t|>a. \end{cases}$ Då är $\displaystyle\int_{-\infty}^{\infty} p_a(t)\,dt = 1,$

och $\displaystyle\widehat{p_a}(\omega) = \frac{\sin a\omega}{a\omega}$ (ur $\mathcal{F}$-tabell).

Så $\displaystyle\frac{\sin\omega}{\omega}\,\frac{\sin(\omega/2)}{\omega/2}\cdots\frac{\sin(\omega/2^n)}{\omega/2^n} = \widehat{p_1}(\omega)\widehat{p_{1/2}}(\omega)\cdots\widehat{p_{1/2^n}}(\omega) =$

$= (p_1 * p_{1/2} * \cdots * p_{1/2^n})^\wedge(\omega)$, och Fouriers inv. formel ger att

$$\int_{-\infty}^{\infty} \frac{\sin\omega}{\omega}\cdots\frac{\sin(\omega/2^n)}{\omega/2^n}\,d\omega = 2\pi(p_1 * p_{1/2} * \cdots * p_{1/2^n})(0) = 2\pi\cdot\frac{1}{2} = \pi,$$

ty $p_1 = \frac{1}{2}$ då $|t|\le 1$, så $p_1 * p_{1/2} = \frac{1}{2}$ då $|t| \le 1 - \frac{1}{2}$, så

$p_1 * p_{1/2} * p_{1/2^2} = \frac{1}{2}$ då $|t| \le 1 - \frac{1}{2} - \frac{1}{2^2}$, och så vidare.

Eftersom $1 - \frac{1}{2} - \cdots - \frac{1}{2^n} > 0$ för alla $n\in\mathbb{N}$ följer

att $(p_1 * p_{1/2} * \cdots * p_{1/2^n})(0) = \frac{1}{2}$ för alla $n\in\mathbb{N}$.