(2) Lokala extrempunkter

$$\frac{\partial^2 f}{\partial x^2}=\frac{3y^2}{(2+xy)^2},\ \frac{\partial^2 f}{\partial x^2}=\frac{3x^2}{(2+xy)^2},\ \frac{\partial^2 f}{\partial x\partial y}=-\frac{3}{2+xy}+\frac{3xy}{(2+xy)^2};$$

$\underline{(x,y)=(1,1)}$: $f''_{xx}=\frac{1}{3}=f''_{yy}$, $f''_{xy}=-\frac{2}{3}$;

$Q(h,k)=\frac{1}{3}(h^2-4hk+k^2)=\frac{1}{3}((h-2k)^2-3k^2)$, indefinit; (1,1) är en sadelpunkt.

$\underline{(x,y)=(2,2)}$: $f''_{xx}=\frac{1}{3}=f''_{yy}$, $f''_{xy}=-\frac{1}{6}$;

$Q(h,k)=\frac{1}{3}(h^2-hk+k^2)=\frac{1}{3}((h-\frac{1}{2}k)^2+\frac{3}{4}k^2)$, positivt definit; (2,2) är en lokal min/pkt.

e) $\underline{f(x,y)=\ln(x^2+y^2)-x-2y}$

(1) Stationära punkter

$$\left.\begin{aligned}\frac{\partial f}{\partial x}&=\frac{2x}{x^2+y^2}-1=0\\ \frac{\partial f}{\partial y}&=\frac{2y}{x^2+y^2}-2=0\end{aligned}\right\}\Rightarrow\begin{cases}y=2x\\ \frac{2x}{x^2+y^2}=1\end{cases}\Leftrightarrow\begin{cases}y=2x\\ \frac{2}{5x}=1\end{cases}\Leftrightarrow\begin{cases}x=\frac{2}{5}\\ y=\frac{4}{5}\end{cases}$$

(2) Lokala extrempunkter

$$\frac{\partial^2 f}{\partial x^2}=\frac{2}{|\mathbf{x}|^2}-\frac{4x^2}{|\mathbf{x}|^4},\ \frac{\partial^2 f}{\partial y^2}=\frac{2}{|\mathbf{x}|^2}-\frac{4y^2}{|\mathbf{x}|^4},\ \frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}=-\frac{4xy}{|\mathbf{x}|^4};$$

$$\left.\begin{aligned}f''_{xx}(\tfrac{2}{5},\tfrac{4}{5})&=\frac{2}{4/5}-\frac{16/5}{16/25}=\frac{5}{2}-5=-\frac{5}{2}\\ f''_{yy}(\tfrac{2}{5},\tfrac{4}{5})&=\frac{2}{4/5}-\frac{64/25}{16/25}=\frac{5}{2}-4=-\frac{3}{2}\\ f''_{xy}(\tfrac{2}{5},\tfrac{4}{5})&=-\frac{32/5^2}{16/25}=-2\end{aligned}\right\}\Rightarrow Q(h,k)=-\frac{5}{2}h^2-$$

$$-4hk-\frac{3}{2}k^2=-\frac{5}{2}(h^2+\frac{8}{5}hk+\frac{3}{5}k^2)=-\frac{5}{2}((h+\frac{4}{5}k)^2-\frac{k^2}{25}),$$

indefinit; $(\frac{2}{5},\frac{4}{5})$ är ingen lokal extr/pkt.

f) $\underline{f(x,y,z)=x^4+y^4+z^4-4xyz}$

(1) $\frac{\partial f}{\partial x}=4x^3-4yz$, $\frac{\partial f}{\partial y}=4y^3-4xz$, $\frac{\partial f}{\partial y}=4z^3-4xy$

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=\frac{\partial f}{\partial z}=0\Rightarrow\begin{cases}x^3=yz\\ y^3=xz\\ z^3=xy\end{cases}\Leftrightarrow\begin{cases}x^4=xyz\\ y^4=xyz\\ z^4=xyz\end{cases}\Leftrightarrow x^4=y^4=$$

$$=z^4\Leftrightarrow\begin{cases}x^4=y^4\\ y^4=z^4\end{cases}\Leftrightarrow\begin{cases}x=\pm y\\ z=\pm y\end{cases}\Leftrightarrow\begin{cases}x=y\\ z=y\end{cases}\vee\begin{cases}x=y\\ z=-y\end{cases}\vee$$

$$\vee\begin{cases}x=-y\\ z=y\end{cases}\vee\begin{cases}x=-y\\ z=-y\end{cases}\Leftrightarrow\begin{cases}x=0\\ y=0\\ z=0\end{cases}\vee\begin{cases}x=1\\ y=1\\ z=1\end{cases}\vee\begin{cases}x=1\\ y=-1\\ z=-1\end{cases}\vee\begin{cases}x=-1\\ y=-1\\ z=1\end{cases}\vee$$

$$\vee\begin{cases}x=-1\\ y=1\\ z=-1\end{cases}\Rightarrow\mathbf{x}=\underline{(0,0,0)},\ \underline{(1,1,1)},\ \underline{(1,-1,-1)},\ \underline{(-1,-1,1)},\ \underline{(-1,1,-1)}.$$

(2) Lokala extrempunkter

$$\frac{\partial^2 f}{\partial x^2}=12x^2,\ \frac{\partial^2 f}{\partial y^2}=12y^2,\ \frac{\partial^2 f}{\partial y^2}=12z^2,\ \frac{\partial^2 f}{\partial x\partial y}=-4z,\ \frac{\partial^2 f}{\partial x\partial z}=-4y,$$
$$\frac{\partial^2 f}{\partial y\partial z}=-4x;$$

$\underline{\mathbf{x}=(0,0,0)}$: Inga slutsatser kan dras, ang. dess karaktär (art).

$\underline{\mathbf{x}=(1,1,1)}$: $f''_{xx}=f''_{yy}=f''_{zz}=12$, $f'_{xy}=f''_{xz}=f''_{yz}=-4$.

$Q(h,k,l) = 12h^2+12l^2+12z^2-8hk-8hl-8kl =$

$$= [h\ k\ l]\begin{bmatrix} 12 & -4 & -4 \\ -4 & 12 & -4 \\ -4 & -4 & 12 \end{bmatrix}\begin{bmatrix} h \\ k \\ l \end{bmatrix} \Rightarrow A = \begin{bmatrix} 12 & -4 & -4 \\ -4 & 12 & -4 \\ -4 & -4 & 12 \end{bmatrix};$$

$|A_1| = |12| = 12$, $|A_2| = \begin{vmatrix} 12 & -4 \\ -4 & 12 \end{vmatrix} = 128 > 0$, $|A_3| = |A| =$
$= 1024 > 0$, så formen Q är positiv definit, dvs $(1,1,1)$ ger lokalt minimum.

På samma sätt visas att de övriga punkterna ger lok/min.

Anm. Egenvärdena till A är positiva.

g) $f(x,y) = x^3+3xy^2-15x-12y$

(1) Stationära punkter

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 3x^2+3y^2-15 = 0 \\ \frac{\partial f}{\partial y} &= 6xy-12 = 0 \end{aligned}\right\} \Rightarrow \begin{cases} x^2+y^2=5 \\ xy=2 \end{cases} \Leftrightarrow \begin{cases} (x+y)^2=9 \\ (x-y)^2=1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x+y=3 \\ x-y=1 \end{cases} \vee \begin{cases} x+y=3 \\ x-y=-1 \end{cases} \vee \begin{cases} x+y=-3 \\ x-y=1 \end{cases} \vee \begin{cases} x+y=-3 \\ x-y=-1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x=2 \\ y=1 \end{cases} \vee \begin{cases} x=1 \\ y=2 \end{cases} \vee \begin{cases} x=-1 \\ y=-2 \end{cases} \vee \begin{cases} x=-2 \\ y=-1 \end{cases}.$$

(2) Lokala extrempunkter

$\frac{\partial^2 f}{\partial x^2} = 6x$, $\frac{\partial^2 f}{\partial y^2} = 6x$, $\frac{\partial^2 f}{\partial x \partial y} = 6y$

$(x,y) = (2,1)$: $f''_{xx} = f''_{yy} = 12$, $f''_{xy} = 6$; $Q(h,k) = 12(h^2+hk+k^2) = 12(h+\frac{1}{2}k)^2+9k^2$, positiv definit; lokalt minimum föreligger.

$(x,y) = (1,2)$: $f''_{xx} = f''_{yy} = 6$, $f''_{xy} = 12 \Rightarrow Q = 6(h^2+4hk+k^2) = 6(h+2k)^2-18k^2$ indefinit; sadelpunkt.

$(x,y) = (-1,-2)$: $f''_{xx} = f''_{yy} = -6$, $f''_{xy} = -12$; $Q = -6(h^2+4hk+k^2) = -6(h+2k)^2+18k^2$, indefinit; ingen extrempunkt (sadelpunkt).

$(x,y) = (-2,-1)$: $f''_{xx} = f''_{yy} = -12$, $f''_{xy} = -6$; $Q = -12(h^2+hk+k^2) = -12(h+\frac{k}{2})^2-9k^2$, negativt definit; lokalt maximum föreligger.

h) $f(x,y) = (x^2+y^2-4)e^{-x-y}$

(1) Stationära punkter

$\frac{\partial f}{\partial x} = (2x+4-x^2-y^2)e^{-x-y}$, $\frac{\partial f}{\partial y} = (2y+4-x^2-y^2)e^{-x-y}$;

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Rightarrow \begin{cases} x^2+y^2=2x+4 \\ x^2+y=2y+4 \end{cases} \Leftrightarrow \begin{cases} x^2+y^2=2x+4 \\ x=y \end{cases} \Leftrightarrow$$

$\Rightarrow x^2 = x+2 \wedge y = x \Leftrightarrow (x,y) = (-1,-1) \vee (x,y) = (2,2).$

(2) $\frac{\partial^2 f}{\partial x^2}=(2-2x+x^2+y^2-2x-4)e^{-x-y}=(x^2+y^2-4x-2)e^{-x-y}$

$\frac{\partial^2 f}{\partial y^2}=(x^2+y^2-4y-2)e^{-x-y}$;

$\frac{\partial^2 f}{\partial x\partial y}=(x^2+y^2-2x-2y-4)e^{-x-y}$;

$\underline{(x,y)=(-1,-1)}$: $f''_{xx}=f''_{yy}=4e^2$; $f''_{xy}=2e^2$;

$Q=4e^2(h^2+hk+k^2)$, positivt definit; minimum föreligger.

$\underline{(x,y)=(2,2)}$: $f''_{xx}=-2e^{-4}=f''_{yy}$, $f''_{xy}=-4e^{-4}$.

$Q=-2e^{-4}(h^2+4hk+k^2)=-2e^{-4}(h+2k)^2+6e^{-4}k^2$,

indefinit; sadelpunkt.

i) $f(x,y)=4x^2+4xy^2+y^4+y^5$

(1) Stationära punkter

$$\begin{cases}\frac{\partial f}{\partial x}=8x+4y^2=0\\ \frac{\partial f}{\partial y}=8xy+4y^3+5y^4=0\end{cases}\Leftrightarrow\begin{cases}x=-\frac{1}{2}y^2\\ 5y^4=0\end{cases}\Leftrightarrow \underline{(x,y)=(0,0)}.$$

(2) Lokala extrempunkter

$\frac{\partial^2 f}{\partial x^2}=8$, $\frac{\partial^2 f}{\partial y^2}=8x+12y^2+20y^3$, $\frac{\partial^2 f}{\partial x\partial y}=8y$;

$\underline{(x,y)=(0,0)}$: $f''_{xx}=0$, $f''_{yy}=f''_{xy}=0$; $Q(h,k)=8k^2$,

positivt semidefinit; ingen slutsats kan dras angående origos art.

j) $f(x,y,z)=x+y^2/4x+z^2/y+2/z$

(1) Stationära punkter

$$\left.\begin{array}{l}\frac{\partial f}{\partial x}=1-y^2/4x^2=0\\ \frac{\partial f}{\partial y}=y/2x-z^2/y^2=0\\ \frac{\partial f}{\partial z}=2z/y-2/z^2=0\end{array}\right\}\Leftrightarrow\begin{cases}y^2=4x^2\\ z^2=y^3/2x \quad (\text{Obs! } x,y\neq 0)\\ y=z^3\end{cases}$$

$$\Leftrightarrow\begin{cases}y=\pm 2x\\ z^2=4x^2\\ y=z^3\end{cases}\Leftrightarrow\begin{cases}y=\pm 2x\\ z=\pm 2x\\ y=y^3\end{cases}\Leftrightarrow\begin{cases}z=\pm 2x\\ y=z^3\\ y=\pm 1\end{cases}\Leftrightarrow\begin{cases}x=\pm\frac{z}{2}\\ y=\pm 1\\ z=\pm 1\end{cases}$$

$\Leftrightarrow (x,y,z)=(\frac{1}{2},1,1) \vee (x,y,z)=(-\frac{1}{2},-1,-1)$.

(2) Lokala extrempunkter

$\frac{\partial^2 f}{\partial x^2}=y^2/2x^3$, $\frac{\partial^2 f}{\partial y^2}=1/2x+2z^2/y^3$, $\frac{\partial^2 f}{\partial z^2}=2/y+4/z^3$;

$\frac{\partial^2 f}{\partial x\partial y}=-y/2x^2$, $\frac{\partial^2 f}{\partial x\partial z}=0$, $\frac{\partial^2 f}{\partial y\partial z}=-2z/y^2$;

$\underline{(x,y,z)=(\frac{1}{2},1,1)}$: $f''_{xx}=4$, $f''_{yy}=3$, $f''_{zz}=6$, $f''_{xy}=-2$,

$f''_{xz}=0$, $f''_{yz}=-2$; $Q=4h^2+3k^2+6l^2-2hk-2kl=$

$$=[h\ k\ l]\begin{bmatrix}4 & -1 & 0\\ -1 & 3 & -1\\ 0 & -1 & 6\end{bmatrix}\begin{bmatrix}h\\ k\\ l\end{bmatrix}\Rightarrow A=\begin{bmatrix}4 & -1 & 0\\ -1 & 3 & -1\\ 0 & -1 & 6\end{bmatrix}=A^t\Rightarrow$$

$\Rightarrow |A_1|=4 \wedge |A_2|=11 \wedge |A_3|=|A|=62>0 \Rightarrow A$ är en positiv matris (alla egenvärden är positiva) $\Rightarrow Q$ positivt definit $\Rightarrow (\frac{1}{2},1,1)$ är en lokal min/pkt.

$f(-x,-y,-z)=-f(x,y,z)$, dvs f är udda, vilket medför att $(-\frac{1}{2},-1,-1)$ är en lokal max/pkt.

Problem 2.72 (Sid. 10)

Lösning

(1) Stationära punkter

$$\frac{\partial f}{\partial x}=\frac{4x^3(x^2+y^2)-2x(x^4+y^4)}{(x^2+y^2)^2}=\frac{2x((x^2+y^2)^2-2y^4)}{(x^2+y^2)^2}=$$

$$=\frac{2x(x^2+(\sqrt{2}+1)y^2)(x^2+(1-\sqrt{2})y^2)}{(x^2+y^2)^2}=0 \Rightarrow y^2=(\sqrt{2}-1)x^2.$$

$\frac{\partial f}{\partial y}=0 \Leftrightarrow y^2=(\sqrt{2}+1)x^2$, pga symmetrin. Stationära punkter saknas.

(2) $f(r\cos\theta, r\sin\theta)=r^2(\cos^4\theta+\sin^4\theta)\geq 0$; så $(0,0)$ är en lokal och global min/pkt.

Problem 2.73 (Sid. 10)

Lösning

Låt oss sätta $A=f''_{xx}(a,b)$, $B=f''_{xy}(a,b)$ och $C=$ $=f''_{yy}(a,b)$. Motsvarande kvadratisk form är

$$Q=Ah^2+2Bhk+Ck^2=A(h^2+2\frac{B}{A}hk+\frac{C}{A}k^2)=$$

$$=A((h+\frac{B}{A}k)^2+\frac{C}{A}-\frac{B^2}{A^2}=A(h+\frac{B}{A}k)^2+\frac{AC-B^2}{A}.$$

Q är positiv definit endast om $AC-B^2>0$. Detta är inte alltid fallet.

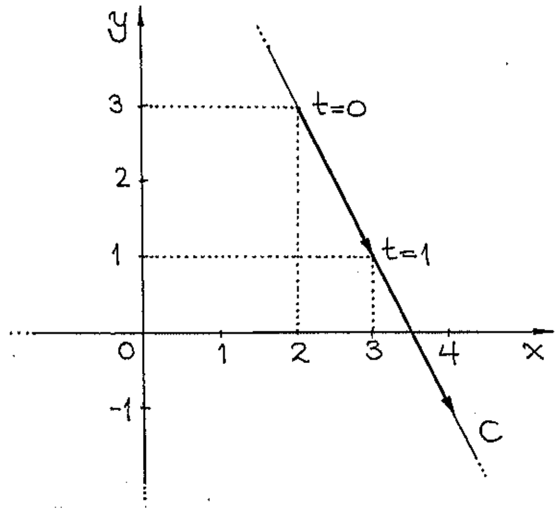
Svar: Nej.

Differentialkalkyl för vektorvärda funktioner

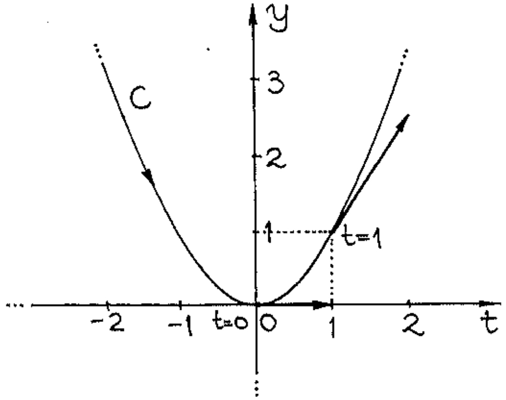
Problem 3.1 (Sid. 10)

Lösning

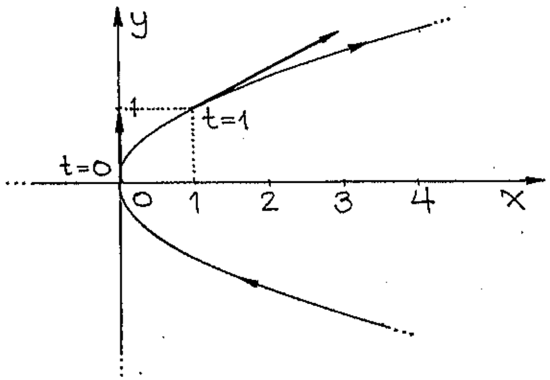
a) $C:\begin{cases} x=2+t \\ y=3-2t \end{cases} \Rightarrow 2x+y=7$ (riktning som i figuren).



b) $C:\begin{cases} x=t \\ y=t^2 \end{cases} \Rightarrow y=x^2$ (riktning som i figuren).



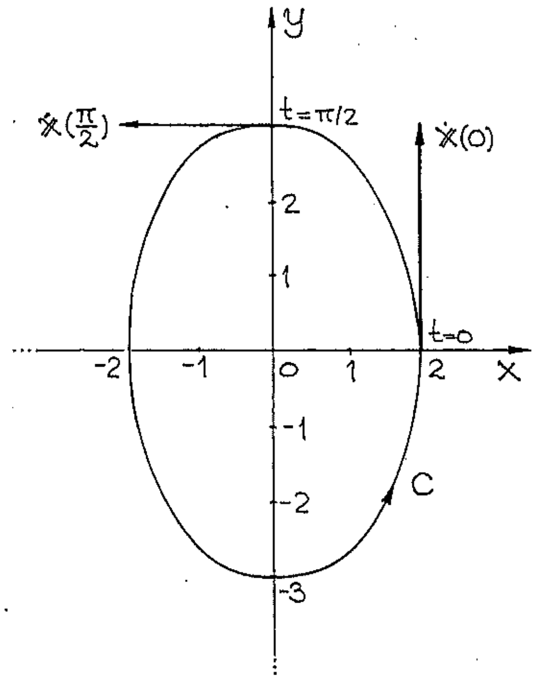
c) $C:\begin{cases} x=t^2 \\ y=t \end{cases} \Rightarrow x=y^2$ (riktning som i figuren).



Problem 3.2 (Sid. 10)

Lösning: a) $(x,y)=(2\cos t, 3\sin t)$, $0\leq t<2\pi$.

$C:\begin{cases} x=2\cos t \\ y=3\sin t \end{cases} \Rightarrow \frac{x^2}{2^2}+\frac{y^2}{3^2}=\cos^2 t+\sin^2 t=1$ (ellips).



$x(t)=(2\cos t, 3\sin t) \Rightarrow \dot{x}(t)=(-2\sin t, 3\cos t) \Rightarrow \begin{cases} \dot{x}(0)=(0,3) \\ \dot{x}(\frac{\pi}{2})=(-2,0) \end{cases}$

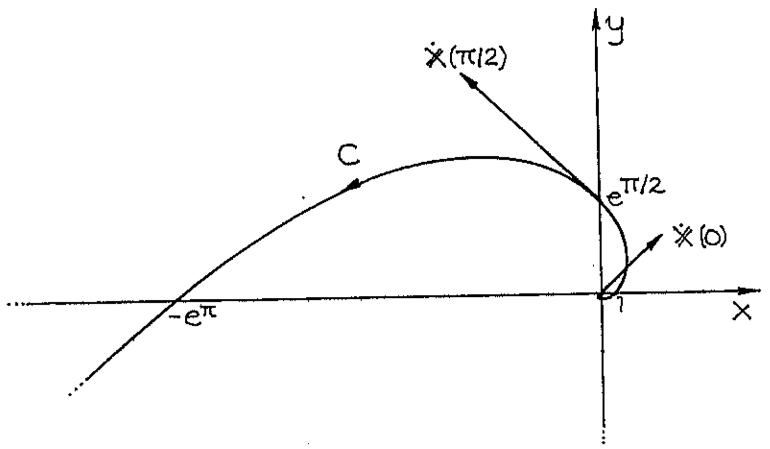
b) $\begin{cases} x=e^t\cos t \Rightarrow dx=e^t(\cos t-\sin t)dt \\ y=e^t\sin t \Rightarrow dy=e^t(\sin t+\cos t)dt \end{cases} \Rightarrow \frac{dy}{dx}=\frac{\sin t+\cos t}{\cos t-\sin t}$

$x(t)=(e^t\cos t, e^t\sin t) \Rightarrow \dot{x}(t)=e^t(\cos t-\sin t, \cos t+\sin t)$;

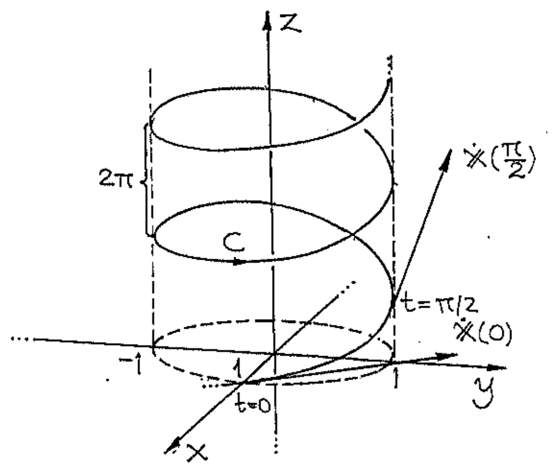
$\dot{x}(0)=(1,1)$, $\dot{x}(\frac{\pi}{2})=e^{\pi/2}(-1,1)$.

En värdetabell är på sin plats...

t	$-\infty$	0,1	0,2	0,3	0,5	1	1,5	2
x	0	1,1	1,2	1,29	1,45	0,54	0,32	-3,07
y	0	0,1	0,24	0,40	0,79	2,29	4,47	6,71



c) Kurvans ortogonala projektion på xy-planet är enhetscirkeln ; $\dot{z}=1$, så kurvan stiger med en fart 1 på cylinderytan $x^2+y^2=1$.



$\dot{x}(t)=(-\sin t, \cos t, 1)$; $\dot{x}(0)=(0,1,1)$, $\dot{x}(\frac{\pi}{2})=(-1,0,1)$

Kurvan kallas spiralkurva (eng. helix).

Problem 3.3 (Sid. 10)

Lösning

$$C:\begin{cases} x=\sin t \\ y=\cos t \\ z=\arctan t \end{cases} \Rightarrow r(t)=(\sin t, \cos t, \arctan t) \Rightarrow$$

$$\Rightarrow \dot{r}(t)=(\cos t, -\sin t, \frac{1}{t^2+1}) \Rightarrow \begin{cases} r(0)=(0,1,0) \\ \dot{r}(0)=(1,0,1) \end{cases} \Rightarrow$$

$$\Rightarrow x(t)=r(0)+s\cdot\dot{r}(0)=(0,1,0)+s\cdot(1,0,1)=(s,1,s)$$

$$\Leftrightarrow \begin{cases} x=s \\ y=1 \\ z=s \end{cases}, (s\in\mathbb{R}) \Leftrightarrow \begin{cases} x=z \\ y=1 \end{cases}$$

Tangenten är skärningslinjen mellan två plan, nämligen $x-z=0$ och $y=1$.

Problem 3.4 (Sid. 10)

Lösning

En bas för tangentplanet är som bekant

$$\mathcal{B}=(\dot{r}_s(1,0), \dot{r}_t(1,0)).$$

$C: r(t)=(se^t-1, \sin(st), 2s+\arcsin t)$, $s\in\mathbb{R}$, $|t|\leq 1$.

$$\left.\begin{array}{l} \frac{\partial r}{\partial s}=(e^t, t\cdot\cos(st), 2) \Rightarrow \dot{r}_s(1,0)=(1,0,2) \\ \frac{\partial r}{\partial t}=(se^t, s\cos(st), 1/(t^2+1)) \Rightarrow \dot{r}_t(1,0)=(1,1,1) \end{array}\right\} \Rightarrow$$

$\Rightarrow \mathbb{x}(s,t) = \mathbb{r}(1,0) + \lambda \mathbb{r}_s(1,0) + \mu \mathbb{r}_t(1,0) = (0,0,2) +$

$+ \lambda(1,0,2) + \mu(1,1,1) = (\lambda+\mu,\ \mu,\ 2+2\lambda+\mu) = (x,y,z)$

$$\Leftrightarrow \begin{cases} x = \lambda+\mu \\ y = \mu \\ z = 2+2\lambda+\mu \end{cases} \Leftrightarrow \begin{cases} x = \lambda+y \\ y = \mu \\ z = 2+2\lambda+y \end{cases} \Rightarrow z = 2+2(x-y)+y$$

$\Leftrightarrow \underline{2x-y-z+2=0.}$

Anm. Linjens och planets ekvation i olika skepnader studeras i den linjära geometrin (och algebran).

Problem 3.3 (Sid. 10)

Lösning

a) $\gamma: \begin{cases} x = \sin t \\ y = \sin 2t \end{cases} \Rightarrow \gamma(t+2\pi) = \gamma(t)$, dvs γ periodisk med grundperiod 2π och jag studerar $\gamma_{2\pi}$, restriktionen av γ till intervallet $[0, 2\pi]$.

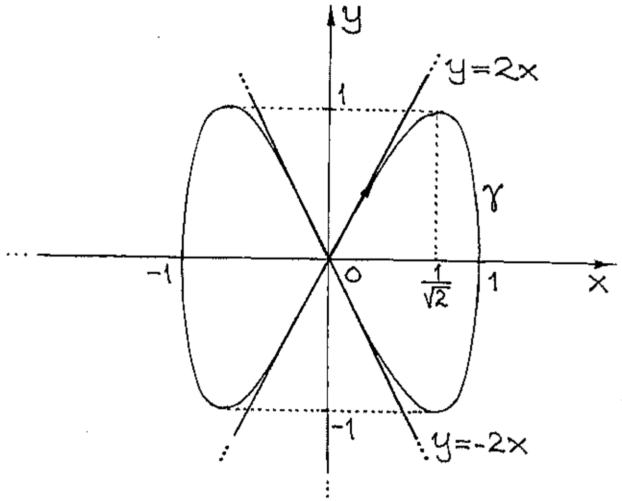
(1) $x = \sin t = 0 \Leftrightarrow t_1 = 0,\ t_2 = \pi,\ t_3 = 2\pi;$

$y = \sin 2t = 0 \Leftrightarrow t_1 = 0,\ t_2 = \frac{\pi}{2},\ t_3 = \pi,\ t_4 = \frac{3\pi}{2},\ t_5 = 2\pi;$

(2) $\dot{x} = \cos t = 0 \Leftrightarrow t_1 = \frac{\pi}{2},\ t_2 = \frac{3\pi}{2};$

$\dot{y} = 2\cos 2t = 0 \Leftrightarrow t_1 = \frac{\pi}{4},\ t_2 = \frac{3\pi}{4},\ t_3 = \frac{5\pi}{4},\ t_4 = \frac{7\pi}{4};$

Jag sammanfattar: Kurvan skär x-axeln i origo samt $(\pm 1, 0)$, enligt (1) ovan, och y-axeln i i origo; tangenten är vertikal i punkterna $(\pm 1, 0)$ och horisontell i $(\pm\frac{1}{\sqrt{2}}, \pm 1)$ (se figur)



Kurvan är en sk. Lissajou /lisa'zu/ figur och kan avbildas (triggas) på oscilloskopsskärmen.

b) $\mathbb{r}(t) = (\sin t, \sin 2t) \Rightarrow t=0$ och $t=\pi$ ger origo.

$\mathbb{r}'(t) = (\cos t, 2\cos 2t) \Rightarrow \mathbb{r}'(0) = (1,2) \wedge \mathbb{r}'(\pi) = (-1,2)$

Tangenterna är $y = \pm 2x$.

c) $f(t) = \sqrt{x^2+y^2} = \sqrt{\sin^2 t + \sin^2 2t} = \sqrt{g(t)}$;

$g'(t) = \sin 2t + 2\sin 4t = \sin 2t(1 + 4\cos 2t) = 0 \Leftrightarrow$

$$\Leftrightarrow \underbrace{\sin 2t = 0}_{\text{förkastas}} \vee 4\cos 2t = -1 \Leftrightarrow 4(2\cos^2 t - 1) = -1 \Leftrightarrow$$

$$\Leftrightarrow \cos^2 t = 3/8 = 1 - \sin^2 t \Leftrightarrow \sin^2 t = 5/8 \Rightarrow x^2 + y^2 = g(t) =$$

$$= \sin^2 t + 4\sin^2 t \cdot \cos^2 t = \frac{5}{8} + 4 \cdot \frac{5}{8} \cdot \frac{3}{8} = \frac{5}{8} \cdot \frac{5}{2} = \left(\frac{5}{4}\right)^2 \Rightarrow$$

$\Rightarrow f_{max} = \frac{5}{4}$ (antas i vilka punkter?).

$x = \sin t = \pm\sqrt{5/8}$, $y = \sin 2t = 2\sin t \cos t = \pm\frac{\sqrt{15}}{4}$, dvs i punkterna $\left(\pm\sqrt{\frac{5}{8}}, \pm\frac{\sqrt{15}}{4}\right)$.

Problem 3.6 (Sid. 10)

Lösning

$$f(t,\varphi) = (x,y) = (3t\cos\psi, 2t\sin t) \Leftrightarrow \begin{cases} x = 3t\cos\psi \\ y = 2t\sin\psi \end{cases} \Rightarrow$$

$$\Rightarrow f'(t,\psi) = \frac{\partial(x,y)}{\partial(t,\psi)} = \begin{bmatrix} 3\cos\psi & -3t\sin\psi \\ 2\sin\psi & 2t\cos\psi \end{bmatrix} \Rightarrow \det f(t,\psi) =$$

$$= \frac{d(x,y)}{d(t,\psi)} = 6t\cos^2\psi + 6t\sin^2\psi = 6t(\cos^2\psi + \sin^2\psi) = 6t.$$

Problem 3.7 (Sid. 11)

Lösning

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 3 & -2 & 1 \\ -1 & -1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Leftrightarrow \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{bmatrix} 1 & 1 & 2 \\ 3 & -2 & 1 \\ -1 & -1 & 5 \end{bmatrix} \Rightarrow \frac{d(u,v,w)}{d(x,y,z)} =$$

$= -10 - 6 - 1 - 4 - 15 + 1 = -35 \neq 0$; avbildningen är 1-1.

Problem 3.8 (Sid. 11)

Lösning

a) $$\begin{cases} u = e^x + y \\ v = 2x + e^y \end{cases} \Rightarrow \frac{d(u,v)}{d(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} e^x & 1 \\ 2 & e^y \end{vmatrix} = e^{x+y} - 2$$

$$\Rightarrow \frac{d(u,v)}{d(x,y)}\bigg|_{(1,0)} = e - 2 \neq 0 \Rightarrow \mathcal{C}^1\text{-invers finns}$$

$$f(x,y) = (e^x + y, 2x + e^y) \Rightarrow f'(x,y) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} e^x & 1 \\ 2 & e^y \end{bmatrix}$$

$$\Rightarrow f'(1,0) = \begin{bmatrix} e & 1 \\ 2 & 1 \end{bmatrix} \Leftrightarrow f^{-1}(e,3) = \begin{bmatrix} e & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{e-2} \begin{bmatrix} 1 & -1 \\ -2 & e \end{bmatrix} =$$

$$= \frac{d(x,y)}{d(u,v)}\bigg|_{(e,3)} \Rightarrow \begin{cases} \frac{\partial x}{\partial u} = \frac{1}{e-2} \\ \frac{\partial x}{\partial v} = -\frac{1}{e-2} \end{cases} \wedge \begin{cases} \frac{\partial y}{\partial u} = -\frac{2}{e-2} \\ \frac{\partial y}{\partial v} = \frac{e}{e-2} \end{cases}$$

b) $$f'(x,y) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} e^x & 1 \\ 2 & e^y \end{bmatrix} \Leftrightarrow f^{-1}(u,v) = \left\{\frac{\partial(x,y)}{\partial(u,v)}\right\} =$$

$$= \left\{\frac{\partial(u,v)}{\partial(x,y)}\right\}^{-1} = \begin{bmatrix} e^x & 1 \\ 2 & e^y \end{bmatrix}^{-1} = \frac{1}{e^{x+y} - 2} \begin{bmatrix} e^y & -1 \\ -2 & e^x \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \frac{\partial x}{\partial u} = \frac{e^y}{e^{x+y} - 2} \wedge \frac{\partial x}{\partial v} = \frac{-1}{e^{x+y} - 2} \wedge \frac{\partial y}{\partial u} = \frac{-2}{e^{x+y} - 2} \wedge \frac{\partial y}{\partial v} =$$

$= \frac{e^x}{e^{x+y} - 2}$ i en punkt $x = x(u,v)$, $y = y(u,v)$.

c) $\begin{cases} e^x+y=u \\ 2x+e^y=v \end{cases} \Rightarrow \begin{cases} \frac{\partial}{\partial u}(e^x+y)=1 \\ \frac{\partial}{\partial u}(2x+e^y)=0 \end{cases} \Leftrightarrow \begin{cases} e^x\frac{\partial x}{\partial u}+\frac{\partial y}{\partial u}=1 \\ 2\frac{\partial x}{\partial u}+e^y\frac{\partial y}{\partial u}=0 \end{cases} \Leftrightarrow$

$$\Leftrightarrow \begin{bmatrix} e^x & 1 \\ 2 & e^y \end{bmatrix}\begin{bmatrix} \partial x/\partial u \\ \partial x/\partial v \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \end{bmatrix} = \frac{1}{e^{x+y}-2}\begin{bmatrix} e^y & -1 \\ -2 & e^x \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\frac{\partial x}{\partial u}=\frac{e^y}{e^{x+y}-2} \wedge \frac{\partial y}{\partial u}=\frac{-2}{e^{x+y}-2}.$$

På samma sätt fås

$$\frac{\partial x}{\partial v}=\frac{-1}{e^{x+y}-2} \wedge \frac{\partial y}{\partial v}=\frac{e^x}{e^{x+y}-2}$$

$$\begin{cases} e^x\frac{\partial x}{\partial u}+\frac{\partial y}{\partial u}=1 \\ 2\frac{\partial x}{\partial u}+e^y\frac{\partial y}{\partial u} \end{cases} \Rightarrow \begin{cases} \frac{\partial}{\partial v}\left(e^x\frac{\partial x}{\partial u}+\frac{\partial y}{\partial u}\right)=0 \\ \frac{\partial}{\partial v}\left(2\frac{\partial x}{\partial u}+e^y\frac{\partial y}{\partial u}\right)=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} e^x\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}+e^x\frac{\partial^2 x}{\partial v\partial u}+\frac{\partial^2 y}{\partial v\partial u}=0 \\ 2\frac{\partial^2 x}{\partial v\partial u}+e^y\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}+e^y\frac{\partial^2 y}{\partial v\partial y}=0 \end{cases} \Rightarrow e^x\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}+$$

$$+e^x\frac{\partial^2 x}{\partial v\partial u}=2\frac{\partial^2 x}{\partial v\partial u}+e^y\frac{\partial y}{\partial u}\frac{\partial y}{\partial v} \Leftrightarrow (e^x-2)\frac{\partial^2 x}{\partial v\partial u}=$$

$$=e^y\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}-e^x\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}=e^y\cdot\frac{-2e^x}{(e^{x+y}-2)^2}-e^x\frac{-e^y}{(e^{x+y}-2)^2}=$$

$$=-\frac{e^{x+y}}{(e^{x+y}-2)^2} \Leftrightarrow \frac{\partial^2 x}{\partial v\partial u}=-\frac{e^{x+y}}{(e^x-2)(e^{x+y}-2)^2} \Rightarrow x''_{uv}(e,3)=$$

$$=-\frac{e^{x+y}}{(e^x-2)(e^{x+y}-2)^2}\bigg|_{(1,0)}=-\frac{e}{(e-2)(e-2)^2}=-\frac{e}{(e-2)^3}.$$

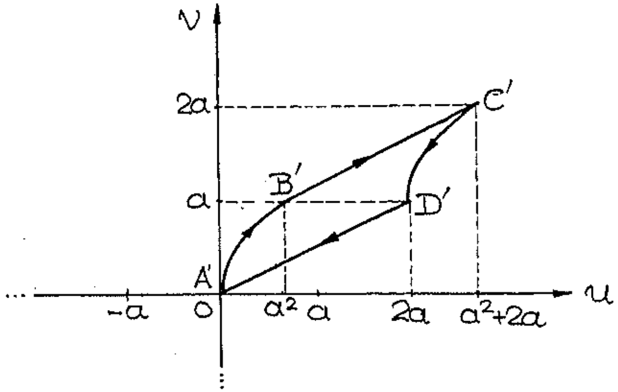
Problem 3.9 (Sid. 11)

Lösning: $f(x,y)=(x^2+2y,\, x+y)=(u,v) \Leftrightarrow \begin{cases} u=x^2+2y \\ v=x+y \end{cases};$

(4) $\overline{DA}=\{(0,y): 0\le y\le a\} \Rightarrow \begin{cases} u=2y \\ v=y \end{cases} \Rightarrow \begin{cases} u=2v \\ 0\le v\le a \end{cases} \Rightarrow$

$\Rightarrow \overline{D'A'}=\{(u,v): u=2v,\ 0\le v\le a\}.$

Kvadraten ABCD i xy-planet avbildas på följande figur i uv-planet:



c) Riktningen på randkurvan har kastats om; determinanten är negativ i origo.

$$|A'B'C'D'|=\int_0^a(2v-v^2)\,dv+\int_a^{2a}\left((v-a)^2+2a-2v+2a-a^2\right)dv$$

$$=\left[v^2-\tfrac{1}{3}v^3\right]_0^a+\left[\tfrac{1}{3}(v-a)^3-v^2+(4a-a^2)v\right]_a^{2a}=$$

$$=a^2-\tfrac{1}{3}a^3+\tfrac{1}{3}a^3-4a^2+8a^2-2a^3+a^2-4a^2+a^3=$$

$=2a^2-a^3\approx 2a^2=2|ABCD|$, försmå a.

Problem 3.10 (Sid. 11)

Lösning

a) $\begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases} \Rightarrow \dfrac{d(u,v)}{d(x,y)} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2+y^2) = 4|\mathbf{x}|^2 > 0;$

b) Avbildningen är inte 1-1, ty punkterna $(\pm 1, 0)$ avbildas på samma punkt $(1,0)$.

c) $u^2+v^2 = (x^2-y^2)^2 + (2xy)^2 = u^2+v^2 \Leftrightarrow x^2+y^2 = \sqrt{u^2+v^2}$

$$\begin{cases} x^2 - y^2 = u \\ x^2+y^2 = \sqrt{u^2+v^2} \end{cases} \Rightarrow 2x^2 = \sqrt{u^2+v^2} + u \overset{!}{\Leftrightarrow} x = \sqrt{\frac{\sqrt{u^2+v^2}+u}{2}}$$

Detta kombineras med $v = 2xy$ och vi får

$$x = \sqrt{\frac{\sqrt{u^2+v^2}+u}{2}} \text{ och } y = \sqrt{\frac{\sqrt{u^2+v^2}-u}{2}}.$$

För $x>0$ tas punkten $(-1,0)$ bort i b) ovan och avbildningen har invers, dvs

$$\frac{d(x,y)}{d(u,v)} = \left\{\frac{d(u,v)}{d(x,y)}\right\}^{-1} = \frac{1}{4\sqrt{x^2+y^2}} = \frac{1}{4\sqrt{u^2+v^2}}.$$

Problem 3.11 (Sid. 11)

Lösning: $\underline{F(x,y) = x^3+y^3+xy-x-y}$

Ekvationen $x^3+y^3+xy = x+y$ är en nivåkurva till $F(x,y)$, $C=0$. (Obs! symmetrin kring $y=x$.)

$$\frac{\partial F}{\partial x} = 3x^2+y-1, \quad \frac{\partial F}{\partial y} = 3y^2+x-1$$

Studera implicita funktionssatsen i läroboken.

a) $F_y'(0,0) = -1 \neq 0 \Rightarrow y = f(x)$ nära $(0,0)$;

$F(0,0) = 0 \Rightarrow f(0) = 0$;

$f'(0) = -\dfrac{F_x'(0,0)}{F_y'(0,0)} = 1.$

b) $F_y'(0,1) = 2 \neq 0 \Rightarrow y = f(x)$ nära $(0,1)$;

$F(0,1) = 0 \Rightarrow f(0) = 1$;

$f'(0) = -\dfrac{F_x'(0,1)}{F_y'(0,1)} = 0.$

c) $F_y'(0,-1) = 2 \neq 0 \Rightarrow y = f(x)$ nära $(0,-1)$;

$F(0,-1) = 0 \Rightarrow f(0) = -1$;

$f'(0) = -\dfrac{F_x'(0,-1)}{F_y'(0,-1)} = 1.$

Problem 3.12 (Sid. 11)

Lösning

$$\underline{F(x,y) = x^3 - 3xy^2 - 1}$$

$$\underline{\frac{\partial F}{\partial x} = 3x^2 - 3y^2}, \quad \underline{\frac{\partial F}{\partial y} = -6xy}; \quad (*)$$

a) $F'_x(1,0)=3\neq 0 \Rightarrow x=x(y)$ nära $(1,0)$; $x'(y)=-\frac{F'_y(x,y)}{F'_x(x,y)}$

$F(1,0)=0 \Rightarrow x(0)=1$;

$x'(0)=-\frac{F'_y(1,0)}{F'_x(1,0)}=0$, enl. (*).

b) $F'_x(x,y)=3(x^2-y^2) \Rightarrow x''(y)=\frac{G(x,y)}{(x^2-y^2)^2}$ och $x\neq \pm y$ i närheten av $(1,0)$, så $x=x(y)$ är C^2 där:

$$x'(y)=\frac{2xy}{x^2-y^2}\Rightarrow x''(y)=\frac{d}{dy}\frac{2xy}{x^2-y^2}=\frac{2x+2yx'(y)}{x^2-y^2}-$$

$$-\frac{2xy}{(x^2-y^2)^2}\cdot(2x\cdot x'(y)-2y)\Rightarrow x''(0)=2. \quad (\text{Obs! } 2\cdot x(0)\cdot 0=0).$$

$x(0)=1$, $x'(0)=0$; $x''(0)>0 \Rightarrow$ minimum föreligger.

c) $F'_x(x,y)=0 \Leftrightarrow 3(x^2-y^2)=0\Leftrightarrow y=x \vee y=-x$;

(1) $F(x,x)=0\Leftrightarrow x^3+1=3x^3 \Leftrightarrow x^3=-\frac{1}{2} \Leftrightarrow x=-1/\sqrt[3]{2}=y$;

(2) $F(x,-x)=0\Leftrightarrow x^3=1+3x^3\Leftrightarrow x^3=-\frac{1}{2} \Leftrightarrow x=-1/\sqrt[3]{2}=-y$;

De punkter som efterfrågas är

$$P_1:\left(-\frac{1}{\sqrt[3]{2}},-\frac{1}{\sqrt[3]{2}}\right),\ P_2:\left(-\frac{1}{\sqrt[3]{2}},\frac{1}{\sqrt[3]{2}}\right).$$

I dessa punkter går det alldeles utmärkt att definiera y som funktion av x; $F'_y(P_1)\neq 0$ och $F'_y(P_2)\neq 0$. I P_1 och P_2 förekommer stationära punkter, ty y' försvinner.

Problem 3.13 (Sid. 11)

Lösning

(1). $F(x,y,z)=xy-(x+y)z^2-\tan 2z+1 = F(1,-1,\pi)$.

$F'_z(x,y,z)=-2(x+y)z-\frac{2}{\cos^2 2z}\Rightarrow F'_z(1,-1,\pi)=-2\neq 0\Rightarrow$

$\Rightarrow$ det går att bestämma $z=z(x,y)$ nära $(1,-1)$.

(2) $F_x(x,y,z)=y-z^2$, $F'_y(x,y,z)=x-z^2$;

$$z'_x(1,-1)=-\frac{F'_x(1,-1,\pi)}{F'_z(1,-1,\pi)}=-\frac{-1-\pi^2}{-2}=-\frac{1+\pi^2}{2};$$

$$z'_y(1,-1)=-\frac{F'_y(1,-1,\pi)}{F_z(1,-1,\pi)}=-\frac{1-\pi^2}{-2}=\frac{1-\pi^2}{2}.$$

Anm. $\frac{1}{\cos^2 2z}=\frac{\cos^2 2z+\sin^2 2z}{\cos^2 2z}=1+\tan^2 2z \Rightarrow$

$\Rightarrow F'_z=-2(x+y)z-2-2\tan^2 2z$ osv.

Problem 3.14 (Sid. 12)

Lösning

a) $y^3+y=e+8$ har exakt en rot, ty $g(y)=y^3+y$ är strängt växande, dvs 1-1.

b) Samma sak gäller för varje fixt x, ty $D_g=\mathbb{R}$.

c) $\frac{d}{dx}(y^3+y)=\frac{d}{dx}(e^x-x+9)\Rightarrow(3y^2+1)y'=e^x-1\Leftrightarrow$

$y'=f'(x)=\frac{e^x-1}{3y^2(x)+1}\Rightarrow f\in C^1$, eftersom $3f(x)^2+1>0$.

d) Se under c) ovan.

e) $HL = e^x - x + 9 = g(x)$ är definierad för alla x, så även $y = f(x)$ är det: $D_f = D_g = \mathbb{R}$.

$\lim_{x\to\infty} g(x) = +\infty \Rightarrow f$ ej uppåt begränsad.

$$g'(x) = e^x - 1 \Rightarrow \begin{cases} x<0 \Rightarrow g'(x)<0 \Rightarrow g \text{ avtagande} \\ x>0 \Rightarrow g'(x)>0 \Rightarrow g \text{ växande} \end{cases} \Rightarrow$$

$$\Rightarrow g(x) \geq g(0) = 10 \Leftrightarrow y^3 + y \geq 10 \Leftrightarrow y = f(x) \geq 2 \Leftrightarrow$$

$$\Leftrightarrow V_f = [2, \infty[.$$

Problem 3.15 (Sid. 12)

Lösning

$t = v - e \cdot \sin v \Leftrightarrow t - v + e \cdot \sin v = 0$ är en nivåyta till funktionen

$$f(t,v) = t - v - e \cdot \sin v.$$

(1) $\frac{\partial f}{\partial v} = -1 - e\cos v < 0 \Rightarrow v = v(t)$, enligt Sats 3 (s. 148).

(2) $t = v - e\sin v \Rightarrow 1 = \frac{d}{dt}(v - e\sin v) = (1 - e\cdot\cos v)\,v'(t)$

$\Leftrightarrow v'(t) = \frac{1}{1 - e\cos v} > 0 \Rightarrow v$ strängt växande.

Anm. $0 \leq e < 1 \Rightarrow 0 \leq e\cos v < \cos v \Rightarrow 1 - e\cos v > 0$.

Problem 3.16 (Sid. 12)

Lösning

$(y^2+z^4)x + x^5 = 1$ är en nivåyta till funktionen

$$f(x,y,z) = (y^2+z^4)x + x^5.$$

$\frac{\partial f}{\partial x} = y^2 + z^4 + 5x^4 > 0 \Rightarrow x = x(y,z)$, enligt implicita funktionssatsen.

$$(y^2+z^4)x + x^5 = 1 \Rightarrow \begin{cases} 2yx + (y^2+z^4)x'_y + 5x^4 x'_y = 0 \\ 4z^3x + (y^2+z^4)x'_z + 5x^4 x'_z = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} (5x^4+y^2+z^4)x'_y = -2yx \\ (5x^4+y^2+z^4)x'_z = -4z^3x \end{cases} \Leftrightarrow \begin{cases} x'_y = -\frac{2yx}{5x^4+y^2+z^4} \\ x'_z = -\frac{4z^3x}{5x^4+y^2+z^4} \end{cases}.$$

Problem 3.17 (Sid. 12)

Lösning

(1) $S_1: x+y+z = 6, \quad S_2: xyz = 6.$

S_1 är nivåytan till $f(x,y,z) = x+y+z$ som går genom $P_0: (1,2,3)$ och S_2 är nivåytan till funktionen $g(x,y,z) = xyz$ genom samma punkt.

$$\frac{\partial(f,g)}{\partial(x,z)} = \begin{bmatrix} 1 & 1 \\ yz & xy \end{bmatrix} \Rightarrow \frac{d(f,g)}{d(x,z)} = y(x-z) \Rightarrow \frac{d(f,g)}{d(x,z)}\Big|_{P_0} = -4 \neq 0$$

$\Rightarrow$ x och z kan i närheten av P_0 framställas som funktioner av y.

(2) $\begin{cases} x+y+z=6 \\ xyz=6 \end{cases} \Rightarrow \begin{cases} \frac{d}{dy}(x+y+z)=0 \\ \frac{d}{dy}(xyz)=0 \end{cases} \Leftrightarrow \begin{cases} \frac{dx}{dy}+1+\frac{dz}{dy}=0 \\ \frac{dx}{dy}yz+xz+xy\frac{dz}{dy}=0 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} x'(2)+1+z'(2)=0 \\ 6x'(2)+1\cdot 3+2z'(2)=0 \end{cases} \Leftrightarrow \begin{cases} x'(2)+z'(2)=-1 \quad -2 \\ 6x'(2)+2z'(2)=-3 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x'(2)+z'(2)=-1 \\ 4x'(2) \qquad =-1 \quad -1 \end{cases} \Leftrightarrow \begin{cases} z'(2)=-1-x'(2) \\ x'(2)=-\frac{1}{4} \end{cases} \Leftrightarrow \begin{cases} x'(2)=-\frac{1}{4} \\ z'(2)=-\frac{3}{4} \end{cases}$.

<u>Anm.</u> $\begin{cases} x'(y)+z'(y)=-1 \\ yzx'(y)+xyz'(y)=-xz \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 1 \\ yz & xy \end{bmatrix}\begin{bmatrix} x' \\ z' \end{bmatrix}=\begin{bmatrix} -1 \\ -xz \end{bmatrix}$

$\Leftrightarrow \begin{bmatrix} x' \\ z' \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ yz & xy \end{bmatrix}^{-1}\begin{bmatrix} -1 \\ -xz \end{bmatrix}=\frac{1}{xy-yz}\begin{bmatrix} xy & -1 \\ -yz & 1 \end{bmatrix}\begin{bmatrix} -1 \\ -xz \end{bmatrix}=$

$=\frac{1}{xy-yz}\begin{bmatrix} xz-xy \\ yz-xz \end{bmatrix} \Leftrightarrow \begin{cases} x'=\frac{xz-xy}{xy-yz}, \\ z'=\frac{yz-xz}{xy-yz} \end{cases}$

<u>Problem 3.18 (Sid. 12)</u>

<u>Lösning</u>

(1) $S_1: x^2+y^2-z^2=2, \quad S_2: x+y=2e^z.$

S_1 är nivåytan till funktionen $f(x,y,z)=x^2+y^2-z^2$ genom $P_0:(1,1,0)$ och S_2 är nivåytan till $g(x,y,z)=x+y-2e^z$ genom samma punkt.

$\frac{d(f,g)}{d(y,z)}=\begin{vmatrix} 2y & -2z \\ 1 & -2e^z \end{vmatrix}=2z-4ye^z \Rightarrow \frac{d(f,g)}{d(y,z)}\Big|_{(1,0)}=-4\neq 0$

$\Rightarrow y=y(x)$ och $z=z(x)$ innanför klotet ifråga.

(2) $\begin{cases} x^2+y^2-z^2=2 \\ x+y-2e^z=0 \end{cases} \Rightarrow \begin{cases} 2x+2yy'-2zz'=0 \\ 1+y'-2e^z z'=0 \end{cases} \Leftrightarrow \begin{cases} yy'-zz'=-x \\ y'-2e^z z'=-1 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{bmatrix} y & -z \\ 1 & -2e^z \end{bmatrix}\begin{bmatrix} y' \\ z' \end{bmatrix}=\begin{bmatrix} -x \\ -1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} y' \\ z' \end{bmatrix}=\begin{bmatrix} y & -z \\ 1 & -2e^z \end{bmatrix}^{-1}\begin{bmatrix} -x \\ -1 \end{bmatrix}=$

$=\frac{1}{2z-4ye^z}\begin{bmatrix} -2e^z & z \\ -1 & y \end{bmatrix}\begin{bmatrix} -x \\ -1 \end{bmatrix}=\frac{1}{2z-4ye^z}\begin{bmatrix} 2xe^z-z \\ x-y \end{bmatrix} \Leftrightarrow$

$\Leftrightarrow \begin{cases} y'(x)=\frac{2xe^z-z}{2z-4ye^z} \\ z'(x)=\frac{x-y}{2z-4ye^z} \end{cases} \Rightarrow \begin{cases} y'(1)=\frac{2\cdot 1e^0-0}{2\cdot 0-4\cdot e^0} \\ z'(1)=\frac{1-1}{2\cdot 0-4e^0} \end{cases} \Leftrightarrow \begin{cases} y'(1)=-\frac{1}{2} \\ z'(1)=0 \end{cases}$

En riktningsvektor för tangenten är $v=(1,-1,0)$.

<u>Anm</u> Jag har satt $x=\frac{1}{2}t$, $y=y(t)$ & $z=z(t)$, så att $u=(\frac{1}{2},-\frac{1}{2},0)=\frac{1}{2}v$, med v som i facit.

<u>Problem 3.19 (Sid. 12)</u>

<u>Lösning</u>: Se nästföljande sida.

$$(1)\quad \begin{cases} \rho = \rho(t, \mathbf{x}(t)) = \rho(t, x(t), y(t), z(t))\,; & \rho \in C^1. \\ \mathbf{v} = \mathbf{v}(t, \mathbf{x}(t)) = \mathbf{v}(t, x(t), y(t), z(t))\,; & \mathbf{v} \in C^1. \end{cases}$$

$$(2)\quad \frac{d\rho}{dt} = \frac{d}{dt}\rho(t,\mathbf{x}) = \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x}\frac{dx}{dt} + \frac{\partial\rho}{\partial y}\frac{dy}{dt} + \frac{\partial\rho}{\partial z}\frac{dz}{dt} =$$

$$= \frac{\partial\rho}{\partial t} + \left(\frac{\partial\rho}{\partial x}, \frac{\partial\rho}{\partial y}, \frac{\partial\rho}{\partial z}\right)\cdot\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) =$$

$$= \frac{\partial\rho}{\partial t} + (\operatorname{grad}\rho)\cdot\frac{d\mathbf{x}}{dt} = \frac{\partial\rho}{\partial t} + (\nabla\rho)\cdot\mathbf{v}\,;$$

$$(3)\quad \frac{d\rho}{dt} + \rho\,\nabla\cdot\mathbf{v} = 0 \overset{(2)}{\Leftrightarrow} \frac{\partial\rho}{\partial t} + (\nabla\rho)\cdot\mathbf{v} + \rho(\nabla\cdot\mathbf{v}) \overset{!}{=} \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\cdot\mathbf{v}) = 0.$$

<u>Anm</u> $\nabla\cdot(\rho\mathbf{v}) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(\rho v_x, \rho v_y, \rho v_z) =$

$$= \frac{\partial}{\partial x}\rho v_x + \frac{\partial}{\partial y}\rho v_y + \frac{\partial}{\partial z}\rho v_z =$$

$$= \frac{\partial\rho}{\partial x}v_x + \frac{\partial\rho}{\partial y}v_y + \frac{\partial\rho}{\partial z}v_z + $$

$$+ \rho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) =$$

$$= \left(\frac{\partial\rho}{\partial x}, \frac{\partial\rho}{\partial y}, \frac{\partial\rho}{\partial z}\right)\cdot(v_x, v_y, v_z) + \rho \operatorname{div}\mathbf{v} =$$

$$= (\nabla\rho)\cdot\mathbf{v} + \rho(\nabla\cdot\mathbf{v}).$$

Ibland skriver man $\operatorname{grad} f(\mathbf{x}) = f'(\mathbf{x}) = \frac{df}{d\mathbf{x}} = \frac{df}{d(x,y,z)} = \nabla_{\mathbf{x}} f.$

4 <u>Optimering</u>

<u>Problem 4.1 (Sid. 12)</u>

<u>Lösning</u>

Sats 4 på sidan 41 och Definition 4 på sidan 15 konsulteras.

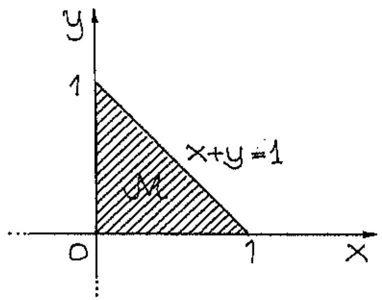
a) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}\,;\ M = \{(x,y) : |x| \leq 1,\ |y| < 1\}.$

M är inte kompakt (en del av randen ingår inte i M), så det är inte säkert att f antar sina extrema i M.

b) $f : \mathbb{R}^3 \longrightarrow \mathbb{R}\,;\ M = \{(x,y,z) :\ x^2+y^2+z^2 \leq 1\}.$

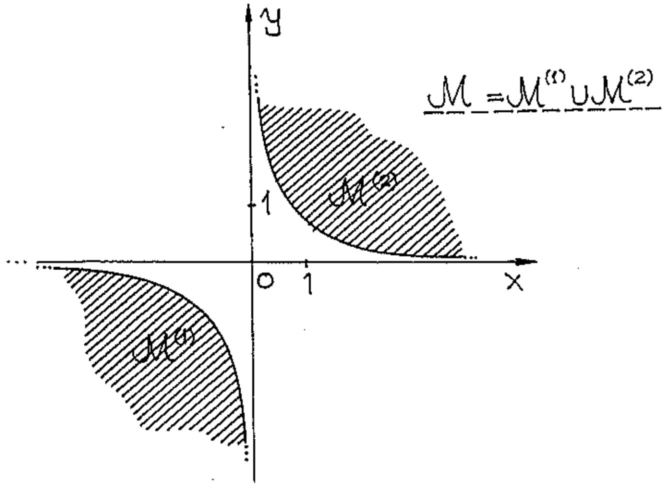
M, enhetsklotet i $\mathbb{R}^3$, är kompakt, så båda extrema antas.

c) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}\,;\ M = \{(x,y) :\ x,y \geq 0,\ x+y \leq 1\}.$



M är kompakt så f:s extrema antas på M.

d) $f:\mathbb{R}^2 \to \mathbb{R}$; $\mathcal{M}=\{(x,y):\ xy\geq 1\}$.



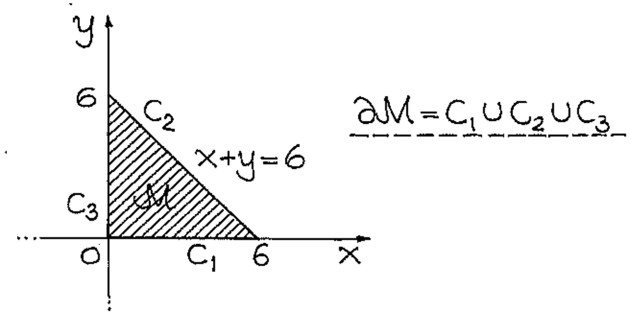
Som synes är $\mathcal{M}$ icke-kompakt så det är inte säkert att f antar sina extrema i $\mathcal{M}$.

Anm. Motexempel ges i facit.

Problem 4.2 (Sid. 12)

Lösning

a) $f(x,y)=xy-x-y$; $\mathcal{M}=\{(x,y):\ x+y\leq 6,\ x,y\geq 0\}$.



(1) $\mathring{\mathcal{M}}=\{(x,y):\ x+y<6,\ x>0,\ y>0\}$.

$\frac{\partial f}{\partial x}=y-1$, $\frac{\partial f}{\partial y}=x-1$;

$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0 \Rightarrow x=1 \wedge y=1 \Rightarrow (x,y)=(1,1)$ stationär.

$f(1,1)=-1$.

(2) $C_1=\{(x,0):\ 0\leq x\, 6\}=[0,6]\times\{0\}$.

$f(x,0)=-x$; $\psi_1(y)=-x,\ 0\leq x\leq 6$, avtagande.

$f(0,0)=0,\ f(6,0)=-6$.

(3) $C_2=\{(x,y):\ y=6-x,\ 0\leq x\leq 6\}$.

$f(x,6-x)=-x^2+6x-6$; $\psi_2(x)=-x^2+6x-6,\ 0\leq x\leq 6$.

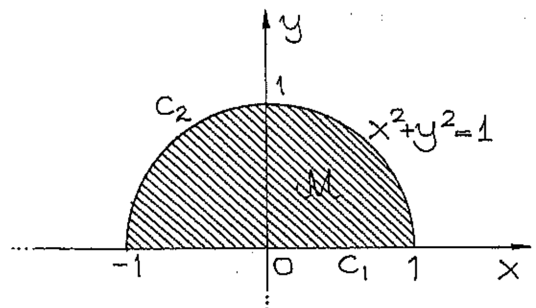
$\psi_2'(x)=-2x+6=0 \Leftrightarrow 2x=6 \Leftrightarrow x=3$;

$f(0,0)=0,\ f(3,3)=3,\ f(6,0)=-6$.

(4) $C_3=\{(0,y):\ 0\leq y\leq 3\}$; Se under (2) ovan!

Resultat: $f_{max}=f(3,3)=3$, $f_{min}=f(6,0)=f(0,6)=-6$.

b) $f(x,y)=x^2+2y^2-x$; $\mathcal{M}=\{(x,y):\ x^2+y^2\leq 1,\ y\geq 0\}$.



$\mathcal{M}$ är kompakt så f_{max} och f_{min} antas på $\mathcal{M}$.

(1) $\underline{\mathring{\mathcal{M}} = \{(x,y) : x^2+y^2<1,\ y>0.}$

$\frac{\partial f}{\partial x} = 2x-1$, $\frac{\partial f}{\partial y} = 4y$;

$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Rightarrow 2x-1=0 \wedge y=0 \Leftrightarrow (x,y) = (\frac{1}{2}, 0) \notin \mathring{\mathcal{M}}$.

Stationära punkter saknas.

(2) $\underline{C_1 = \{(x,0) : -1 \le x \le 1\}.}$

$f(x,0) = x^2 - x$; $\phi_1(x) = x^2 - x$, $-1 \le x \le 1$;

$\phi_1'(x) = 2x-1 = 0 \Leftrightarrow x = \frac{1}{2}$;

$\underline{f(-1,0) = 2,\ f(\frac{1}{2},0) = -\frac{1}{4},\ f(1,0) = 0.}$

(3) $\underline{C_2 = \{(x,y) : y = \sqrt{1-x^2},\ -1 \le x \le 1\}.}$

$f(x, \sqrt{1-x^2}) = 2 - x - x^2$; $\phi_2(x) = 2 - x - x^2$, $-1 \le x \le 1$;

$\phi_2'(x) = -1 - 2x = 0 \Leftrightarrow x = -\frac{1}{2}$;

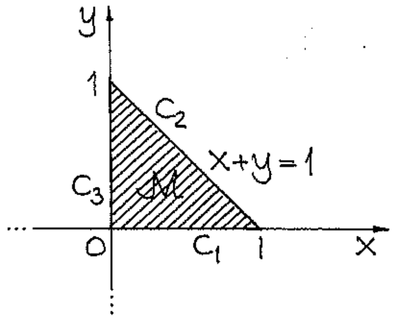
$\underline{f(-1,0) = 2,\ f(-\frac{1}{2},0) = \frac{9}{4},\ f(1,0) = 0.}$

Resultat: $f_{max} = f(-\frac{1}{2}, 0) = \frac{9}{4}$, $f_{min} = f(\frac{1}{2}, 0) = -\frac{1}{4}$.

c) $\mathcal{M}$ är samma mängd som i Problem 4.1 c).

Den finns uppritad på nästföljande sida.

$f(x,y) = x^2 - 2xy + 4y^2 - 2y$; $\mathcal{M} = \{(x,y) : x+y \le 1;\ x,y \ge 0\}$



(1) $\underline{\mathring{\mathcal{M}} = \{(x,y) : x+y<1,\ x>0,\ y>0\}}$

$\frac{\partial f}{\partial x} = 2x - 2y$; $\frac{\partial f}{\partial y} = -2x + 8y - 2$;

$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Rightarrow \begin{cases} 2x - 2y = 0 \\ -2x + 8y - 2 = 0 \end{cases} \Leftrightarrow \begin{cases} x = y \\ 6y = 2 \end{cases} \Leftrightarrow \begin{cases} x = \frac{1}{3} \\ y = \frac{1}{3} \end{cases}$;

$\underline{f(\frac{1}{3}, \frac{1}{3}) = -\frac{1}{3}}$;

(2) $\underline{C_1 = \{(x,0) : 0 \le x \le 1\}.}$

$f(x,0) = x^2$; $\psi_1(x) = x^2$, $0 \le x \le 1$, är växande;

$\underline{f(0,0) = 0,\ f(1,0) = 1.}$

(3) $\underline{C_2 = \{(x,y) : y = 1-x,\ 0 \le x \le 1\}.}$

$f(x, 1-x) = 7x^2 - 8x + 2$; $\psi_2(x) = 7x^2 - 8x + 2$, $0 \le x \le 1$.

$\psi_2'(x) = 14x - 8 = 0 \Leftrightarrow x = \frac{4}{7}$;

$\underline{f(0,1) = 2,\ f(\frac{4}{7}, \frac{3}{7}) = -\frac{2}{7},\ f(1,0) = 1.}$

(4) $\underline{C_3 = \{(0,y) : 0 \le y \le 1\}}$

forts

$f(0,y)=4y^2-2y;\quad \psi_3(y)=4y^2-2y,\ 0\le y\le 1.$

$\psi_3'(y)=8y-2=0 \Leftrightarrow y=\frac{1}{4};$

$\underline{f(0,0)=0,\ f(0,\tfrac{1}{4})=-\tfrac{1}{4},\ f(0,1)=2.}$

<u>Resultat</u>: $f_{max}=f(0,1)=2,\quad f_{min}=f(\frac{1}{3},\frac{1}{3})=-\frac{1}{3}.$

d) $\underline{f(x,y)=x^2+y^2+y;\quad M=\{(x,y):\ x^2+y^2<1\}.}$

$\frac{\partial f}{\partial x}=2x,\ \frac{\partial f}{\partial y}=2y+1;$

$\frac{\partial f}{\partial x}=0=\frac{\partial f}{\partial y} \Leftrightarrow (x,y)=(0,-\frac{1}{2}) \Rightarrow f(0,-\frac{1}{2})=\underline{-\frac{1}{4}}.$

<u>Anm.</u> $f(x,y)=x^2+(y+\frac{1}{2})^2-\frac{1}{4}\geq -\frac{1}{4}.$

$f_{min}=f(0,-\frac{1}{2})=-\frac{1}{4},\ f_{max}$ <u>antas inte i M.</u>

e) $\underline{f(x,y)=(x+2y)e^{-(x^2+y^2)};\quad M=\{(x,y):\ x^2+y^2\le 1\}.}$

(1) $\underline{\mathring{M}=\{(x,y):\ x^2+y^2<1\}.}$

$$\left.\begin{aligned}\frac{\partial f}{\partial x}&=(1-2x(x+2y))e^{-x^2-y^2}=0\\ \frac{\partial f}{\partial y}&=(2-2y(x+2y))e^{-x^2-y^2}=0\end{aligned}\right\} \Leftrightarrow \begin{cases}2x(x+2y)=1\\ 2y(x+2y=2\end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases}y=2x\\ 10x^2=1\end{cases} \Leftrightarrow \begin{cases}x=\pm\frac{1}{\sqrt{10}}\\ y=2x\end{cases} \Leftrightarrow \begin{cases}x=\frac{1}{\sqrt{10}}\\ y=\frac{2}{\sqrt{10}}\end{cases} \vee \begin{cases}x=-\frac{1}{\sqrt{10}}\\ y=-\frac{2}{\sqrt{10}}\end{cases};$$

$\underline{f(\tfrac{1}{\sqrt{10}},\tfrac{2}{\sqrt{10}})=\tfrac{5}{\sqrt{10}}e^{-1/2}};\ \underline{f(-\tfrac{1}{\sqrt{10}},-\tfrac{2}{\sqrt{10}})=-\tfrac{5}{\sqrt{10}}e^{-1/2}}$

(2) $\underline{\partial M=\{(x,y):\ x^2+y^2=1\}}$

$f(\cos t,\sin t)=(\cos t+2\sin t)e^{-1}=\phi(t),\ 0\le t\le 2\pi.$

$\cos t+2\sin t=\sqrt{5}\cos(t-\mathrm{atn}\,2) \Rightarrow -\frac{\sqrt{5}}{e}\le\phi(t)\le\frac{\sqrt{5}}{e} \Leftrightarrow$

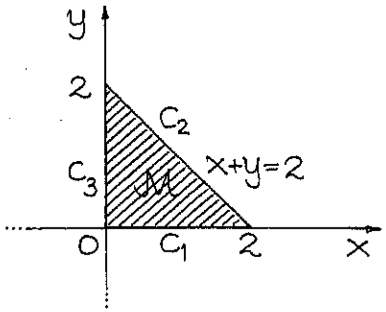
$\Leftrightarrow -\frac{\sqrt{5}}{e}\le f(x,y)\le\frac{\sqrt{5}}{e}.$

(3) $-\frac{5}{\sqrt{10}}e^{-1/2}<-\sqrt{5}e^{-1}<\sqrt{5}e^{-1}<\frac{5}{\sqrt{10}}e^{-1/2}$

<u>Resultat</u>: $f_{max}=f(\frac{1}{\sqrt{10}},\frac{2}{\sqrt{10}})=\frac{5}{\sqrt{10}}e^{-1/2}.$

$f_{min}=f(-\frac{1}{\sqrt{10}}\ -\frac{2}{\sqrt{10}})=-\frac{5}{\sqrt{10}}e^{-1/2}.$

f) $\underline{f(x,y)=(x+y)e^{-x^2-y^2};\quad M=\{(x,y):\ x+y\le 2,\ x,y\ge 0\}.}$



(1) $\underline{\mathring{M}=\{(x,y):\ x+y<2,\ x,y>0\}}$

$\frac{\partial f}{\partial x}=(1-2x(x+y))e^{-x^2-y^2};\ \frac{\partial f}{\partial y}=(1-2y(x+y))e^{-x^2-y^2};$

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0 \Rightarrow \begin{cases}1-2x(x+y)=0\\ 1-2y(x+y)=0\end{cases} \Leftrightarrow \begin{cases}1-4x^2=0\\ y=x\end{cases} \Leftrightarrow \begin{cases}x=\frac{1}{2}\\ y=\frac{1}{2}\end{cases};$$

$(x,y)=(\frac{1}{2},\frac{1}{2})$ är stationär; $\underline{f(\tfrac{1}{2},\tfrac{1}{2})=e^{-1/2}}.$

(2) $\underline{C_1=\{(x,0):\ 0\le x\le 2\}.}$

$f(x,0)=xe^{-x^2}$; $\phi_1(x)=xe^{-x^2}$, $0\le x\le 2$.

$\phi_1'(x)=(1-2x^2)e^{-x^2}=0 \Leftrightarrow 1-2x^2=0 \Leftrightarrow x=1/\sqrt{2}$;

$\phi_1(0)=\underline{f(0,0)=0}$, $\phi_1(\frac{1}{\sqrt{2}})=\underline{f(\frac{1}{\sqrt{2}},0)=\frac{1}{\sqrt{2e}}}$, $\phi_1(2)=\underline{f(2,0)=\frac{2}{e^4}}$.

(3) $\underline{C_2=\{(x,2-x): 0\le x\le 2\}}$.

$f(x,2-x)=2e^{-2x^2+4x-4}=\phi_2(x)$, $0\le x\le 2$.

$\phi_2'(x)=2(-4x+4)e^{-2x^2+4x-4}=0 \Leftrightarrow -4x+4=0 \Leftrightarrow x=1$.

$\phi_2(0)=\underline{f(0,2)=\frac{2}{e^4}}$, $\phi_2(1)=\underline{f(1,1)=\frac{2}{e^2}}$, $\phi_2(2)=\underline{f(2,0)=\frac{2}{e^4}}$.

(4) $\underline{C_3=\{(0,y): 0\le y\le 2\}}$.

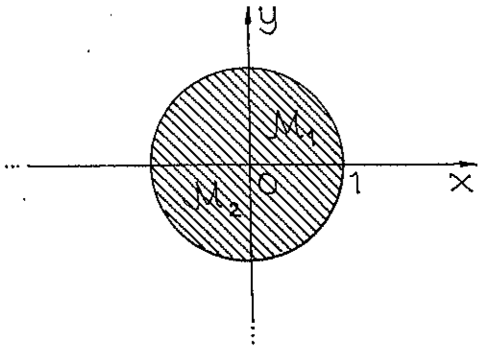
Samma som under (2) härovan: $f(x,y)=f(y,x)$.

(5) Jag sammanfattar: $f(0,0)=0$, $f(\frac{1}{\sqrt{2}},0)=f(0,\frac{1}{\sqrt{2}})=\frac{1}{\sqrt{2e}}$,

$f(2,0)=f(0,2)=\frac{2}{e^4}$, $f(\frac{1}{2},\frac{1}{2})=\frac{1}{\sqrt{e}}$, $f(1,1)=\frac{2}{e^2}$.

Resultat: $f_{max}=f(\frac{1}{2},\frac{1}{2})=e^{-1/2}$, $f_{min}=f(0,0)=0$.

g) $\underline{f(x,y)=x^2+2y^2+|x|;\ M=\{(x,y): x^2+y^2\le 1\}}$.



forts

$M_1=\{(x,y)\in M: x\ge 0\}$, $M_2=\{(x,y)\in M: x\le 0\}$.

(1) $\underline{f(x,y)=x^2+2y^2+x,\ M_1: 0\le x\le\sqrt{1-y^2}, -1\le y\le 1}$.

$\frac{\partial f}{\partial x}=2x+1>0$, så stationära saknas.

$f(\sqrt{1-y^2},y)=1+y^2+\sqrt{1-y^2}=\phi_1(y)$, $-1\le y\le 1$

$\phi_1'(y)=2y-\frac{y}{\sqrt{1-y^2}}=0 \Leftrightarrow y=0 \vee y=\frac{\sqrt{3}}{2}$;

$\phi_1(-1)=f(0,-1)=2$, $\phi_1(0)=f(1,0)=2$, $\phi_1(\frac{\sqrt{2}}{2})=f(\frac{\sqrt{3}}{2},\frac{1}{2})=\frac{9}{4}$,

$\phi_1(1)=f(0,1)=2$.

(2) $\underline{f(x,y)=x^2+2y^2-x,\ M_2: -\sqrt{1-y^2}\le x\le 0, -1\le y\le 1}$.

$\frac{\partial f}{\partial x}=2x-1<0$; stationära punkter saknas.

$f(-\sqrt{1-y^2},y)=\phi_2(y)=\phi_1(y)$. (Se under (1)).

$$f_{max}=f(\frac{\sqrt{3}}{2},\frac{1}{2})=\frac{9}{4}=f(-\frac{\sqrt{3}}{2},-\frac{1}{2}).$$

$$f_{min}=f(0,0)=0.$$

Problem 4.3 (Sid. 13)

Lösning

a) $\underline{f(x,y,z)=x^2-2x+y^2+z^2-4z}$; $\underline{K: x^2+y^2\le z\le 4}$.

(1) $\underline{\mathring{K}: x^2+y^2<z<4}$ (det inre av K).

$\frac{\partial f}{\partial x}=2x-2$, $\frac{\partial f}{\partial y}=2y$, $\frac{\partial f}{\partial z}=2z-4$; kritisk punkt är $P_0:(1,0,2)$; $\underline{f(1,0,2)=-5.}$

(2) $\partial K = S_1 \cup S_2$, där $\underline{S_1: z=x^2+y^2,\ z\le 4}$; $\underline{S_2: x^2+y^2\le 4,\ z=4}$

(3) $\underline{S_1: z=x^2+y^2,\ z\le 4.}$

$\phi(x,y)=f(x,y,x^2+y^2)=x^2-2x+y^2+(x^2+y^2)^2-4(x^2+y^2).$

$$\left.\begin{aligned}\frac{\partial\phi}{\partial x}&=2x-2+4x(x^2+y^2)-8x=4x(x^2+y^2)-6x-2=0\\ \frac{\partial\phi}{\partial y}&=2y+4y(x^2+y^2)-8y=4y(x^2+y^2)-6y=0\end{aligned}\right\}\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}2x(x^2+y^2)=3x+1\\ 2y(x^2+y^2)=3y\end{cases}\Leftrightarrow\begin{cases}2x(x^2+y^2)=3x+1\\ y=0 \vee x^2+y^2=\frac{3}{2}\end{cases}\Leftrightarrow$$

$$\Leftrightarrow\begin{cases}2x^3=3x+1\\ y=0\end{cases}\vee\begin{cases}2x^3+3=3x+1\\ y^2=3/2\end{cases}\Leftrightarrow\begin{cases}x=-1 \vee x=\frac{1\pm\sqrt{3}}{2}\\ y=0\end{cases}$$

$\Leftrightarrow P_1:(-1,0,1)$, $P_2:(\frac{1+\sqrt{3}}{2},0,1+\frac{\sqrt{3}}{2})$, $P_3:(\frac{1-\sqrt{3}}{2},0,1-\frac{\sqrt{3}}{2})$

$\underline{f(P_1)=0}$, $\underline{f(P_2)=-\frac{9+2\sqrt{3}}{4}}$, $\underline{f(P_3)=-\frac{9-2\sqrt{3}}{4}}$.

(4) $\underline{S_2: x^2+y^2\le 4,\ z=4.}$

$\psi(x,y)=f(x,y,4)=x^2+y^2-2x$, $x^2+y^2\le 4$.

$\frac{\partial\psi}{\partial x}=2x-2$, $\frac{\partial\psi}{\partial y}=2y$; $\frac{\partial\psi}{\partial x}=\frac{\partial\psi}{\partial y}=0 \Rightarrow P_4:(1,0,4)$

$\underline{f(P_4)=0.}$

$x^2+y^2=4$, $z=4$: $\xi(x)=f(x,\pm\sqrt{4-x^2},4)=4-2x$, $\underline{|x|\le 2.}$

$P_5:(-2,0,4)$, $P_6:(2,0,4)$; $\underline{f(P_5)=8,\ f(P_6)=0.}$

$\underline{\text{Resultat}}$: $f_{max}=f(-2,0,4)=8$; $f_{min}=f(1,0,2)=-5.$

b) $\underline{f(x,y,z)=3x+xy+z^2}$, $\underline{x^2+y^2+z^2\le 9,\ z\ge 0.}$

$\underline{K: x^2+y^2+z^2\le 9,\ z\ge 0}$, är kompakt.

(1) $\underline{\mathring{K}: x^2+y^2+z^2<9,\ z>0}$, det inre av $\mathring{K}$.

$\frac{\partial f}{\partial x}=3+y$, $\frac{\partial f}{\partial y}=x$, $\frac{\partial f}{\partial z}=2z$; $(0,-3,0)\notin\mathring{K}$, dvs kritiska punkter saknas.

(2) $S=S_1\cup S_2=\partial K$ = randen till K (se nedan).

(3) $\underline{S_1: x^2+y^2+z^2=9,\ x^2+y^2<9.}$

$\phi(x,y)=f(x,y,\sqrt{9-x^2-y^2})=3x+xy-x^2-y^2+9$

$$\left.\begin{aligned}\frac{\partial\phi}{\partial x}&=3+y-2x=0\\ \frac{\partial\phi}{\partial y}&=x-2y=0\end{aligned}\right\}\Rightarrow\begin{cases}x=2y\\ 3y=3\end{cases}\Leftrightarrow\begin{cases}x=2\\ y=1\end{cases}\Rightarrow\begin{cases}x=2\\ y=1\\ z=2\end{cases}\Rightarrow$$

$\Rightarrow P_1:(2,1,2)$; $\underline{f(P_1)=12.}$

(4) $\underline{\mathring{S}_2: x^2+y^2<9,\ z=0}$ (det inre av S_2).

$f(x,y,0)=3x+xy=\psi(x,y) \Rightarrow \frac{\partial\psi}{\partial x}=3+y$, $\frac{\partial\psi}{\partial y}=x$;

$(0,-3)\notin\mathring{S}$ dock.

(5) $\underline{\partial S_2: x^2+y^2=9,\ z=0.}$

$f(x,y,0)=f(3\cos t, 3\sin t, 0)=9\cos t+9\sin t\cos t$;

$g(t)=9(\cos t+\frac{1}{2}\sin 2t)\Rightarrow g'(t)=9(-\sin t+\cos 2t)=$

$= 9(-\sin t + 1 - 2\sin^2 t)$; $g'(t) = 0 \Rightarrow 2\sin^2 t + \sin t = 1$

$\Leftrightarrow \sin t = -1 \vee \sin t = \frac{1}{2} \Leftrightarrow t = \frac{3\pi}{2} \vee t = \frac{\pi}{6} \vee t = \frac{5\pi}{6}$

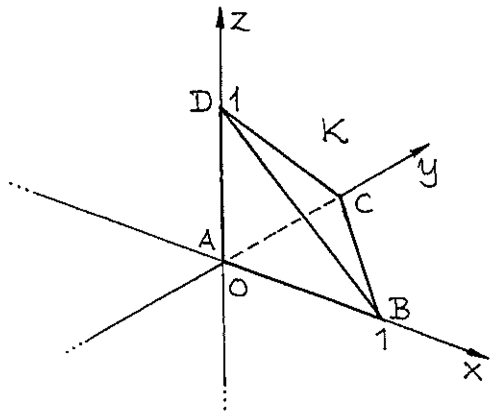
$\Leftrightarrow P_2: (0,-3,0),\ P_3: (\frac{3\sqrt{3}}{2}, \frac{3}{2}, 0),\ P_4: (-\frac{3\sqrt{3}}{2}, \frac{3}{2}, 0) \Rightarrow$

$\Rightarrow \underline{f(P_1) = 0},\ \underline{f(P_3) = \frac{27}{4}\sqrt{3}},\ \underline{f(P_4) = -\frac{27}{4}\sqrt{3}}.$

Resultat: $f_{max} = f(2,1,2) = 12$;

$f_{min} = f(-\frac{3\sqrt{3}}{2}, \frac{3}{2}, 0) = -\frac{27}{4}\sqrt{3}.$

c) $\underline{f(x,y,z) = (1-x)^3 + (1-y)^3 + (1-z)^3}$; $\underline{K: x+y+z \le 1,\ x,y,z \ge 0}$



K är en (solid) tetraeder i den första kvadranten.

(1) $\underline{\mathring{K}: x+y+z<1,\ x,y,z>0}$, det inre av K.

$\frac{\partial f}{\partial x} = -3(1-x)^2 < 0,\ \frac{\partial f}{\partial y} = -3(1-y)^2 < 0,\ \frac{\partial f}{\partial z} = -3(1-z)^2 < 0.$

Kritiska punkter saknas. forts

(2) $\operatorname{grad} f(x,y,z) = -3((1-x)^2, (1-y)^2, (1-z)^2)$ pekar i den riktning f avtar snabbast. Pga symmetrin ligger både f_{max} och f_{min} på strålen $x = y = z \ge 0$; intressanta punkter är $P_1: (0,0,0)$ och $P_2: (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$; båda ligger på randen.

$f(P_1) = 3,\ f(P_2) = 3 \cdot (\frac{2}{3})^3 = \frac{8}{9}.$

Resultat: $f_{max} = f(0,0,0) = 3,\ f_{min} = f(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = \frac{8}{9}.$

d) $\underline{f(x,y,z) = (x+y+z)e^{-xyz}}$; $\underline{D: 0 \le x,y,z \le 1.}$

(1) $\underline{\mathring{D}: 0<x,y,z<1}$; (det inre av D).

$\frac{\partial f}{\partial x} = (1 - yz(x+y+z))e^{-xyz},\ \frac{\partial f}{\partial y} = (1 - xz(x+y+z))e^{-xyz},$

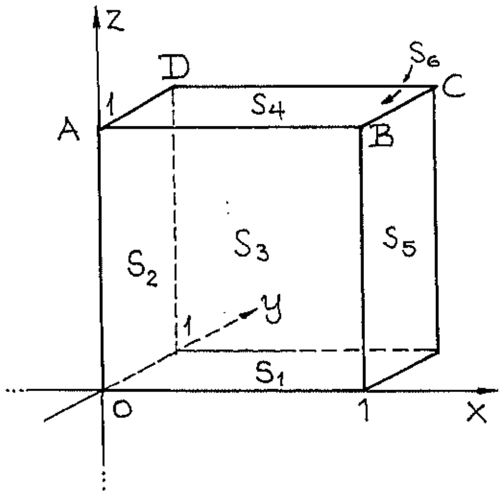
$\frac{\partial f}{\partial z} = (1 - xy(x+y+z))e^{-xyz};$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0 \Rightarrow \begin{cases} 1 - xy(x+y+z) = 0 \\ 1 - xz(x+y+z) = 0 \\ 1 - yz(x+y+z) = 0 \end{cases} \Leftrightarrow x = y = z =$$

$= xyz(x+y+z) \Rightarrow x^2 \cdot 3x = 1 \Rightarrow x = 1/\sqrt[3]{3} = y = z$;

$\underline{f(\frac{1}{\sqrt[3]{3}}, \frac{1}{\sqrt[3]{3}}, \frac{1}{\sqrt[3]{3}}) = 3^{2/3}e^{-1/3} \approx 1{,}49.}$

(2) Randen ∂D består av 6 kvadrater i den första kvadranten, som i figuren.



$\underline{S_1: \ 0 \leq x, y \leq 1, \ z = 0.}$

$$f(x,y;0) = x+y = f_1(x,y) \Rightarrow 0 \leq f_1(x,y) \leq 2 \Rightarrow$$

$$\Rightarrow (f_1)_{min} = f_1(0,0) = 0 \text{ och } (f_1)_{max} = f_1(1,1) = 2.$$

$\underline{S_2: \ x=0, \ 0 \leq y, z \leq 1}$

$$f_2(y,z) = y+z = f(0,y,z) \Rightarrow (f_2)_{min} = f_2(0,0) = 0 \text{ och}$$

$(f_2)_{max} = f_2(1,1) = 2$ (visas med nivåkurvor).

$\underline{S_3: \ 0 \leq x, z \leq 1, \ y = 0}$

$f_3(x,z) = f(x,0,z) = x+z$ ger på samma sätt

$$(f_3)_{min} = f(0,0) = 0 \text{ och } (f_3)_{max} = f_3(1,1) = 2.$$

$\underline{S_4: \ 0 \leq x, y \leq 1, \ z = 1.}$

$$f_4(x,y) = f(x,y,1) = (x+y+1)e^{-xy}, \ \underline{0 \leq x \leq 1, \ 0 \leq y \leq 1}$$

$$\frac{\partial f_4}{\partial x} = (1 - y(x+y+1))e^{-xy}, \ \frac{\partial f_4}{\partial y} = (1 - x(x+y+1))e^{-xy}$$

$$\frac{\partial f_4}{\partial x} = \frac{\partial f_4}{\partial y} = 0 \Rightarrow x = y \Rightarrow 1 = x(2x+1) \Leftrightarrow 2x^2 + x = 1 \Leftrightarrow$$

$$\Leftrightarrow x = \frac{1}{2} = y \Rightarrow \underline{f_4\left(\frac{1}{2}, \frac{1}{2}\right) = 2e^{-1/4}}$$

$\underline{\overline{AB}: \ x = x, \ y = 0, \ z = 1}$ $\quad (0 \leq x \leq 1)$

$$f(x,0,1) = x+1 = \phi(x), \ 0 \leq x \leq 1 \Rightarrow 1 \leq \phi(x) \leq 2.$$

$\underline{\overline{AD}: \ x = 0, \ y = y, \ z = 1}$ $\quad (0 \leq y \leq 1)$

$$f(0,y,1) = y+1 = \psi(y), \ 0 \leq y \leq 1; \quad 1 \leq \psi(y) \leq 2.$$

$\underline{\overline{BC}: \ x = 1, \ y = y, \ z = 1}$ $\quad (0 \leq y \leq 1)$

$$f(1,y,1) = (2+y)e^{-y} = \omega(y), \ 0 \leq y \leq 1.$$

$$\omega'(y) = -(1+y)e^{-y} < 0 \Rightarrow \omega(1) \leq \omega(y) \leq \omega(0) \Rightarrow \frac{3}{e} \leq \omega(y) \leq 2.$$

På samma sätt visas att

$$f(x,y,z) < 2, \text{ för } x+y+z > 1.$$

<u>Resultat</u>: $f_{min} = f(0,0,0) = 0$

$$f_{max} = f(1,1,0) = f(1,0,1) = f(0,1,1) = 2.$$

e) $\underline{f(x,y,z,w) = (x+2y+3z+4w)e^{-x^2-2y^2-3z^2-4w^2};}$

$\underline{M = [0,1]^4 = \{(x,y,z,w): \ 0 \leq x,y,z,w \leq 1\}}$ (enhetskub)

(1) $f(x,y,z,w) \geq 0$ och $\underline{f(0,0,0,0) = 0} \Rightarrow f_{min} = 0.$

(2) Symmetrin kräver att f_{max} antas på "diagonalen" $x=y=z=w=t$, $0\le t\le 1$;

$\phi(t)=f(t,t,t,t)=10te^{-10t^2}$, $0\le t\le 1$, studeras.

$\phi'(t)=10(1-20t^2)e^{-10t^2}=0 \Leftrightarrow t=\frac{1}{\sqrt{20}} \Rightarrow \phi(\frac{1}{\sqrt{20}})=\underline{\sqrt{5}e^{-1/2}}$;

Observera att $\phi(1)=10e^{-10}<\sqrt{5}e^{-1/2}$.

Resultat: $f_{min}=0$, $f_{max}=f(\frac{1}{\sqrt{20}},\frac{1}{\sqrt{20}},\frac{1}{\sqrt{20}},\frac{1}{\sqrt{20}})=\sqrt{\frac{5}{e}}$

Problem 4.4 (Sid. 13)

Lösning

a) $f(x,y)=\arctan(x^2+2y^2)$, $(x,y)\in\mathbb{R}^2$.

$g(x,y)=x^2+2y^2$ är en elliptis paraboloid med toppen i origo och oändlig utsträckning uppåt.
arctan-funktionen är strängt växande, s.a.

$$\text{atn}\,0\le\text{atn}(g(x,y))\le\text{atn}(+\infty)\Leftrightarrow 0\le f(x,y)<\frac{\pi}{2};$$

Resultat: $f_{min}=f(0,0)=0$, f_{max} saknas.

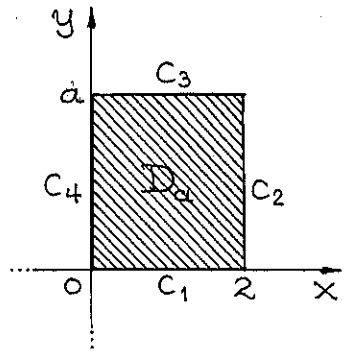
b) $f(x,y)=x^2ye^{-xy}$, $D=\{(x,y): 0\le x\le 2, y\ge 0\}$.

(1) $\forall (x,y)\in\mathring{D}=]0,2[\times\mathbb{R}_+$: $\frac{\partial f}{\partial x}=xy(2-xy)e^{-xy}$ och

$\frac{\partial f}{\partial y}=x^2(1-xy)e^{-xy}$; $\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0\Rightarrow xy=2\wedge xy=1$;

isolerade stationära punkter saknas alltså.

(2) D är inte kompakt så jag "kompaktifierar" den som i figuren nedan; mot slutet låter jag $a\to+\infty$ och ser vad som händer;



(3) På C_1 och C_2 är f identisk lika med 0.

$C_2=\{(2,y): 0\le y\le a\}$.

$f(2,y)=4ye^{-2y}=\phi(y)$, $0\le y\le a$, studeras.

$\phi'(y)=4(1-2y)e^{-2y}=0\Leftrightarrow y=\frac{1}{2}$ ($a>\frac{1}{2}$ antas).

$\phi(0)=f(2,0)=0$, $\phi(\frac{1}{2})=f(2,\frac{1}{2})=2e^{-1}$, $\phi(a)=4ae^{-2a}$;

(4) $C_3=\{(x,a): 0\le x\le 2\}$

$f(x,a)=ax^2e^{-ax}=\psi(x)$, $0\le x\le 2$.

$\psi'(x)=a(2x-ax^2)e^{-ax}=0\Leftrightarrow x=\frac{2}{a}$ ($a>1$ antas)

$\psi(0)=f(0,a)=0$, $\phi(\frac{2}{a})=f(\frac{2}{a},a)=4/e^2a$, $\psi(2)=4ae^{-2a}$;

(5) $\lim_{a\to\infty} 4ae^{-2a} = \lim_{a\to\infty} 4e^{-2}\cdot\frac{1}{a} = 0$ och vi får följande:

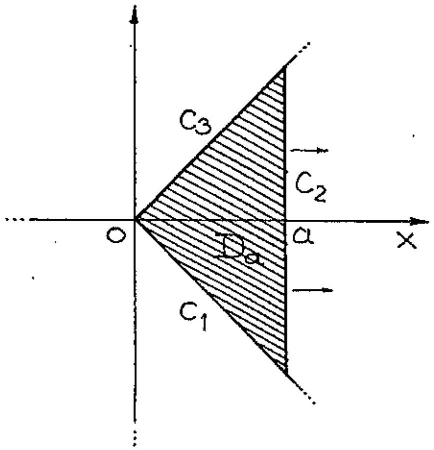
Resultat: $f_{max} = f(2,\frac{1}{2}) = 2/e$; $f_{min} = 0$ antas på randens del på axlarna.

c) $f(x,y) = \frac{e^{y^2-x^2}}{1+y^2}$, $D = \{(x,y) : |y| \leq x\}$.

(1) $|y| \leq x \Leftrightarrow y^2 \leq x^2 \Leftrightarrow y^2 - x^2 \leq 0 \Leftrightarrow e^{y^2-x^2} \leq 1 \Rightarrow$

$\Rightarrow 0 < f(x,y) \leq 1$.

(2) D ligger i det högra halvplanet (ty $x \geq 0$) och är obegränsad, dvs icke-kompakt. Jag stänger den (kompaktifierar den) som i figuren nedan.



C_2 är liksom rörlig så D_a växer med a.

(1) $\mathring{D}_a : |y| < x < a$

$\frac{\partial f}{\partial x} = -2x f(x,y)$, $\frac{\partial f}{\partial y} = 2y\cdot f(x,y) - \frac{2y}{1+y^2} f(x,y)$;

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Rightarrow \begin{cases} -2x = 0 \\ 2y(1 - \frac{1}{1+y^2}) = 0 \end{cases} \text{(saknar lösning)}$$

Inga stationära punkter således.

(2) $\partial D_a = C_1 \cup C_2 \cup C_3$ = randen av D_a.

C_1: $y = -x$, $0 \leq x \leq a$.

$f(x,-x) = \phi(x) = \frac{1}{1+x^2}$, $0 \leq x \leq a$; avtagande.

$\phi(0) = f(0,0) = 1$; $\phi(a) = f(a,-a) = \frac{1}{1+a^2} \xrightarrow[a\to\infty]{} 0$.

C_2: $x = a$, $-a \leq y \leq a$.

$\psi(y) = f(a,y) = \frac{e^{y^2-a^2}}{1+y^2}$, $-a \leq y \leq a$.

$\psi'(y) = 2y\,\psi(y) - \frac{2y}{1+y^2}\psi(y) = 0 \Leftrightarrow y = 0$;

$\psi(-a) = f(a,-a) = \frac{1}{1+a^2} \xrightarrow[a\to\infty]{} 0$, $\psi(0) = f(a,0) = e^{-a^2} \xrightarrow[a\to\infty]{} 0$

$\psi(a) = f(a,a) = \frac{1}{1+a^2} \xrightarrow[a\to\infty]{} 0$.

C_3: $y = x$, $0 \leq x \leq a$.

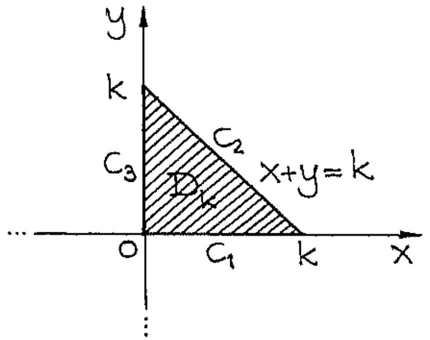
$\omega(x) = f(x,x) = \frac{1}{1+x^2}$, $0 \leq x \leq a$; samma som C_1.

Resultat: $f_{max} = f(0,0) = 1$

f_{min} saknas (antas inte).

d) $$\underline{f(x,y)=\frac{xy}{4+(x+y)^3},\ x\geq 0,\ y\geq 0}$$

D_k: $x+y\leq k$, $x\geq 0, y\geq 0$, är en uttömmande svit till första kvadranten $(k=1,2,3,\dots)$.



(1) $\underline{\mathring{D}_k:\ x+y<k,\ x,y>0.}$

$$\frac{\partial f}{\partial x}=\frac{y}{4+(x+y)^3}-\frac{3xy(x+y)^2}{(4+(x+y)^3)^2}=\frac{y(4+(x+y)^3)-3xy(x+y)^2}{(4+(x+y)^3)^2};$$

$$\frac{\partial f}{\partial y}=\frac{x}{4+(x+y)^2}-\frac{3xy(x+y)^2}{(4+(x+y)^3)^2}=\frac{x(4+(x+y)^3)-3xy(x+y)^2}{(4+(x+y)^3)^2};$$

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0\Rightarrow x=y\Rightarrow x(4+8x^3)-3x^2\cdot 4x^2=0\Leftrightarrow$$

$$\Leftrightarrow x(4+8x^3-12x^3)=4x(1-x^3)=0\Leftrightarrow x=1=y\quad (k>1).$$

$P_0:(1,1)$ är $\underline{\text{stationär}}$; $\underline{f(1,1)=\frac{1}{12}}$.

(2) $\underline{\partial D_k=C_1\cup C_2\cup C_3.}$

$\underline{C_1:\ y=0,\ 0\leq x\leq k.}$

$f(x,0)\equiv 0$;

$\underline{C_2:\ y=k-x,\ 0\leq x\leq k.}$

$$f(x,k-x)=\frac{x(k-x)}{4+k^3}=\phi(x),\ 0\leq x\leq k.$$

$$\phi'(x)=\frac{k-2x}{4+k^3}=0\Rightarrow k-2x=0\Leftrightarrow x=\frac{k}{2};\ f(\tfrac{k}{2},\tfrac{k}{2})=$$

$$=\frac{k^2}{4(4+k^3)}\xrightarrow[k\to\infty]{}0.$$

$\underline{\text{Resultat}}$: $f_{max}=f(1,1)=\frac{1}{12}$, $f_{min}=0$.

e) $$\underline{f(x,y,z)=(x-y+z)e^{-x^2-y^2-z^2},\ D=\mathbb{R}^2.}$$

$$\frac{\partial f}{\partial x}=(1-2x(x-y+z))e^{-|\mathbf{x}|^2},\ \frac{\partial f}{\partial y}=(-1-2y(x-y+z))e^{-|\mathbf{x}|^2},$$

$$\frac{\partial f}{\partial z}=(1-2z(x-y+z))e^{-|\mathbf{x}|^2}.$$

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=\frac{\partial f}{\partial z}=0\Rightarrow\begin{cases}2x(x-y+z)=1\\2y(x-y+z)=-1\\2z(x-y+z)=1\end{cases}\Leftrightarrow x=z=-y\Rightarrow$$

$$\Rightarrow -2y(-3y)=1\Leftrightarrow y^2=1/6\Leftrightarrow y=\pm 1/\sqrt{6}.$$

Stationära är $P_1:(\frac{1}{\sqrt6},-\frac{1}{\sqrt6},\frac{1}{\sqrt6})$ & $P_2:(-\frac{1}{\sqrt6},\frac{1}{\sqrt6},-\frac{1}{\sqrt6})$.

$$f(P_1)=\frac{3}{\sqrt6}e^{-1/2}=3/\sqrt{6e}\quad,\ f(P_2)=-\frac{3}{\sqrt6}e^{-1/2}=-3/\sqrt{6e};$$

$$|f(\mathbf{x})|=|x-y+z|e^{-|\mathbf{x}|^2}\leq(|x|+|y|+|z|)e^{-|\mathbf{x}|^2}\leq 3|\mathbf{x}|e^{-|\mathbf{x}|^2}\xrightarrow[|\mathbf{x}|\to\infty]{}0.$$

$\underline{\text{Resultat}}$: $f_{max}=f(\frac{1}{\sqrt6},-\frac{1}{\sqrt6},\frac{1}{\sqrt6})=\frac{3}{\sqrt{6e}}$;

$f_{min}=f(-\frac{1}{\sqrt6},\frac{1}{\sqrt6},-\frac{1}{\sqrt6})=-\frac{3}{\sqrt{6e}}$.

f) $$\underline{f(x,y)=(x+y)e^{-x^2-y^2},\ D:\ x\geq 0,\ y\geq 0}$$

(1) $\underline{\mathring{D}:\ x>0,\ y>0.}$

forts

$$\frac{\partial f}{\partial x}=(1-2x(x+y))e^{-x^2-y^2},\quad \frac{\partial f}{\partial y}=(1-2y(x+y))e^{-x^2-y^2};$$

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0 \Rightarrow \begin{cases} 2y(x+y)=1 \\ 2x(x+y)=1 \end{cases} \Leftrightarrow \begin{cases} y=x \\ 4x^2=1 \end{cases} \Leftrightarrow \begin{cases} x=1/2 \\ y=1/2 \end{cases} \Rightarrow$$

$$\Rightarrow f(\tfrac{1}{2},\tfrac{1}{2})=\underline{e^{-1/2}}$$

(2) $\underline{\partial D=\{(x,0): x\geq 0\}\cup\{(0,y): y\geq 0\}=C_1\cup C_2}.$

På C_1: $y=0, x\geq 0$ har vi (C_2 är "liknande")

$f(x,0)=xe^{-x^2}=\phi(x),\ x\geq 0;$

$\phi'(x)=(1-2x^2)e^{-x^2}=0 \Leftrightarrow x=1/\sqrt{2}$;

$\phi(0)=f(0,0)=0,\ \phi(\frac{1}{\sqrt{2}})=f(\frac{1}{\sqrt{2}},0)=\frac{1}{\sqrt{2}}e^{-1/2}=\frac{1}{\sqrt{2e}}<\frac{1}{\sqrt{e}}.$

(3) $|f(x,y)|=|x+y|e^{-|\mathbf{x}|^2}\leq(|x|+|y|)e^{-|\mathbf{x}|^2}\leq 2|\mathbf{x}|e^{-|\mathbf{x}|^2}\xrightarrow[|\mathbf{x}|\to\infty]{} 0.$

Resultat: $f_{max}=f(\frac{1}{2},\frac{1}{2})=1/\sqrt{e}.$

$f_{min}=f(0,0)=0.$

Problem 4.5 (Sid. 13)

Lösning

$\underline{f(x,y)=x^2+xy+y^2-3y,\quad D: x^2+y^2\leq 9.}$

(1) $\underline{\mathring{D}: x^2+y^2<9.}$

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0 \Rightarrow \begin{cases} 2x+y=0 \quad -2 \\ x+2y=3 \end{cases} \Leftrightarrow \begin{cases} y=-2x \\ -3x=3 \end{cases} \Leftrightarrow \begin{cases} x=-1 \\ y=2 \end{cases} \Rightarrow$$

$\underline{f(-1,2)=-3.}$

(2) $\underline{\partial D: x^2+y^2=9} \Leftrightarrow (x,y)=(3\cos t, 3\sin t),\ 0\leq t\leq 2\pi.$

$f(3\cos t, 3\sin t)=9(1+\frac{1}{2}\sin 2t-\sin t)=\phi(t),\ 0\leq t\leq 2\pi;$

$\phi'(t)=9(-\cos t+\cos 2t)=0 \Leftrightarrow \cos 2t-\cos t=0 \Leftrightarrow$

$\Leftrightarrow 2\cos^2 t-\cos t-1=0 \Leftrightarrow \cos t=1 \vee \cos t=-\frac{1}{2} \Leftrightarrow \sin t=$

$=0 \vee \sin t=\pm\frac{\sqrt{3}}{2} \Rightarrow P_1:(3,0),\ P_2:(-\frac{3}{2},\frac{3\sqrt{3}}{2})$ och

$P_3:(-\frac{3}{2},-\frac{3\sqrt{3}}{2})$; $\underline{f(3,0)=9}$, $\underline{f(-\frac{3}{2},\frac{3\sqrt{3}}{2})=\frac{9}{4}(4-3\sqrt{3})}$ och

$\underline{f(-\frac{3}{2},-\frac{3\sqrt{3}}{2})=\frac{9}{4}(4+3\sqrt{3})}.$

Resultat: $f_{max}=f(-\frac{3}{2},-\frac{3\sqrt{3}}{2})=\frac{9}{4}(4+3\sqrt{3}).$

$f_{min}=f(-1,2)=-3.$

Problem 4.6 (Sid. 13)

Lösning

(1) $f(x,y)=x^2+y^2$ (=kvadraten på avståndet).

(2) $g(x,y)=x^4+2y^4-6=0.$

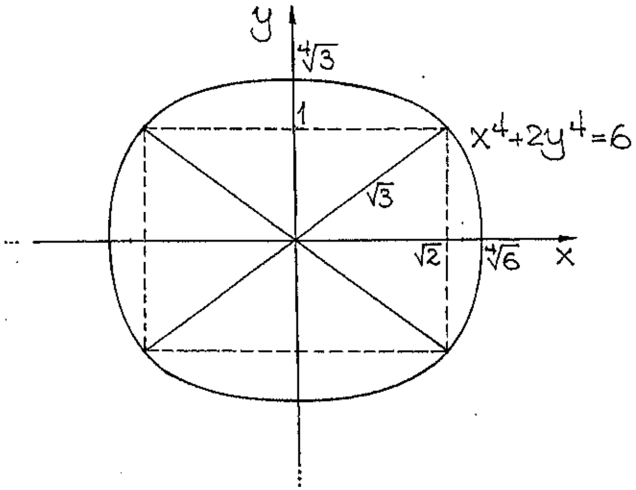
$\operatorname{grad} f(x,y)=(2x,2y)=\lambda(4x^3,8y^3)=\lambda\cdot\operatorname{grad} g(x,y)\Rightarrow$

$\Rightarrow \frac{4x^3}{2x}=\frac{8y^3}{2y} \Leftrightarrow x^2=2y^2 \overset{(2)}{\Leftrightarrow} 6y^4=6 \Leftrightarrow y=\pm 1 \Rightarrow x=\pm\sqrt{2}$

(3) $d^2=f(\pm\sqrt{2},\pm 1)=2+1=3 \Leftrightarrow d=\sqrt{3}=d_{max}.$

(4) Kurvan $x^4+2y^4=6 \Leftrightarrow (\frac{x}{\sqrt[4]{6}})^4+(\frac{y}{\sqrt[4]{3}})^4=1$ är en

hyperellips; den är spegelsymmetrisk m.a.p. koordinataxlarna. Det minimala avståndet är $\sqrt[4]{3}$, vilket syns i figuren nedan.



Problem 4.7 (Sid. 13)

Lösning

$$f(x,y)=(x+y)e^{-(x^2/7)-y^2},\quad D: x^2+7y^2\le 7.$$

(1) $\mathring{D}: x^2+7y^2<7$

$\frac{\partial f}{\partial x}=(1-\frac{2}{7}x(x+y))e^{-(x^2/7)-y^2}$, $\frac{\partial f}{\partial y}=(1-2y(x+y))e^{-(x^2/7)-y^2}$;

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0 \Rightarrow \begin{cases} 2x(x+y)=7 \\ 2y(x+y)=1 \end{cases} \Leftrightarrow \begin{cases} x=7y \\ 16y^2=1 \end{cases} \Leftrightarrow \begin{cases} x=\pm 7/4 \\ y=\pm 1/4 \end{cases} \Rightarrow$$

$P_1:(\frac{7}{4},\frac{1}{4})$, $P_2:(\frac{7}{4},-\frac{1}{4})$, $P_3:(-\frac{7}{4},\frac{1}{4})$, $P_4:(-\frac{7}{4},-\frac{1}{4})$ stationära.

$f(P_1)=2e^{-1/2}$, $f(P_2)=\frac{3}{2}e^{-1/2}$, $f(P_3)=-\frac{3}{2}e^{-1/2}$, $f(P_4)=-2e^{-1/2}$.

(2) $\partial D: \frac{x^2}{7}+y^2=1 \Leftrightarrow (x,y)=(\sqrt{7}\cos t, \sin t)$, $0\le t\le 2\pi$.

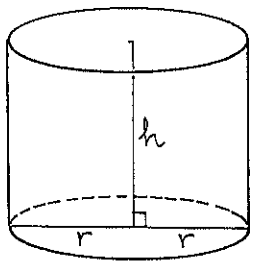
$f(\sqrt{7}\cos t, \sin t)=(\sqrt{7}\cos t+\sin t)e^{-1}=\phi(t)$, $0\le t\le 2\pi$.

$\phi(t)=\sqrt{8}e^{-1}\cos(t-\mathrm{atn}\frac{1}{\sqrt{7}}) \Rightarrow -\frac{\sqrt{8}}{e}\le\phi(t)\le\frac{\sqrt{8}}{e}<2e^{-1}$

Resultat: $f_{max}=f(\frac{7}{4},\frac{1}{4})=2e^{-1/2}$ $f_{min}=f(-\frac{7}{4},-\frac{1}{4})=-2e^{-1/2}$.

Övning 4.8 (Sid. 13)

Lösning



(le)

S = den totala arean; V_0, volymen, är given.

(1) $\pi r^2h=\pi(\frac{d}{2})^2h=\frac{1}{4}\pi d^2h=V_0 \Rightarrow g(d,h)=\frac{\pi}{4}d^2h-V_0=0$.

(2) $S=2\pi rh+2\pi r^2=\pi dh+\frac{\pi}{2}d^2=f(d,h)$.

(3) $\mathrm{grad} f(d,h) \parallel \mathrm{grad} g(d,h) \Rightarrow (\pi h+\pi d, \pi d)=\lambda\cdot(\frac{\pi dh}{2},\frac{\pi d^2}{4})$

$$\Rightarrow \frac{\pi(d+h)}{\pi dh/2}=\frac{\pi d}{\pi d^2/4} \Leftrightarrow \frac{2(d+h)}{dh}=\frac{4}{d} \Leftrightarrow 1+\frac{h}{d}=2 \Leftrightarrow \frac{h}{d}=1.$$

Anm. $V_0 = \frac{\pi}{4} d^2 h \Leftrightarrow h = \frac{4V_0}{\pi d^2}$

$S = \pi d h + \frac{\pi}{2} d^2 = \pi d \cdot \frac{4V_0}{\pi d^2} + \pi d^2 = \frac{4V_0}{d} + \pi d^2$

$S' = -\frac{4V_0}{d^2} + 2\pi d$, $S'' = \frac{8V_0}{d^3} + 2\pi > 0 \Rightarrow$ minimum föreligger.

Problem 4.9 (Sid. 13)

Lösning

$(x-y)^2 - x - y + 1 = 0$ är en nivåkurva till

$$f(x,y) = (x-y)^2 - x - y + 1.$$

I vilka punkter är tangenten parallell med x-axeln? I de punkter där $y' = -\frac{\partial f/\partial x}{\partial f/\partial y} = 0$!

$\frac{\partial f}{\partial x} = 0 \Rightarrow 2(x-y) - 1 = 0 \Leftrightarrow x - y = \frac{1}{2} \Leftrightarrow \underline{x = y + 1/2}$;

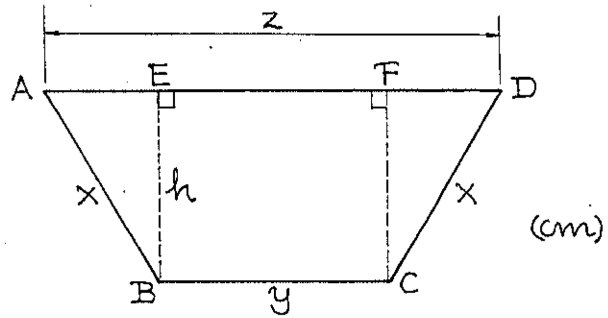
Detta kombineras med kurvans ekvation, vilket leder till: $\frac{1}{4} - y - \frac{1}{2} - y + 1 = 0 \Leftrightarrow y = \frac{3}{8} \Rightarrow x = \frac{7}{8}$.

Svar: $(\frac{7}{8}, \frac{3}{8})$.

Anm. Lagranges multiplikatormetod används ofta i tillämpningarna; teorin finns på sidorna 174-177 i läroboken.

Problem 4.10 (Sid. 13)

Lösning



(1) $AE = FD = \frac{z-y}{2} \Rightarrow h = BE = CF = \sqrt{x^2 - (z-y)^2/4}$.

(2) Arean $= \frac{AD+BC}{2} \cdot BE = \underline{\frac{1}{4}(y+z)\sqrt{4x^2-(z-y)^2} = f(x,y,z)}$.

(3) $2x + y = 60 \Leftrightarrow \underline{g(x,y,z) = 2x + y - 60 = 0}$.

(4) Låt oss sätta $k(\mathbf{x}) = \sqrt{4x^2 - (z-y)^2}$.

$\operatorname{grad} f(\mathbf{x}) = \left(\frac{x(y+z)}{k(\mathbf{x})}, \frac{z^2-y^2}{4k(\mathbf{x})} + \frac{1}{4}k(\mathbf{x}), \frac{y^2-z^2}{4k(\mathbf{x})} + \frac{1}{4}k(\mathbf{x})\right)$;

$\operatorname{grad} g(\mathbf{x}) = (2,1,0)$;

$\operatorname{grad} f(\mathbf{x}) \parallel \operatorname{grad} g(\mathbf{x}) \Leftrightarrow \operatorname{grad} f(\mathbf{x}) = \lambda \cdot \operatorname{grad} g(\mathbf{x}) \Leftrightarrow$

$$\Leftrightarrow \begin{cases} \frac{x(y+z)}{k(\mathbf{x})} = 2\lambda \\ \frac{z^2-y^2}{4k(\mathbf{x})} + \frac{1}{4}k(\mathbf{x}) = \lambda \\ \frac{y^2-z^2}{4k(\mathbf{x})} + \frac{1}{4}k(\mathbf{x}) = 0 \end{cases} \Leftrightarrow \begin{cases} 2\lambda = \frac{x(y+z)}{k(\mathbf{x})} \\ 2\lambda = \frac{z^2-y^2}{k(\mathbf{x})} \quad \text{(2:a–3:e ekv.)} \\ \frac{y^2-z^2}{4k(\mathbf{x})} + \frac{1}{4}k(\mathbf{x}) = 0 \end{cases}$$

$$\Rightarrow \frac{x(y+z)}{k(\mathbf{x})} = \frac{z^2-y^2}{k(\mathbf{x})} \Leftrightarrow x(z+y) - (z+y)(z-y) = 0 \Leftrightarrow$$

$\Leftrightarrow (z+y)(x-(z-y))=0 \Leftrightarrow x=z-y \Leftrightarrow \underline{z=x+y.} \Rightarrow$

$\Rightarrow h=\frac{\sqrt{3}}{2}x$ (ΔABE är en halv liksidig) $\Rightarrow AE=\frac{x}{2}$.

(5). $g(\mathbf{x})=0 \Leftrightarrow 2x+y=60 \Leftrightarrow \underline{y=60-2x} \Rightarrow \underline{z=60-x}$;

$f(x,y,z)=f(x,60-2x,60-x)=\cdots=\frac{3\sqrt{3}}{4}(40x-x^2)=\phi(x)$;

$\phi'(x)=\frac{3\sqrt{3}}{2}(20-x)=0 \Leftrightarrow x=y=20 \Rightarrow \phi(20)=300\sqrt{3}$.

Resultat: Arean kan bli maximalt $300\sqrt{3}\,cm^2$.

Problem 4.11 (Sid. 13)

Lösning

$$\underline{f(x,y)=x^2(1+y)^3+y^2.}$$

(1) $\frac{\partial f}{\partial x}=2x(1+y)^3$, $\frac{\partial f}{\partial y}=3x^2(1+y)^2+2y$;

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0 \Rightarrow \begin{cases} x(1+y)^3=0 \\ 3x^2(1+y)^2+2y=0 \end{cases} \Leftrightarrow \begin{cases} x=0 \\ y=0 \end{cases} \Rightarrow \underline{\mathbf{x}=(0,0)}$$

är en stationär punkt.

(2) $\frac{\partial^2 f}{\partial x^2}=2(1+y)^3$, $\frac{\partial^2 f}{\partial x\partial y}=6x(1+y)^2$, $\frac{\partial^2 f}{\partial y^2}=6x^2(1+y)+2$;

$f''_{xx}(0,0)=2=f''_{yy}(0,0)$, $f''_{xy}(0,0)=0$;

$Q(h,k)=2h^2+2k^2$, positiv definit $\Rightarrow (0,0)$ ger ett lokalt minimum.

(3) $f(2,y)=4(1+y)^3+y^2 \xrightarrow{y\to-\infty} -\infty \Rightarrow$ minimum saknas.

Problem 4.12 (Sid. 13)

Lösning

$$\underline{f(x,y,z)=xy+xz,\quad g(x,y,z)=x^2+y^2+z^2-1=0.}$$

(1) $\operatorname{grad} f(\mathbf{x})=(y+z,x,x)$, $\operatorname{grad} g(\mathbf{x})=(2x,2y,2z)$;

$\operatorname{grad} f(\mathbf{x}) \parallel \operatorname{grad} g(\mathbf{x}) \Leftrightarrow (y+z,x,x)=k(x,y,z) \Leftrightarrow$

$$\Leftrightarrow \begin{cases} y+z=kx \\ x=ky \\ x=kz \end{cases} \Leftrightarrow \begin{cases} 2y=kx \\ x=ky \\ y=z \end{cases} \Leftrightarrow \begin{cases} x^2=2y^2 \\ z=y \end{cases} \Leftrightarrow \begin{cases} x=\pm\sqrt{2}y \\ z=y \end{cases};$$

(2) $g(\pm\sqrt{2}y,y,y)=4y^2-1=0 \Leftrightarrow y=\pm\frac{1}{2}=z$, vilket ger

$P_1:(\frac{\sqrt{2}}{2},\frac{1}{2},\frac{1}{2})$, $P_2:(-\frac{\sqrt{2}}{2},\frac{1}{2},\frac{1}{2})$, $P_3:(-\frac{\sqrt{2}}{2},-\frac{1}{2},-\frac{1}{2})$, $P_4:(\frac{\sqrt{2}}{2},-\frac{1}{2},-\frac{1}{2})$;

$f(P_1)=f(P_3)=\frac{\sqrt{2}}{2}=f_{max}$; $f(P_2)=f(P_4)=-\frac{\sqrt{2}}{2}=f_{min}$.

Problem 4.13 (Sid. 13)

Lösning

a) $\underline{f(x,y,z)=2z^2+y^2-x^2-x}$; $\underline{K: x^2+y^2+z^2\le 1.}$

$\underline{\mathring{K}: x^2+y^2+z^2<1}$ (det inre av K).

$\frac{\partial f}{\partial x}=-2x-1$, $\frac{\partial f}{\partial y}=2y$, $\frac{\partial f}{\partial z}=4z$; $P_0:(-\frac{1}{2},0,0)$ kritisk.

$\underline{f(-\frac{1}{2},0,0)=\frac{1}{4}.}$

$\underline{\partial K: x^2+y^2+z^2=1.}$

forts

$z^2 = 1-x^2-y^2 \Leftrightarrow z = \pm\sqrt{1-x^2-y^2}$;

$\phi(x,y) = f(x,y,\pm\sqrt{1-x^2-y^2}) = 2(1-x^2-y^2)+y^2-x^2-x =$

$= 2-3x^2-y^2-x$, $\quad D: x^2+y^2 \le 1$.

$\mathring{D}: x^2+y^2<1$ (det inre av D).

$$\left.\begin{array}{l} \frac{\partial\phi}{\partial x} = -6x-1 = 0 \\ \frac{\partial\phi}{\partial y} = -2y = 0 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} x = -\frac{1}{6} \\ y = 0 \end{array}\right. \Rightarrow P_1: \left(-\frac{1}{6}, 0, -\frac{\sqrt{35}}{6}\right) \text{ och}$$

$P_2: \left(-\frac{1}{6}, 0, \frac{\sqrt{35}}{6}\right)$; $\quad f(P_1) = \frac{75}{36} = f(P_2)$.

$\partial D: x^2+y^2=1$ (randen av D).

$\psi(x) = \phi(x, \pm\sqrt{1-x^2}) = 2-3x^2-1+x^2-x = 1-x-2x^2$, $|x|\le 1$.

$\psi'(x) = -1-4x = 0 \Leftrightarrow x = -\frac{1}{4} \Rightarrow y = \pm\frac{\sqrt{15}}{4}$;

$P_3: \left(-\frac{1}{4}, \frac{\sqrt{15}}{4}, 0\right) \Rightarrow f(P_3) = \frac{9}{8}$; $P_4: \left(-\frac{1}{4}, -\frac{\sqrt{15}}{4}, 0\right) \Rightarrow f(P_4) = \frac{9}{8}$.

$f(-1,0,0)=0$, $f(1,0,0)=-2$.

Resultat: $f_{max} = f\left(-\frac{1}{6}, 0, \pm\frac{\sqrt{35}}{6}\right) = \frac{25}{12}$;

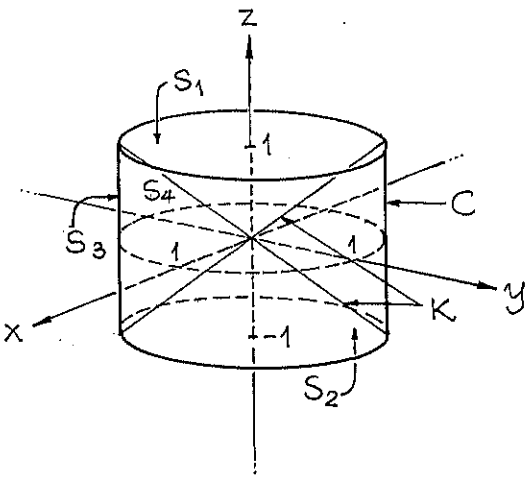
$f_{min} = f(1,0,0) = -2$.

b) $f(x,y,z) = 2z^2+y^2-x^2-x$; $C: x^2+y^2\le 1$, $z^2 \le 1$.

$\mathring{C}: x^2+y^2<1$, $-1<z<1$; (det inre av)

$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0 \Rightarrow P_0: \left(-\frac{1}{2}, 0, 0\right)$; $f(P_0) = \frac{1}{4}$.

$\partial C = S_1 \cup S_2 \cup S_3$ = randen av C (se figur).



$S_1: x^2+y^2\le 1$, $z=1$.

$\mathring{S}_1: x^2+y^2<1$, $z=1$ (det inre av S_1)

$g(x,y) = f(x,y,1) = y^2-x^2-x+2$, $\quad x^2+y^2<1$

$$\left.\begin{array}{l} \frac{\partial g}{\partial x} = -2x-1 = 0 \\ \frac{\partial g}{\partial y} = 2y = 0 \end{array}\right\} \Rightarrow P_0: \left(-\frac{1}{2}, 0, 1\right); \ f(P_0) = 9/4.$$

$\partial S_1: x^2+y^2=1$, $z=1$ (randen till S_1)

$h(x) = f(x, \pm\sqrt{1-x^2}, 1) = 3-x-2x^2$, $-1\le x\le 1$.

$h'(x) = -1-4x = 0 \Leftrightarrow x = -1/4$;

$h(-1) = f(-1,0,1) = 2$, $\quad h\left(\frac{1}{4}\right) = f\left(-\frac{1}{4}, \pm\frac{\sqrt{15}}{4}, 1\right) = 25/8$,

$h(1) = f(1,0,1) = 0$.

$S_2: x^2+y^2\le 1$, $z=-1$

forts.

Räkningarna går som under S_1 och vi får
$f(-\frac{1}{2},0,-1)=9/4$, $f(-1,0,-1)=2$, $f(-\frac{1}{4},\pm\frac{\sqrt{15}}{4},-1)=25/8$,
$f(1,0,-1)=0$.

Resultat: $f_{max}=f(-\frac{1}{4},\pm\frac{\sqrt{15}}{4},\pm 1)=\frac{25}{8}$

$f_{min}=f(1,0,\pm 1)=0$.

c) $f(x,y,z)=2z^2+y^2-x^2-x$; $D: x^2+y^2\leq z^2\leq 1$.

$D: \sqrt{x^2+y^2}\leq |z|\leq 1$.

Kritiska punkter saknas i D; $(-\frac{1}{2},0,0)\notin \mathring{D}$.

$\partial D=S_1\cup S_2\cup S_4$: $S_4: z^2=x^2+y^2,\ x^2+y^2\leq 1$.

$f(x,y,\pm\sqrt{x^2+y^2})=2(x^2+y^2)+y^2-x^2-x=x^2+3y^2-x$

$g(x,y)=x^2+3y^2-x,\ x^2+y^2\leq 1$

$\frac{\partial g}{\partial x}=2x-1=0=6y=\frac{\partial g}{\partial y}\Rightarrow P_0:(\frac{1}{2},0,\pm\frac{1}{2})$; $f(P_0)=-\frac{1}{4}$.

S_1 och S_2 ingår även i C.

Resultat: $f_{max}=f(-\frac{1}{4},\pm\frac{\sqrt{15}}{4},\pm 1)=\frac{25}{8}$

$f_{min}=f(\frac{1}{2},0,\pm\frac{1}{2})=-\frac{1}{4}$.

d) Klotet K och konen D är delmängder av C. Svaret är negativt.

Anm. Ett typiskt tentatal... Pedagogik?

Problem 4.14 (Sid. 13)

Lösning

$f(\mathbf{x})=x^2+y^2+z^2$; $g(\mathbf{x})=3x^2+y^2+3z^2+2xy+yz-3=0$.

$\nabla f(\mathbf{x})=(2x,2y,2z)$; $\nabla g(\mathbf{x})=(6x+2y,2y+2x+z,6z+y)$

$\nabla f(\mathbf{x}) \parallel \nabla g(\mathbf{x}) \Leftrightarrow (6x+2y,\ 2x+2y+z,\ y+6z)=\lambda\cdot(x,y,z)$

$$\Leftrightarrow \begin{cases} 6x+2y=\lambda x \\ 2x+2y+z=\lambda y \\ y+6z=\lambda z \end{cases} \Leftrightarrow \begin{cases} \frac{2x+2y+z}{6x+2y}=\frac{\lambda y}{\lambda x} \\ \frac{y+6z}{6x+2y}=\frac{\lambda z}{\lambda x} \end{cases} \Leftrightarrow \begin{cases} \frac{2x+2y+z}{6x+2y}=\frac{y}{x} \\ \frac{y+6z}{6x+2y}=\frac{z}{x} \end{cases}$$

$$\Leftrightarrow x(2x+2y+z)=y(6x+2y) \wedge x(y+6z)=z(6x+2y) \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 2x^2+2xy+xz=6xy+2y^2 \\ xy+\cancel{6xz}=\cancel{6xz}+2yz \end{cases} \Leftrightarrow \begin{cases} 2x^2-4xy-2y^2+xz=0 \\ xy-2yz=0 \end{cases}$$

$$\Leftrightarrow \begin{cases} 2x^2-4xy-2y^2+xz=0 \\ y(x-2z)=0 \end{cases} \Leftrightarrow \begin{cases} 2x^2-4xy-2y^2+xz=0 \\ y=0 \vee x=2z \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 2x^2+xz=0 \\ y=0 \end{cases} \vee \begin{cases} 10z^2-8yz-2y^2=0 \\ x=2z \end{cases} \Leftrightarrow \begin{cases} x(2x+z)=0 \\ y=0 \end{cases}$$

$$\vee \begin{cases} y=-5z \vee y=z \\ x=2z \end{cases} \Leftrightarrow \begin{cases} x=2z \\ y=-5z \end{cases} \vee \begin{cases} x=2z \\ y=z \end{cases};$$

(1) $g(2z,-5z,z)=12z^2+25z^2+3z^2-20z^2-5z^2-3=0 \Leftrightarrow$

$\Leftrightarrow 15z^2=3 \Leftrightarrow z^2=\frac{1}{5} \Rightarrow x^2=\frac{4}{5} \wedge y^2=5 \Rightarrow f(\mathbf{x})=6$

(2) $g(2z,z,z) = 12z^2+z^2+3z^2+4z^2+z^2-3 = 0 \Leftrightarrow z^2 = \frac{1}{7} \Rightarrow$

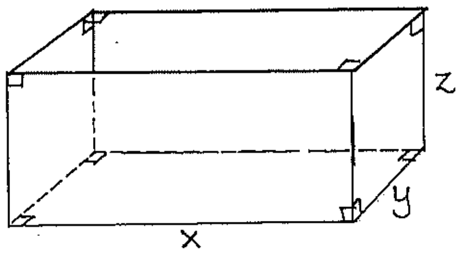
$\Rightarrow x^2 = \frac{4}{7} \wedge y^2 = \frac{1}{7} \Rightarrow f(\mathbf{x}) = \frac{6}{7}.$

Svar: Det största avståndet är $\sqrt{6}$ och är till punkterna $\pm(\frac{2}{\sqrt{5}}, \sqrt{5}, \frac{1}{\sqrt{5}})$; det minsta är $\sqrt{\frac{6}{7}}$ och är till punkterna $\pm(\frac{2}{\sqrt{7}}, \frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}})$.

Problem 4.15 (Sid. 13)

Lösning

Dimensionerna kallas x, y och z (se figur).



$f(x,y,z) = 2xz+2yz+xy, \quad g(x,y,z) = xyz - V_0 = 0.$

$\nabla f(\mathbf{x}) = (y+2z, x+2z, 2x+2y), \ \nabla g(\mathbf{x}) = (yz, xz, xy);$

$\nabla f(\mathbf{x}) \parallel \nabla g(\mathbf{x}) \Rightarrow (y+2z, x+2z, 2x+2y) = \lambda(yz, xz, xy)$

$$\Leftrightarrow \begin{cases} y+2z = \lambda yz \\ x+2z = \lambda xz \\ 2x+2y = \lambda xy \end{cases} \Leftrightarrow \begin{cases} \frac{x+2z}{y+2z} = \frac{\lambda xz}{\lambda yz} \\ \frac{2x+2y}{y+2z} = \frac{\lambda xy}{\lambda yz} \end{cases} \Leftrightarrow \begin{cases} \frac{x+2z}{y+2z} = \frac{x}{y} \\ \frac{2x+2y}{y+2z} = \frac{x}{z} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y(x+2z) = x(y+2z) \\ z(2x+2y) = x(y+2z) \end{cases} \Leftrightarrow \begin{cases} \cancel{xy}+2yz = \cancel{xy}+2xz \\ 2\cancel{xz}+2yz = xy+2\cancel{xz} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y = x \\ z = x/2 \end{cases} \Rightarrow g(x,x,\tfrac{x}{2}) = \frac{x^3}{2} - V_0 = 0 \Leftrightarrow x = \sqrt[3]{2V_0} \Rightarrow$$

$$\Rightarrow y = \sqrt[3]{2V_0} \wedge z = \tfrac{1}{2}\sqrt[3]{2V_0} \Rightarrow f(\sqrt[3]{2V_0}, \sqrt[3]{2V_0}, \tfrac{1}{2}\sqrt[3]{2V_0}) = \underline{3(\sqrt[3]{2V_0})^2}.$$

Anm. f är uttrycket för väggarean.

Problem 4.16 (Sid. 13)

Lösning

$f(x,y,z) = xyz; \quad g(x,y,z) = 2(xy+yz+xz) - A_0 = 0.$

$\text{grad} f(\mathbf{x}) = (yz, xz, xy); \ \text{grad} g(\mathbf{x}) = (2y+2z, 2x+2z, 2x+2y)$

$\nabla g(\mathbf{x}) \parallel \nabla f(\mathbf{x}) \Leftrightarrow (yz, xz, xy) = \lambda(y+z, x+y, x+y) \Leftrightarrow$

$$\Leftrightarrow \begin{cases} yz = \lambda(y+z) \\ xz = \lambda(x+z) \\ xy = \lambda(x+y) \end{cases} \Leftrightarrow \begin{cases} \frac{xz}{yz} = \frac{\lambda(x+z)}{\lambda(y+z)} \\ \frac{xy}{yz} = \frac{\lambda(x+y)}{\lambda(y+z)} \end{cases} \Leftrightarrow \begin{cases} \frac{x}{y} = \frac{x+z}{y+z} \\ \frac{x}{z} = \frac{x+y}{y+z} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x(y+z) = y(x+z) \\ x(y+z) = z(x+y) \end{cases} \Leftrightarrow \begin{cases} \cancel{xy}+xz = \cancel{xy}+yz \\ xy+\cancel{xz} = \cancel{xz}+yz \end{cases} \begin{cases} x = y \\ x = z \end{cases} \Leftrightarrow$$

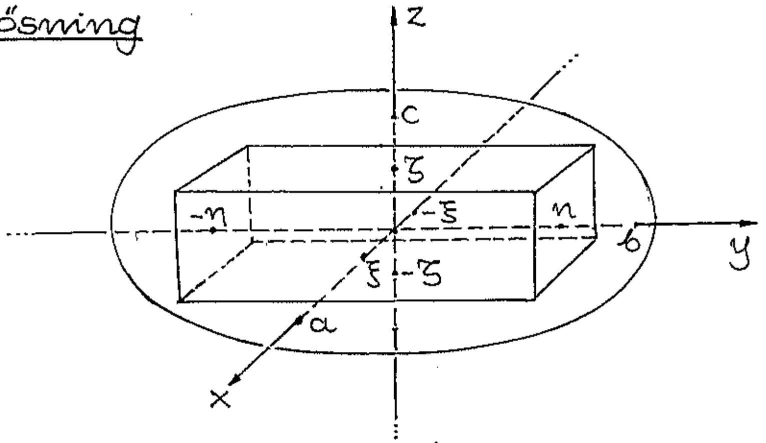
$\Leftrightarrow x=y=z \Rightarrow f(x,x,x) = 6x^2 - A_0 = 0 \Leftrightarrow x=y=z=\sqrt{\frac{A_0}{6}};$

$f(\sqrt{\frac{A_0}{6}}, \sqrt{\frac{A_0}{6}}, \sqrt{\frac{A_0}{6}}) = (\frac{A_0}{6})^{3/2} = V_{max}.$

Anm Lådan med den maximala volymen är kubisk (sida $\sqrt{\frac{A_0}{6}}$).

Problem 4.17 (Sid. 13)

Lösning



Parallellepipedens volym är $2\xi\cdot 2\eta\cdot 2\zeta = 8\xi\eta\zeta$.

$f(\mathbf{x}) = 8xyz, \quad g(\mathbf{x}) = \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-1=0; \quad x,y,z>0.$

$\operatorname{grad} f(\mathbf{x}) = 8(yz, xz, xy), \ \operatorname{grad} g(\mathbf{x}) = 2\left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right)$

$\nabla f(\mathbf{x}) \parallel \nabla g(\mathbf{x}) \Leftrightarrow (yz, xz, xy) = \lambda\left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right) \Leftrightarrow$

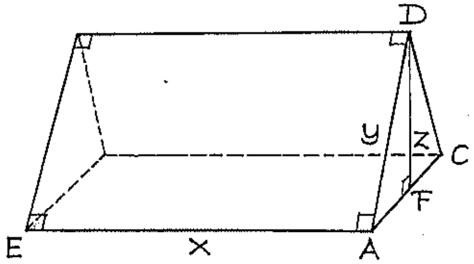
$$\Leftrightarrow \begin{cases} yz = \lambda x/a^2 \\ xz = \lambda y/b^2 \\ xy = \lambda z/c^2 \end{cases} \Leftrightarrow \begin{cases} xyz = \lambda x^2/a^2 \\ xyz = \lambda y^2/b^2 \\ xyz = \lambda z^2/c^2 \end{cases} \Leftrightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} \Rightarrow$$

$$\Rightarrow (g(\mathbf{x})=0) \Rightarrow \begin{cases} 3x^2/a^2 = 1 \\ 3y^2/b^2 = 1 \\ 3z^2/c^2 = 1 \end{cases} \Leftrightarrow \begin{cases} x^2 = a^2/3 \\ y^2 = b^2/3 \\ z^2 = c^2/3 \end{cases} \Leftrightarrow \begin{cases} x = a/\sqrt{3} \\ y = b/\sqrt{3} \\ z = c/\sqrt{3} \end{cases};$$

Resultat: $V_{max} = \frac{abc}{3\sqrt{3}}$.

Problem 4.18 (Sid. 13)

Lösning



Tältet betraktas som ett liggande prisma med basen ΔACD och höjden AE.

(1) ΔADF: Pythagoras' sats $\Rightarrow (AF)^2 + (FD)^2 = (AD)^2$

$\Rightarrow (AF)^2 = y^2 - z^2 \Rightarrow AC = 2\cdot AF = 2\sqrt{y^2-z^2} \Rightarrow V = xz\sqrt{y^2-z^2}.$

(2) Den totala tältytan är $S = 2xy + 2z\sqrt{y^2-z^2}$.

(3) $f(\mathbf{x}) = 2xy + 2z\sqrt{y^2-z^2}, \quad g(\mathbf{x}) = xz\sqrt{y^2-z^2} - V_0 = 0;$

$$\operatorname{grad} f(\mathbf{x}) = \left(2y,\ 2x + \frac{2yz}{\sqrt{y^2-z^2}},\ \frac{2(y^2-2z^2)}{\sqrt{y^2-z^2}}\right);$$

$$\operatorname{grad}\ (\mathbf{x}) = \left(z\sqrt{y^2-z^2},\ \frac{xyz}{\sqrt{y^2-z^2}},\ \frac{x(y^2-2z^2)}{\sqrt{y^2-z^2}}\right);$$

$$\nabla f(\mathbf{x}) \parallel \nabla g(\mathbf{x}) \Leftrightarrow \nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}) \Rightarrow \frac{2(y^2-2z^2)}{\sqrt{y^2-z^2}} = \frac{\lambda x(y^2-2z^2)}{\sqrt{y^2-z^2}}$$

(z-komponenterna betraktas)

$\Leftrightarrow \lambda x = 2 \ \vee \ y^2 = 2z^2 \Rightarrow y = \sqrt{2}z$ ($\lambda x = 2$ förkastas).

(4) Basen $= \Delta ACD$: $|\Delta ACD| = z\sqrt{y^2-z^2} = z\sqrt{2z^2-z^2} = z^2$.

Volymen $= V_0 = xz^2$ (fås ur $g(\mathbf{x})=0$) $\Leftrightarrow x = \frac{V_0}{z^2}$;

(5) $f(\mathbf{x}) = f\left(\frac{V_0}{z^2}, \sqrt{2}z, z\right) = 2z^2 + \frac{2\sqrt{2}V_0}{z} = h(z)$

$h'(z) = 4z - \frac{2\sqrt{2}V_0}{z^2} = 0 \Leftrightarrow z^3 = \frac{V_0}{\sqrt{2}} \Leftrightarrow z = 2^{-1/6} \cdot V_0^{-1/3}$.

Resultat: Höjden ska vara $2^{-1/6} \cdot V_0^{-1/3}$; längden ska vara $2^{1/3} V_0^{5/3}$; bredden ska vara $2^{5/6} V_0^{-1/3}$.

Anm. $h''(z) = 4 + \frac{4\sqrt{2}V_0}{z^3} > 0 \Rightarrow$ minimum föreligger.

Problem 4.19 (Sid. 14)

Lösning

$f(\mathbf{x}) = \sin x \sin y \sin z$, $g(\mathbf{x}) = x + y + z - \pi = 0$.

$$\begin{cases} \nabla f(\mathbf{x}) = (\cos x \sin y \sin z, \sin x \cos y \sin z, \sin x \sin y \cos z); \\ \nabla g(\mathbf{x}) = (1,1,1); \end{cases}$$

Med Lagranges multiplikatormetod fås:

$$\nabla f(\mathbf{x}) \parallel \nabla g(\mathbf{x}) \Leftrightarrow \nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}) \Leftrightarrow \begin{cases} \cos x \sin y \sin z = \lambda \\ \sin x \cos y \sin z = \lambda \\ \sin x \sin y \cos z = \lambda \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \frac{\sin x \cos y \cancel{\sin z}}{\cos x \sin y \cancel{\sin z}} = \frac{\lambda}{\lambda} \\ \frac{\sin x \cancel{\sin y} \cos z}{\cos x \cancel{\sin y} \sin z} = \frac{\lambda}{\lambda} \end{cases} \Leftrightarrow \begin{cases} \tan x = \tan y \\ \tan x = \tan z \end{cases} \Leftrightarrow x = y = z \Rightarrow$$

$$\Rightarrow (g(\mathbf{x}) = 0) \Rightarrow 3x = \pi \overset{!}{\Leftrightarrow} x = y = z = \frac{\pi}{3} \text{ (liksidig).}$$

Svar: Det största värdet är $\left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{8}$.

Problem 4.20 (Sid. 14)

Lösning

$f(\mathbf{x}) = e^{-x^2 - 2y^2 - 3z^2}$, $xyz \geq 6$, $x, y, z > 0$

(1) $\nabla f(\mathbf{x}) = f(\mathbf{x}) \cdot (-2x, -4y, -6z) \neq 0$; stationära saknas.

(2) $f(\mathbf{x}) = e^{-x^2 - 2y^2 - 3z^2}$, $g(\mathbf{x}) = xyz - 6 = 0$.

$\operatorname{grad} f(\mathbf{x}) \parallel \operatorname{grad} g(\mathbf{x}) \Leftrightarrow (-2x, -4y, -6z) \cdot f(\mathbf{x}) = \lambda (yz, xz, xy)$

$$\Leftrightarrow \begin{cases} -2x f(\mathbf{x}) = \lambda yz \\ -4y f(\mathbf{x}) = \lambda xz \\ -6z f(\mathbf{x}) = \lambda xy \end{cases} \Leftrightarrow \begin{cases} \frac{-4y f(\mathbf{x})}{-2x f(\mathbf{x})} = \frac{\lambda xz}{\lambda yz} \\ \frac{-6z f(\mathbf{x})}{-2x f(\mathbf{x})} = \frac{\lambda xy}{\lambda yz} \end{cases} \Leftrightarrow \begin{cases} 2\frac{y}{x} = \frac{x}{y} \\ 3\frac{z}{x} = \frac{x}{z} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} y^2 = x^2/2 \\ z^2 = x^2/3 \end{cases} \Leftrightarrow \begin{cases} y = \frac{x}{\sqrt{2}} \\ z = \frac{x}{\sqrt{3}} \end{cases} \Rightarrow (g(\mathbf{x}) = 0) \Rightarrow \frac{x^3}{\sqrt{6}} = 6 \Leftrightarrow x = \sqrt{6}$$

$$\Rightarrow y = \sqrt{3} \wedge z = \sqrt{2} \Rightarrow f(\sqrt{6}, \sqrt{3}, \sqrt{2}) = e^{-18}.$$

Svar: $f_{max} = f(\sqrt{6}, \sqrt{3}, \sqrt{2}) = e^{-18}$; f_{min} saknas. $(f(\mathbf{x}) > 0)$.

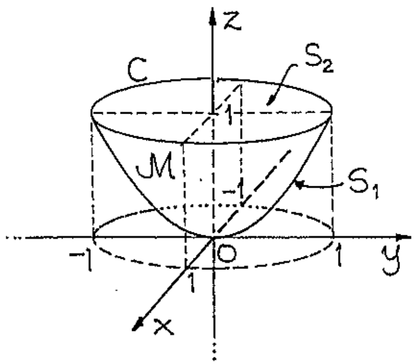
Problem 4.21 (Sid. 14)

Lösning

$f(x,y,z) = x + y + z$; $M = \{(x,y,z) : x^2 + y^2 \leq z \leq 1\}$.

(1) $\mathring{M}: x^2 + y^2 < z < 1$

$\operatorname{grad} f(\mathbf{x}) = (1,1,1) \neq 0$; kritiska punkter saknas.



$\partial M = S_1 \cup S_2.$

(2) $S_1:\ x^2+y^2=z\leq 1$

$$\phi(x,y)=f(x,y,x^2+y^2)=x+y+x^2+y^2,\quad x^2+y^2\leq 1.$$

$\mathring{S}_1:\ x^2+y^2=z<1$ (det inre av S_1)

$$\frac{\partial\phi}{\partial x}=\frac{\partial\phi}{\partial y}=0 \Rightarrow 2x+1=0=1+2y \Leftrightarrow P_0:\left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)$$

$$f\left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)=-\frac{1}{2}.$$

(3) $S_2:\ x^2+y^2\leq 1,\ z=1$

$$f(x,y,1)=x+y+1 \Rightarrow f(r\cos\theta, r\sin\theta, 1)=r(\cos\theta+$$
$$+\sin\theta)+1=1+\sqrt{2}\,r\sin\left(\theta-\frac{\pi}{4}\right),\quad 0\leq r\leq 1,\ 0\leq\theta\leq 2\pi.$$

$$\Rightarrow 1-\sqrt{2}\leq 1+r(\cos\theta+\sin\theta)\leq 1+\sqrt{2};$$

$1-\sqrt{2}=1+r(\cos\theta+\sin\theta)\Rightarrow r=1\wedge\theta=\frac{5\pi}{4};\ P_1:\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},1\right).$

$1+\sqrt{2}=1+r(\cos\theta+\sin\theta)\Rightarrow r=1\wedge\theta=\frac{\pi}{4};\ P_2:\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},1\right).$

Svar: $f_{max}=f\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},1\right)=1+\sqrt{2};\quad f_{min}=f\left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)=-\frac{1}{2}.$

Problem 4.22 (Sid. 14)

Lösning

$f(\mathbf{x})=xyz\ ;\ D:\ x+y+z=1,\ x^2+y^2+z^2\leq 1.$

(1) $\mathring{D}:\ x+y+z=1,\ x^2+y^2+z^2<1.$

$$f(\mathbf{x})=xyz,\quad g(\mathbf{x})=x+y+z-1=0,\quad h(\mathbf{x})=x^2+y^2+z^2-1<0.$$

$$\operatorname{grad}f(\mathbf{x})=\lambda\operatorname{grad}g(\mathbf{x}) \Leftrightarrow (yz,xz,xy)=\lambda(1,1,1)\Leftrightarrow$$

$$\Leftrightarrow\begin{cases} yz=\lambda\\ xz=\lambda\\ xy=\lambda\end{cases}\Leftrightarrow x=y=z \Rightarrow (g(\mathbf{x})=0)\Rightarrow x=y=z=\frac{1}{3};$$

$$f\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)=\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}=\frac{1}{27}$$

(2) $f(\mathbf{x})=xyz,\ g(\mathbf{x})=x+y+z-1=0,\ h(\mathbf{x})=x^2+y^2+z^2-1=0$

$$\nabla f(\mathbf{x})=(yz,xz,xy);\ \nabla g(\mathbf{x})=(1,1,1);\ \nabla h(\mathbf{x})=2(x,y,z);$$

$$\nabla f(\mathbf{x})\cdot\nabla g(\mathbf{x})\times\nabla h(\mathbf{x})=\begin{vmatrix} yz & 1 & 2x\\ xz & 1 & 2y\\ xy & 1 & 2z\end{vmatrix} -1 =\begin{vmatrix} yz & 1 & 2x\\ (x-y)z & 0 & 2(y-x)\\ (x-z)y & 0 & 2(z-x)\end{vmatrix}=$$

$$=-\begin{vmatrix}(x-y)z & 2(y-x)\\ (x-z)y & 2(z-x)\end{vmatrix}=(y-x)(x-z)\begin{vmatrix} z & -2\\ y & -2\end{vmatrix}=2(y-x)(x-z)(y-z)=$$

$$=0\Leftrightarrow x=y\ \vee\ x=z\ \vee\ y=z.$$

$$x=y:\ \begin{cases} g(\mathbf{x})=0 \Rightarrow 2x+z=1\\ h(\mathbf{x})=0\Rightarrow 2x^2+z^2=1\end{cases}\Leftrightarrow\begin{cases} x=\frac{2}{3}\\ y=\frac{2}{3}\\ z=-1/3\end{cases}\Rightarrow f\left(\frac{2}{3},\frac{2}{3},-\frac{1}{3}\right)=-\frac{4}{27}$$

$\underline{x=z}:\begin{cases} g(\mathbf{x})=0 \Rightarrow y+2z=1 \\ h(\mathbf{x})=0 \Rightarrow y^2+2z^2=1 \end{cases} \Leftrightarrow \begin{cases} x=2/3 \\ y=-1/3 \\ z=2/3 \end{cases} \Rightarrow f(\tfrac{2}{3},-\tfrac{1}{3},\tfrac{2}{3})=-\tfrac{4}{27}.$

$\underline{y=z}:\begin{cases} g(\mathbf{x})=0 \Rightarrow x+2y=1 \\ h(\mathbf{x})=0 \Rightarrow x^2+2y^2=1 \end{cases} \Leftrightarrow \begin{cases} x=-1/3 \\ y=2/3 \\ z=2/3 \end{cases} \Rightarrow f(-\tfrac{1}{3},\tfrac{2}{3},\tfrac{2}{3})=-\tfrac{2}{27}.$

Svar: $f_{max}=f(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3})=\tfrac{1}{27}.$

$f_{min}=f(\tfrac{2}{3},\tfrac{2}{3},-\tfrac{1}{3})=f(\tfrac{2}{3},-\tfrac{1}{3},\tfrac{2}{3})=f(-\tfrac{1}{3},\tfrac{2}{3},\tfrac{2}{3})=-\tfrac{4}{27}.$

Problem 4.23 (Sid. 14)

Lösning

(1) $f(\mathbf{x})=3x+2y+z,\ g(\mathbf{x})=x^2+y^2+z^2-1=0,\ h(\mathbf{x})=x+y+z-1=0$

$\nabla f(\mathbf{x})=(3,2,1),\ \nabla g(\mathbf{x})=2(x,y,z),\ \nabla h(\mathbf{x})=(1,1,1);$

$$\nabla f(\mathbf{x})\cdot\nabla g(\mathbf{x})\times\nabla h(\mathbf{x})=0 \Rightarrow \begin{vmatrix} 3 & x & 1 \\ 2 & y & 1 \\ 1 & z & 1 \end{vmatrix}=-x+2y-z=0;$$

Detta villkor kombineras med bivillkoren:

$$\begin{cases} x^2+y^2+z^2=1 \\ x+y+z=1 \\ x-2y+z=0 \ (-1) \end{cases} \Leftrightarrow \begin{cases} x^2+y^2+z^2=1 \\ 3y=1 \\ x-2y+z=0 \end{cases} \Leftrightarrow \begin{cases} x^2+z^2=8/9 \\ y=1/3 \\ x+z=2/3 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} z^2-2z/3=2/9 \\ y=1/3 \\ x=2/3-z \end{cases} \Leftrightarrow \begin{cases} z=(1\pm\sqrt{3})/3 \\ y=1/3 \\ x=2/3-z \end{cases} \Leftrightarrow \begin{cases} P_1:(\frac{1+\sqrt{3}}{3},\frac{1}{3},\frac{1-\sqrt{3}}{3}) \\ P_2:(\frac{1-\sqrt{3}}{3},\frac{1}{3},\frac{1+\sqrt{3}}{3}) \end{cases} \Rightarrow$$

$\Rightarrow \underline{f(P_1)=\tfrac{2}{3}(3+\sqrt{3})},\ \underline{f(P_2)=\tfrac{2}{3}(3-\sqrt{3})}.$

(2) $\underline{f(\mathbf{x})=3x+2y+z}$; $\underline{M: x^2+y^2+z^2=1,\ x+y+z>1.}$

Antag att planet $3x+2y+z=C$ tangerar enhetssfären i punkten $Q:(\alpha,\beta,\gamma)$.

$$\nabla f(Q)\parallel\nabla g(Q) \Leftrightarrow (3,2,1)=k\cdot(2\alpha,2\beta,2\gamma) \Leftrightarrow \begin{cases} 2k\alpha=3 \\ 2k\beta=2 \\ 2k\gamma=1 \end{cases}$$

$\Leftrightarrow (\alpha,\beta,\gamma)=(\frac{3}{2k},\frac{1}{k},\frac{1}{2k}) \Rightarrow (g(\mathbf{x})=0) \Rightarrow$

$\Rightarrow (\frac{9}{4}+1+\frac{1}{4})\frac{1}{k^2}=1 \Leftrightarrow k^2=\frac{14}{4} \Leftrightarrow k=\frac{\sqrt{14}}{2}$, ty $\alpha+\beta+\gamma>0$.

$f(\alpha,\beta,\gamma)=f(\frac{3}{\sqrt{14}},\frac{2}{\sqrt{14}},\frac{1}{\sqrt{14}})=\frac{9+4+1}{\sqrt{14}}=\underline{\sqrt{14}}.$

Svar: Största värdet $\sqrt{14}$ antas i $(\frac{3}{\sqrt{14}},\frac{2}{\sqrt{14}},\frac{1}{\sqrt{14}})$; minsta värdet $\frac{6-2\sqrt{3}}{3}$ antas i $(\frac{1-\sqrt{3}}{3},\frac{1}{3},\frac{1+\sqrt{3}}{3})$.

Problem 4.24 (Sid. 14)

Lösning

$\underline{f(\mathbf{x})=x^2+y^2+z^2}$; $\underline{g(\mathbf{x})=x^2+y^2+2z^2-4=0}$, $\underline{h(\mathbf{x})=x+y+z-1=0}.$

$\nabla f(\mathbf{x})=2(x,y,z),\ \nabla g(\mathbf{x})=2(x,y,2z),\ \nabla h(\mathbf{x})=(1,1,1);$

$\nabla f(\mathbf{x})\cdot\nabla g(\mathbf{x})\times\nabla h(\mathbf{x})=0 \Leftrightarrow 4z(y-x)=0 \Leftrightarrow \underline{y=x} \vee \underline{z=0}.$

Varje fall behandlas separat.

(1) $\underline{z=0} \Rightarrow \begin{cases} g(\mathbf{x})=0 \Rightarrow x^2+y^2=4 \\ h(x)=0 \Rightarrow x+y=1 \end{cases} \Leftrightarrow \begin{cases} x^2+(x-1)^2=4 \\ y=1-x \end{cases} \Leftrightarrow \begin{cases} 2x^2-2x=3 \\ y=1-x \end{cases} \Leftrightarrow$

$$\Leftrightarrow \begin{cases} x=\frac{1\pm\sqrt{17}}{2}\\ y=1-x\end{cases} \Leftrightarrow \begin{cases}P_1:\left(\frac{1+\sqrt{17}}{2},\frac{1-\sqrt{17}}{2}\right)\\ P_2:\left(\frac{1-\sqrt{17}}{2},\frac{1+\sqrt{17}}{2}\right)\end{cases} \Rightarrow f(P_1)=f(P_2)=4.$$

(2) $\underline{y=x} \Rightarrow \begin{cases} x^2+z^2=2\\ 2x+z=1\end{cases} \Leftrightarrow \begin{cases}5x^2-4x-1=0\\ z=1-2x\end{cases} \Leftrightarrow \begin{cases}P_3:(1,1,-1)\\ P_4:\left(-\frac{1}{5},-\frac{1}{5},\frac{7}{5}\right)\end{cases} \Rightarrow$

$$\Rightarrow f(P_3)=3 \text{ och } f(P_4)=\frac{51}{25}.$$

Svar: Största avståndet 2 antas i $\left(\frac{1+\sqrt{17}}{2},\frac{1-\sqrt{17}}{2}\right)$ och $\left(\frac{1-\sqrt{17}}{2},\frac{1+\sqrt{17}}{2}\right)$; minsta avståndet $\frac{\sqrt{51}}{5}$ antas i $\left(-\frac{1}{5},-\frac{1}{5},\frac{7}{5}\right)$.

Problem 4.25 (Sid. 14)

Lösning

$\underline{f(\mathbf{x})=2x-2y+z}$; $\underline{D: x^2+y^2+z^2\le 2,\ z\ge x^2+y^2.}$

(1) $\operatorname{grad} f(\mathbf{x})=(2,-2,1)\neq 0$; båda extrema antas på ∂D.

(2) $\underline{C: x^2+y^2+z^2=2,\ z=x^2+y^2.}$

$f(\mathbf{x})=2x-2y+z$; $g(\mathbf{x})=x^2+y^2+z^2-2=0$; $h(\mathbf{x})=x^2+y^2-z=0$

$\nabla f(\mathbf{x})=(2,-2,1)$, $\nabla g(\mathbf{x})=2(x,y,z)$, $\nabla h(\mathbf{x})=(2x,2y,-1)$;

$$\nabla f(\mathbf{x})\cdot\nabla g(\mathbf{x})\times\nabla h(\mathbf{x})=0 \Rightarrow \begin{vmatrix} 2 & 2x & 2x\\ -2 & 2y & 2y\\ 1 & 2z & -1\end{vmatrix}=\begin{vmatrix} 2 & 2x & 0\\ -2 & 2y & 0\\ 1 & 2z & -2z-1\end{vmatrix}=$$

$$=(-2z-1)(4y+4x)=0 \Leftrightarrow z=-\tfrac{1}{2} \vee y=-x;$$

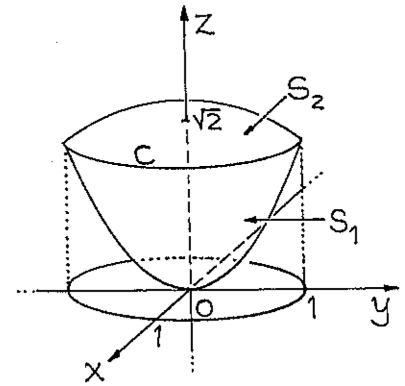
Varje fall behandlas separat:

$$z=-\tfrac{1}{2}\Rightarrow\begin{cases} g(\mathbf{x})=0\Rightarrow x^2+y^2=\frac{7}{4}\\ h(\mathbf{x})=0\Rightarrow x^2+y^2=-\frac{1}{2}\end{cases};\ \text{lösning saknas.}$$

$$y=-x\Rightarrow\begin{cases} g(\mathbf{x})=0\Rightarrow 2x^2+z^2=2\\ h(\mathbf{x})=0\Rightarrow 2x^2=z\end{cases}\Leftrightarrow\begin{cases} z^2+z=2\\ y=-x\\ z=2x^2\end{cases}\Leftrightarrow\begin{cases} x=\pm 1/\sqrt{2}\\ y=-x\\ z=1\end{cases}$$

$$\Leftrightarrow P_1:\left(\tfrac{1}{\sqrt{2}},-\tfrac{1}{\sqrt{2}},1\right)\vee P_2:\left(-\tfrac{1}{\sqrt{2}},\tfrac{1}{\sqrt{2}},1\right);\ \underline{f(P_1)=1+\sqrt{8}},\ \underline{f(P_2)=1-\sqrt{8}}$$

(3)



$\underline{S_1: z=x^2+y^2,\ z\le 1.}$

$f(x,y,x^2+y^2)=2x-2y+x^2+y^2$; $x^2+y^2<1$.

$f_1(x,y)=2x-2y+x^2+y^2 \Rightarrow \frac{\partial f_1}{\partial x}=2+2x$, $\frac{\partial f_1}{\partial y}=-2+2y$;

Stationära punkter saknas.

(4) $\underline{S_2: x^2+y^2+z^2=2,\ z>1.}$

$f(\mathbf{x})=2x-2y+z$, $g(\mathbf{x})=x^2+y^2+z^2-2=0$, $z>1$.

$\operatorname{grad} f(\mathbf{x}) \parallel \operatorname{grad} g(\mathbf{x}) \Leftrightarrow (2x, 2y, 2z) = \lambda\cdot(2,-2,1) \Leftrightarrow$

$\Leftrightarrow (x,y,z) = (\lambda, -\lambda, \frac{1}{2}\lambda) \Rightarrow (g(\mathbf{x})=0) \Rightarrow \frac{9}{4}\lambda^2 = 2 \Leftrightarrow$

$\Leftrightarrow \lambda^2 = \frac{8}{9} < 1$; $z > 1$ så detta förkastas.

Svar: $f_{max} = f(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1) = 1+\sqrt{8}$, $f_{min} = f(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1) = 1-\sqrt{8}$.

Anm. Punkten $(-1,1,2)$ ligger inte i

$M: x^2+y^2+z^2 \leq 2,\ z \geq x^2+y^2.$

Problem 4.26 (Sid. 14)

Lösning

$f(\mathbf{x}) = z;\ g(\mathbf{x}) = x^2+xz+y^2+2z^2-9=0,\ h(\mathbf{x}) = x+y+z-1=0.$

$\operatorname{grad} f(\mathbf{x}) = (0,0,1)$, $\operatorname{grad} g(\mathbf{x}) = (2x+z, 2y, x+4z)$, $\nabla h(\mathbf{x}) = (1,1,1)$.

$$\nabla f(\mathbf{x})\cdot \nabla g(\mathbf{x}) \times \nabla h(\mathbf{x}) = 0 \Rightarrow \begin{vmatrix} 0 & 2x+z & 1 \\ 0 & 2y & 1 \\ 1 & x+4z & 1 \end{vmatrix} = 2x-2y+z=0\ ;$$

$$\begin{cases} x^2+xz+y^2+2z^2=9 \\ x+y+z=1 \quad (-1) \\ 2x-2y+z=0 \end{cases} \Leftrightarrow \begin{cases} x^2+xz+y^2+2z^2=9 \\ x+y+z=1 \\ x-3y=-1 \quad (-1) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x^2+xz+y^2+2z^2=9 \\ 4y+z=2 \\ x-3y=-1 \end{cases} \Leftrightarrow \begin{cases} x^2+xz+y^2+2z^2=9 \\ z=2-4y \\ x=-1+3y \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 15y^2-14y=1 \\ z=2-4y \\ x=-1+3y \end{cases} \Leftrightarrow \begin{cases} y=1 \vee y=-1/15 \\ z=2-4y \\ x=-1+3y \end{cases} \Leftrightarrow \begin{cases} P_1: (2,1,-2) \\ P_2: (-\frac{6}{5}, -\frac{1}{15}, \frac{34}{15}) \end{cases}$$

Svar: Högsta punkten är $(-\frac{6}{5}, -\frac{1}{15}, \frac{34}{15})$ och lägsta $(2,1,-2)$.

Problem 4.27 (Sid. 14)

Lösning

(1) $S: x^2+2y^2+3z^2=1$ är en nivåyta till funktionen

$$f(\mathbf{x}) = x^2+2y^2+3z^2.$$

Tangeringspunkten kallas $P_0: (\alpha, \beta, \gamma)$.

En normalvektor till tangentplanet i P_0 är, som bekant, $\operatorname{grad} f(P_0) = (2\alpha, 4\beta, 6\gamma) = 2(\alpha, 2\beta, 3\gamma)$.

Om π är tangentplanet och $P: (x,y,z) \in \pi$ så är

$\pi: (\alpha, 2\beta, 3\gamma)\cdot(x-\alpha, y-\beta, z-\gamma) = \alpha(x-\alpha) + 2\beta(y-\beta) +$
$+ 3\gamma(z-\gamma) = \alpha x + 2\beta y + 3\gamma z - \alpha^2 - 2\beta^2 - 3\gamma^2 = \alpha x + 2\beta y + 3\gamma z - 1 =$
$= 0 \Leftrightarrow \underline{\pi: \alpha x + 2\beta y + 3\gamma z = 1}$, ty $\alpha^2+2\beta^2+3\gamma^2 = 1$.

(2) Tangentplanet π skär axlarna i $\frac{1}{\alpha}, \frac{1}{2\beta}$ resp. $\frac{1}{3\gamma}$.

Volymen av tetraedern ifråga är $V = \frac{1}{36}\cdot\frac{1}{\alpha\beta\gamma}$.

Att bestämma V_{min} är detsamma som att bestämma f_{max} för $f(\alpha,\beta,\gamma) = \alpha\beta\gamma$, alt. $F(\mathbf{x}) = xyz$.

$F(\mathbf{x}) = xyz$; $G(\mathbf{x}) = x^2+2y^2+3z^2-1 = 0$.

$$\operatorname{grad}F(\mathbf{x}) \parallel \operatorname{grad}G(\mathbf{x}) \Leftrightarrow (yz, xz, xy) = \lambda(2x, 4y, 6z) \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} yz = 2\lambda x \\ xz = 4\lambda y \\ xy = 6\lambda z \end{cases} \Leftrightarrow \begin{cases} \frac{xz}{yz} = \frac{4\lambda y}{2\lambda x} \\ \frac{xy}{yz} = \frac{6\lambda z}{2\lambda x} \end{cases} \Leftrightarrow \begin{cases} \frac{x}{y} = \frac{2y}{x} \\ \frac{x}{z} = \frac{3z}{x} \end{cases} \Leftrightarrow \begin{cases} y^2 = \frac{x^2}{2} \\ z^2 = \frac{x^2}{3} \end{cases} \Rightarrow$$

$$\Rightarrow (G(\mathbf{x}) = 0) \Rightarrow 3x^2 = 1 \Leftrightarrow x^2 = \frac{1}{3} \Rightarrow y^2 = \frac{1}{6} \wedge z^2 = \frac{1}{9} \Rightarrow$$

$$\Rightarrow x^2 \cdot y^2 \cdot z^2 = \frac{1}{162} \Leftrightarrow \frac{1}{xyz} = 9\sqrt{2} \Rightarrow V_{min} = \frac{9\sqrt{2}}{36} = \frac{\sqrt{2}}{4}$$

Problem 4.28 (Sid. 14)

Lösning

a) $\operatorname{grad}f(\mathbf{x}) \parallel \operatorname{grad}g(\mathbf{x}) \Leftrightarrow \operatorname{grad}f(\mathbf{x}) = -\lambda \operatorname{grad}g(\mathbf{x}) \Leftrightarrow$

$\Leftrightarrow \operatorname{grad}(f(\mathbf{x}) + \lambda g(\mathbf{x})) = \mathbf{0} \Leftrightarrow \nabla_{\mathbf{x}} L = \mathbf{0}$ (1)

$g(\mathbf{x}) = 0 \Rightarrow \frac{\partial}{\partial\lambda} L = \nabla_\lambda L = 0 \Rightarrow \nabla L = (\nabla_{\mathbf{x}}, \nabla_\lambda) L = \mathbf{0}$

Anm. $\operatorname{grad}g(\mathbf{x}) = \mathbf{0} \Rightarrow \operatorname{grad}f(\mathbf{x}) = \mathbf{0}$.

b) $\operatorname{grad}f(\mathbf{x})$, $\operatorname{grad}g_1(\mathbf{x})$ och $\operatorname{grad}g_2(\mathbf{x})$ är linjärt beroende om $\operatorname{grad}f(\mathbf{x}) = -\lambda_1 \operatorname{grad}g_1(\mathbf{x}) - \lambda_2 \operatorname{grad}g_2(\mathbf{x})$

$\Leftrightarrow \operatorname{grad}(f(\mathbf{x}) + \lambda_1 g_1(\mathbf{x}) + \lambda_2 g_2(\mathbf{x})) = \mathbf{0} \Leftrightarrow \nabla_{\mathbf{x}} L = \mathbf{0}$. (*)

$$\left.\begin{matrix} g_1(\mathbf{x}) = 0 \Rightarrow \frac{\partial L}{\partial\lambda_1} = 0 \\ g_2(\mathbf{x}) = 0 \Rightarrow \frac{\partial L}{\partial\lambda_2} = 0 \end{matrix}\right\} \overset{(*)}{\Rightarrow} \nabla L = (\nabla_{\mathbf{x}}, \nabla_{\lambda_1}, \nabla_{\lambda_2}) L = \mathbf{0}$$

Anm. $\nabla_{\mathbf{x}} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, $\nabla_{\lambda_1} = \frac{\partial}{\partial\lambda_1}$ och $\nabla_{\lambda_2} = \frac{\partial}{\partial\lambda_2}$.

Problem 4.29 (Sid. 14)

Lösning

$f(\mathbf{x}) = x^4 + y^4 + z^4$; $g(\mathbf{x}) = x^2 + y^2 + z^2 - 1 = 0$, $h(\mathbf{x}) = x + y + z = 0$.

$$\left.\begin{matrix} \operatorname{grad}f(\mathbf{x}) = 4(x^3, y^3, z^3) \\ \operatorname{grad}g(\mathbf{x}) = 2(x, y, z) \\ \operatorname{grad}h(\mathbf{x}) = (1, 1, 1) \end{matrix}\right\} \Rightarrow \begin{vmatrix} x^3 & x & 1 \\ y^3 & y & 1 \\ z^3 & z & 1 \end{vmatrix} = 0 \Rightarrow$$

$$\Leftrightarrow \begin{vmatrix} x^3 & x & 1 \\ y^3 - x^3 & y - x & 0 \\ z^3 - x^3 & z - x & 0 \end{vmatrix} = (y-x)(z-x) \begin{vmatrix} x^3 & x & 1 \\ y^2 + xy + x^2 & 1 & 0 \\ z^2 + xz + x^2 & 1 & 0 \end{vmatrix} =$$

$$= (y-x)(z-x) \begin{vmatrix} y^2 + xy + x^2 & 1 \\ z^2 + xz + x^2 & 1 \end{vmatrix} =$$

$$= (y-x)(z-x) \begin{vmatrix} y^2 + xy + x^2 & 1 \\ y^2 - z^2 + xz - xy & 0 \end{vmatrix} =$$

$$= (y-x)(z-x)(z^2 - y^2 + xy - xz) = (y-x)(z-x)(z-y)(z+y-x) = 0$$

$$\Leftrightarrow \underline{y = x} \vee \underline{z = x} \vee \underline{z = y} \vee \underline{x = y + z}$$

(1) $y = x \Rightarrow \begin{cases} g(x, x, z) = 2x^2 + z^2 - 1 = 0 \\ h(x, x, z) = 2x + z = 0 \end{cases} \Leftrightarrow \begin{cases} z = -2x \\ 6x^2 = 1 \end{cases} \Leftrightarrow$

$$\Rightarrow x^2 = y^2 = \frac{1}{6} \wedge z^2 = \frac{2}{3} \Rightarrow f(\mathbf{x}) = \frac{1}{36} + \frac{1}{36} + \frac{4}{9} = \frac{1}{2}.$$

(2) $z = x$ och $z = y$ ger samma resultat: $f_{max} = \frac{1}{2}$.

6. # Integralkalkyl

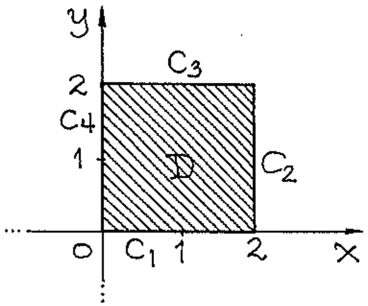
Dubbelintegraler

Problem 6.1 (Sid. 15)

Lösning

a) $f(x,y)=1/(3+x^2-y)$; $D: 0\leq x\leq 2, 0\leq y\leq 2$.

$g(\mathbf{x})=3+x^2-y$, $\mathbf{x}\in D$



(1) $\mathring{D}: 0<x,y<2$ (det inre av D).

$\frac{\partial g}{\partial x}=2x>0 \wedge \frac{\partial g}{\partial y}=-1<0$; kritiska pkr saknas.

(2) $C_1: x=t, y=0; 0\leq t\leq 2$.

$g(t,0)=3+t^2=\phi_1(t), 0\leq t\leq 2$; strängt växande.

$\phi_1(0)=g(0,0)=3$, $\phi_1(2)=g(2,0)=7$;

(3) $C_2: x=2, y=t; 0\leq t\leq 2$.

$g(2,t)=7-t=\phi_2(t), 0\leq t\leq 2$; strängt avtagande.

$\phi_2(0)=g(2,0)=7$, $\phi_2(2)=g(2,2)=5$.

(3) $C_3: x=t, y=2, 0\leq t\leq 2$.

$g(t,2)=1+t^2=\phi_3(t), 0\leq t\leq 2$; strängt växande;

$\phi_3(0)=f(0,2)=1$, $\phi_3(2)=f(2,2)=5$.

(4) $C_4: x=0, y=t; 0\leq t\leq 2$.

$g(0,t)=3-t=\phi_4(t), 0\leq t\leq 2$; strängt avtagande.

$\phi_4(0)=g(0,0)=3$, $\phi_4(2)=g(0,2)=1$.

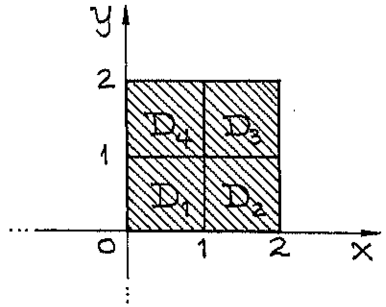
(5) $\mathbf{x}\in D \Rightarrow 1\leq g(\mathbf{x})\leq 7 \Leftrightarrow \frac{1}{7}\leq\frac{1}{g(\mathbf{x})}\leq 1 \Leftrightarrow \frac{1}{7}\leq f(\mathbf{x})\leq 1. \Rightarrow$

$\Rightarrow \iint_D \frac{1}{7}\,dxdy \leq \iint_D f(x,y)\,dxdy \leq \iint_D 1\cdot dxdy \Leftrightarrow$

$\Leftrightarrow \frac{1}{7}\mu(D)\leq \iint_D f(x,y)\,dxdy\leq 1\cdot\mu(D)$, $\mu(D)=|D|$,

$\Leftrightarrow \frac{4}{7}\leq \iint_D f(x,y)\,dxdy\leq 4$.

b)



$f(0,0)=\frac{1}{3}$, $f(1,0)=\frac{1}{4}$, $f(0,1)=\frac{1}{2}$, $f(2,0)=\frac{1}{7}$,

$f(0,2)=1$, $f(1,1)=\frac{1}{3}$, $f(1,2)=\frac{1}{2}$, $f(2,1)=\frac{1}{6}$, $f(2,2)=\frac{1}{5}$.

$x\in D_1 \Rightarrow \frac{1}{4}\leq f(x)\leq \frac{1}{2}$; $x\in D_2\Rightarrow \frac{1}{7}\leq f(x)\leq \frac{1}{3}$;

$x\in D_3 \Rightarrow \frac{1}{6}\leq f(x)\leq \frac{1}{2}$; $x\in D_4\Rightarrow \frac{1}{3}\leq f(x)\leq 1$;

$\underline{\frac{1}{4}\leq \iint_{D_1} f\leq \frac{1}{2}}$; $\underline{\frac{1}{7}\leq \iint_{D_2} f\leq \frac{1}{2}}$; $\underline{\frac{1}{6}\leq \iint_{D_3} f\leq \frac{1}{2}}$, $\underline{\frac{1}{3}\leq \iint_{D_4} f\leq 1}$.

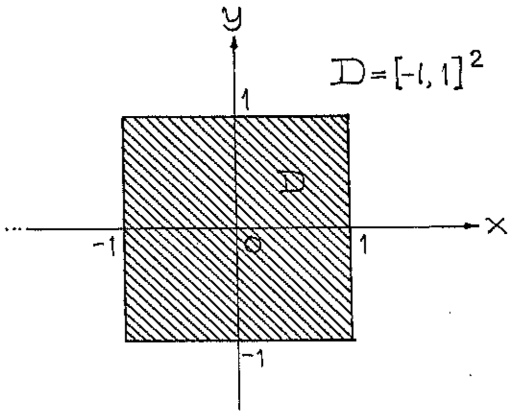
$$\frac{1}{4}+\frac{1}{7}+\frac{1}{6}+\frac{1}{3}\leq \left(\iint_{D_1}+\iint_{D_2}+\iint_{D_3}+\iint_{D_4}\right)f\leq \frac{1}{2}+\frac{1}{3}+\frac{1}{2}+1 \iff$$

$$\iff \underline{\frac{25}{28}\leq \iint_D f(x,y)\,dxdy\leq \frac{7}{3}}.$$

Problem 6.2 (Sid. 15)

Lösning

a)



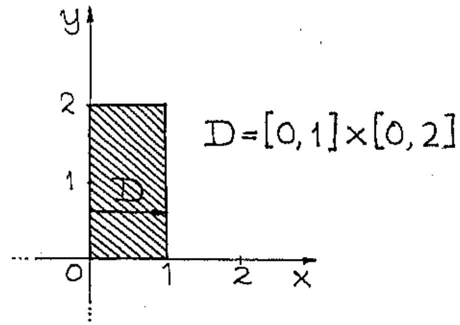
$$\iint_D (x+y)^2\,dxdy=\int_{-1}^{1}dx\int_{-1}^{1}(x+y)^2dy=\int_{-1}^{1}\left(\left[\frac{(x+y)^3}{3}\right]_{-1}^{1}\right)dx=$$

$$=\frac{1}{3}\int_{-1}^{1}((x+1)^3-(x-1)^3)\,dx=\frac{1}{3}\int_{-1}^{1}(6x^2+2)\,dx=\frac{1}{3}\left[2x^3+2x\right]_{-1}^{1}=$$

$$=\frac{1}{3}(4+4)=\frac{8}{3}.$$

Anm $\iint_D f(x,y)\,dxdy=\int_a^b dx\int_c^d f(x,y)\,dy.$

b)



$$\iint_D \frac{dxdy}{1+x+y}=\int_0^2 dy\int_0^1\frac{dx}{1+x+y}=\int_0^2\left(\left[\ln(1+x+y)\right]_{x=0}^{x=1}\right)dy=$$

$$=\int_0^2(\ln(2+y)-\ln(1+y))\,dy=$$

$$=\left[(2+y)\ln(2+y)-(1+y)\ln(1+y)\right]_0^2=$$

$$=4\ln4-3\ln3-2\ln2+1\ln1=$$

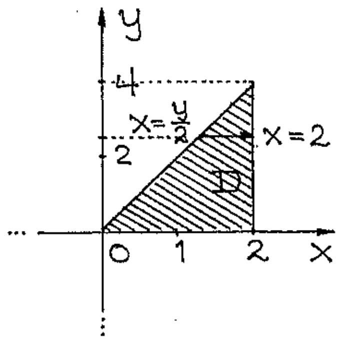
$$=8\ln2-3\ln3-2\ln2=$$

$$=6\ln2-3\ln3.$$

Problem 6.3 (Sid. 15)

Lösning

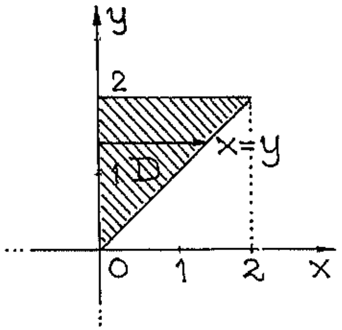
a) $\underline{f(x,y)=xy+y^2,\ D: 0\leq y\leq 2x\leq 4.}$



forts

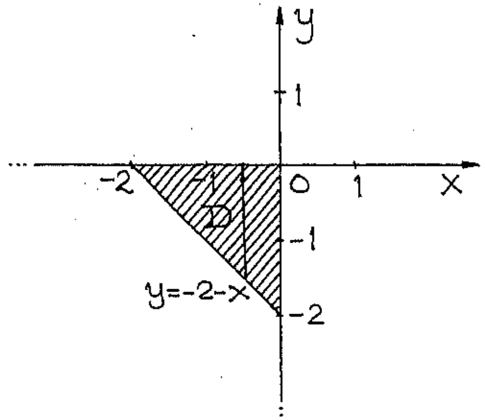
$$\iint_D f(x,y)\,dx\,dy=\int_0^4 dy\int_{y/2}^{2}(xy+y^2)\,dx=$$
$$=\int_0^4\left(\left[\frac{x^2}{2}y+y^2x\right]_{x=\frac{y}{2}}^{x=2}\right)dy=$$
$$=\int_0^4\left(2y+2y^2-\frac{y^3}{8}-\frac{y^3}{2}\right)dy=$$
$$=\int_0^4\left(2y+2y^2-\frac{5}{8}y^3\right)dy=$$
$$=\left[y^2+\frac{2}{3}y^3-\frac{5}{32}y^4\right]_0^4=$$
$$=16\left(1+\frac{8}{3}-\frac{5}{2}\right)=16\cdot\frac{7}{6}=\underline{\underline{\frac{56}{3}}}$$

b) $f(x,y)=(y-x)e^{x+y}$; $D: 0\le x\le y\le 2.$



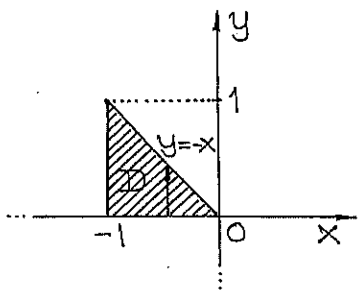
$$\iint_D f(x,y)\,dx\,dy=\int_0^2 dy\int_0^y (y-x)e^{x+y}\,dx=$$
$$=\int_0^2 dy\,e^y\int_0^y (y-x)e^x\,dx=$$
$$=\int_0^2\left(\left[(y-x+1)e^x\right]_{x=0}^{x=y}\right)e^y\,dy=$$
$$=\int_0^2\left(e^y-(y+1)e^0\right)e^y\,dy=$$
$$=\int_0^2\left(e^{2y}-(y+1)e^y\right)dy=$$
$$=\left[\frac{1}{2}e^{2y}-ye^y\right]_0^2=\underline{\frac{e^4-1}{2}-2e^2}.$$

c) $f(x,y)=2+x+y$; $D: x+y\ge -2,\ x\le 0,\ y\le 0.$



$$\iint_D f(x,y)\,dx\,dy=\int_{-2}^{0}dx\int_{-2-x}^{0}(2+x+y)\,dy=$$
$$=\int_{-2}^{0}\left(\left[(2+x)y+\frac{y^2}{2}\right]_{-2-x}^{y=0}\right)dx=$$
$$=\int_{-2}^{0}\left((2+x)^2-\frac{1}{2}(x+2)^2\right)dx=$$
$$=\int_{-2}^{0}\frac{1}{2}(x+2)^2\,dx=\frac{2^3}{6}=\frac{4}{3}.$$

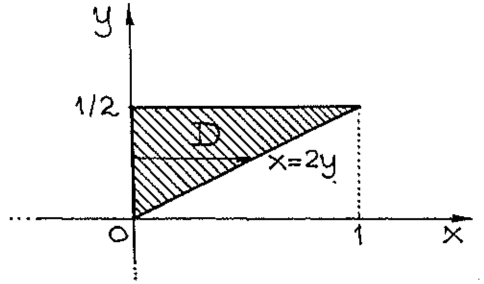
d) $f(x,y)=e^{x^2}$; $D: 0\le y\le -x,\ -1\le x\le 0.$



forts

$$\iint_D f(x,y)\,dx\,dy = \int_{-1}^{0} dx\, e^{x^2} \int_0^{-x} dy = \int_{-1}^{0} (-x) e^{x^2} dx =$$
$$= \left[-\tfrac{1}{2} e^{x^2}\right]_{-1}^{0} = \tfrac{1}{2}(e-1).$$

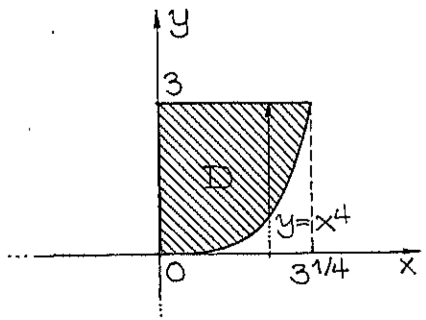
e) $f(x,y) = \dfrac{x^3}{1+y^5}$; $D: 0 \le x \le 2y \le 1.$



$$\iint_D \frac{x^3}{1+y^5}\,dx\,dy = \int_0^{1/2} dy\, \frac{1}{1+y^5} \int_{x=0}^{2y} x^3\,dx = \int_0^{1/2} \left(\frac{1}{1+y^5} \left[\frac{x^4}{4}\right]_0^{2y} \right) dy =$$
$$= \int_0^{1/2} \frac{4y^4}{1+y^5}\,dy =$$
$$= 4\int_0^{1/2} \frac{y^4}{1+y^5}\,dy = \left[\begin{array}{l|l} t = 1+y^5 & y = 1/2 \Rightarrow t = \frac{33}{32} \\ dt = 5y^4 dy & y = 0 \Rightarrow t = 1 \end{array} \right] =$$
$$= 4 \cdot \frac{1}{5} \int_1^{33/32} \frac{dt}{t} = \frac{4}{5} \ln \frac{33}{32}.$$

Problem 6.4 (Sid. 15)

Lösning

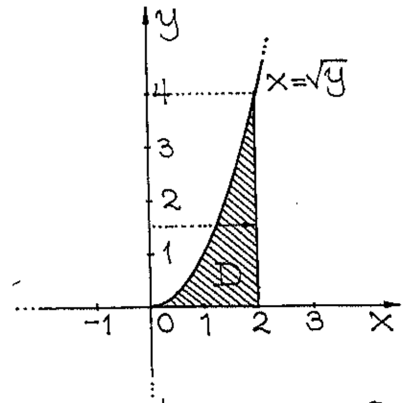


forts

$f(x,y) = x^3 y$, $D: 0 \le x \le 3^{1/4},\ x^4 \le y \le 3.$

$$\iint_D f(x,y)\,dx\,dy = \int_0^{3^{1/4}} dx\, x^3 \int_{x^4}^{3} y\,dy =$$
$$= \int_0^{3^{1/4}} \left(\left[\frac{y^2}{2}\right]_{y=x^4}^{y=3} \right) x^3\,dx =$$
$$= \frac{1}{2} \int_0^{3^{1/4}} (9 - x^8) x^3\,dx =$$
$$= \frac{1}{2} \int_0^{3^{1/4}} (9x^3 - x^{11})\,dx =$$
$$= \frac{1}{2} \left[\frac{9}{4} x^4 - \frac{x^{12}}{12} \right]_0^{3^{1/4}} = \frac{1}{2}\left(\frac{9}{4} \cdot 3 - \frac{3^3}{12} \right) =$$
$$= \frac{1}{2}\left(\frac{27}{4} - \frac{27}{12} \right) = \frac{1}{2} \frac{27}{4} \left(1 - \frac{1}{3}\right) = \frac{9}{4}.$$

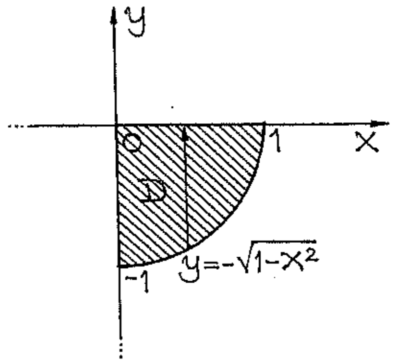
b) $f(x,y) = x \cos\sqrt{y}$; $D = \{(x,y) : \sqrt{y} \le x \le 2\}.$



$$\iint_D f(x,y)\,dx\,dy = \int_0^4 dy \cos\sqrt{y} \int_{\sqrt{y}}^{2} x \cdot dx =$$
$$= \int_0^4 \left(\left[\tfrac{1}{2} x^2 \right]_{\sqrt{y}}^{2} \right) \cos\sqrt{y}\,dy =$$
$$= \frac{1}{2} \int_0^4 (4-y) \cos\sqrt{y}\,dy = \left[\begin{array}{l|l} y = t^2 & y = 4 \Rightarrow t = 2 \\ dy = 2t\,dt & y = 0 \Rightarrow t = 0 \end{array} \right] =$$

$$= \frac{1}{2}\int_0^2 (4-t^2)\cos t\cdot 2t\,dt = \int_0^2 (4t-t^3)\cos t\,dt =$$
$$= \left[(4t-t^3)\sin t\right]_0^2 + \int_0^2 (3t^2-4)\sin t\,dt =$$
$$= \left[(4-3t^2)\cos t\right]_0^2 + 6\int_0^2 t\cos t\,dt =$$
$$= -8\cos 2 - 4 + 6\left[t\sin t\right]_0^2 - 6\int_0^2 \sin t\,dt =$$
$$= -8\cos 2 - 4 + 12\sin 2 + 6(\cos 2 - 1) =$$
$$= 12\sin 2 - 2\cos 2 - 10.$$

c)



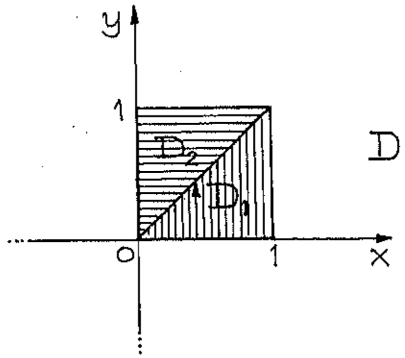
$f(x,y)=xy$, $D: -\sqrt{1-x^2}\le y\le 0,\ 0\le x\le 1.$

$$\iint_D f(x,y)\,dx\,dy = \int_0^1 dx\, x\int_{-\sqrt{1-x^2}}^{0} y\,dy =$$
$$= \int_0^1 \left(\left[\tfrac{1}{2}y^2\right]_{-\sqrt{1-x^2}}^{0}\right) x\,dx =$$
$$= \int_0^1 \tfrac{1}{2}(-x)(1-x^2)\,dx =$$
$$= \frac{1}{2}\int_0^1 (x^3-x)\,dx = \frac{1}{2}\left(\frac{1}{4}-\frac{1}{2}\right) =$$
$$= -1/8.$$

Problem 6.5 (Sid. 15)

Lösning

a) $D_1 = \{(x,y)\in D:\ y\le x\}$, $D_2 = \{(x,y)\in D:\ y\ge x\}$.



$D = D_1 \cup D_2$.

$$f(x,y) = |x-y| = \begin{cases} x-y, & (x,y)\in D_1 \\ y-x, & (x,y)\in D_2 \end{cases};$$

$$\iint_D |x-y|\,dx\,dy = \iint_{D_1}(x-y)\,dx\,dy + \iint_{D_2}(y-x)\,dx\,dy =$$
$$= \int_0^1 dx\int_0^x (x-y)\,dy + \int_0^1 dy\int_0^y (y-x)\,dx =$$
$$= \int_0^1\left(\left[-\tfrac{1}{2}(x-y)^2\right]_0^x\right)dx + \int_0^1\left(\left[-\tfrac{1}{2}(y-x)^2\right]_0^y\right)dy =$$
$$= \int_0^1 \tfrac{1}{2}x^2\,dx + \int_0^1 \tfrac{1}{2}y^2\,dy = \int_0^1 x^2\,dx = \left[\tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{3}.$$

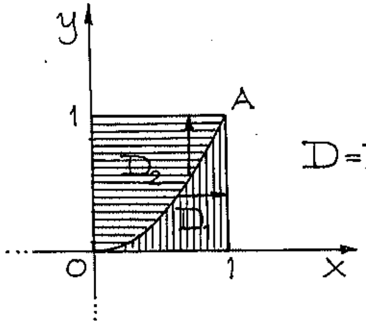
Anm. $f(x,y)=f(y,x)$, dvs man kan räkna över D_1 och multiplicera med 2.

b) $f(x,y)=\max\{x^2, y\}$; $D=[0,1]^2$.

$$f(x,y)=\max\{x^2,y\}=\begin{cases} y, & y\geq x^2 \\ x^2, & y\leq x^2 \end{cases};$$

$\overset{\frown}{OA}: y=x^2$

$\overset{\frown}{OA}: x=\sqrt{y}$



$D=D_1\cup D_2$

(1) $D=\{(x,y)\in\mathbb{R}^2:\ 0\leq x\leq 1,\ 0\leq y\leq 1\}=[0,1]\times[0,1]=[0,1]^2.$

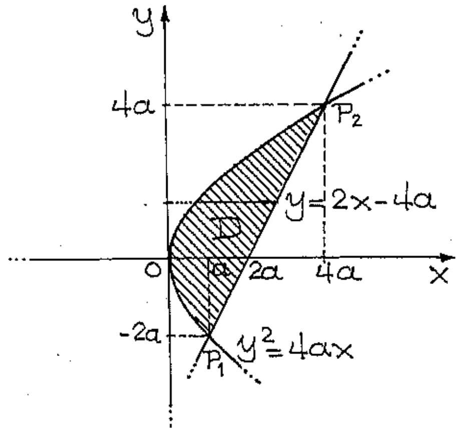
$D_1=\{(x,y)\in D:\ y\leq x^2\}$, $D_2=\{(x,y)\in D:\ y\geq x^2\}$.

$$\iint_D f(x,y)\,dxdy=\iint_{D_1}x^2\,dxdy+\iint_{D_2}y\,dxdy=$$
$$=\int_0^1 dy\int_{\sqrt{y}}^1 x^2\,dx+\int_0^1 dx\int_{x^2}^1 y\,dy=$$
$$=\int_0^1\left(\left[\tfrac{1}{3}x^3\right]_{\sqrt{y}}^1\right)dy+\int_0^1\left(\left[\tfrac{1}{2}y^2\right]_{x^2}^1\right)dx=$$
$$=\int_0^1\tfrac{1}{3}(1-y^{3/2})\,dy+\int_0^1\tfrac{1}{2}(1-x^4)\,dx=$$
$$=\tfrac{1}{3}\left[y-\tfrac{2}{5}y^{5/2}\right]_0^1+\tfrac{1}{2}\left[x-\tfrac{1}{5}x^5\right]_0^1=$$
$$=\tfrac{1}{3}(1-\tfrac{2}{5})+\tfrac{1}{2}(1-\tfrac{1}{5})=\tfrac{1}{3}\tfrac{3}{5}+\tfrac{1}{2}\tfrac{4}{5}=\tfrac{3}{5}.$$

Problem 6.6 (Sid. 15)

Lösning

$D=\{(x,y):\ y\geq 2x-4a,\ 4ax\geq y^2\}$. (Se figur.)



$$D=\left\{(x,y):\ \frac{y^2}{4a}\leq x\leq\frac{y+4a}{2}\right\}.$$

Anm. y-koordinaterna lika $\Rightarrow 4ax=y^2=(2x-4a)^2$

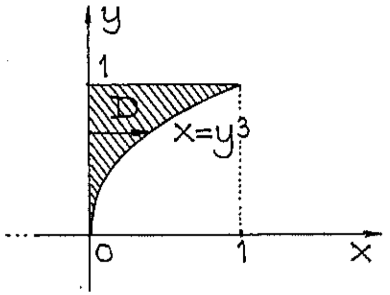
$\Leftrightarrow x=a \lor x=4a \Rightarrow P_1:(a,-2a),\ P_2:(4a,4a).$

$$\iint_D xy\,dxdy=\int_{-2a}^{4a}dy\,y\int_{y^2/4a}^{y/2+2a}x\,dx=\int_{-2a}^{4a}\left(\left[\frac{x^2}{2}\right]_{y^2/4}^{\frac{y}{2}+2a}\right)y\,dy=$$
$$=\frac{1}{2}\int_{-2a}^{4a}\left(\frac{1}{4}(y+4a^2)^2-\frac{y^4}{16a^2}\right)y\,dy=$$
$$=\frac{1}{8}\int_{-2a}^{4a}\left(y^3+8ay^2+16a^2y-\frac{y^5}{4a^2}\right)dy=$$
$$=\frac{1}{8}\left[\frac{y^4}{4}+\frac{8ay^3}{3}+8a^2y^2-\frac{y^6}{24a^2}\right]_{-2a}^{4a}=$$
$$=\frac{1}{8}\cdot 180a^4=22{,}5a^4.$$

Problem 6.7 (Sid. 15)

Lösning

a) $f(x,y)=1/\sqrt{1+y^8}$; $D:\ \sqrt[3]{x}\leq y\leq 1,\ 0\leq x\leq 1$.

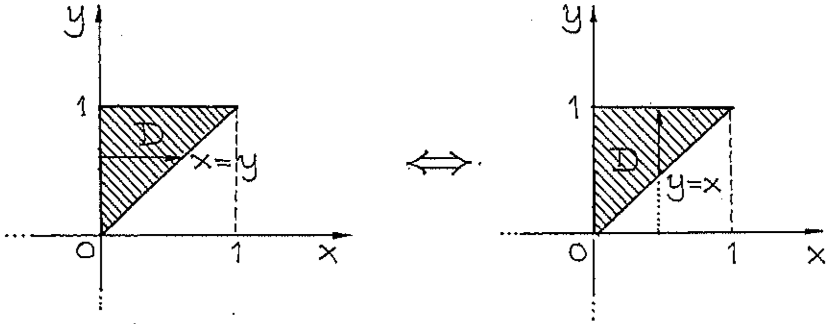


$$\iint_D f(x,y)\,dx\,dy = \int_0^1 dy\,\frac{1}{\sqrt{1+y^8}}\int_0^{y^3} dx =$$

$$= \int_0^1 \frac{y^3}{\sqrt{1+y^8}}\,dy = \left[\begin{array}{l|l} t=y^4 & 1\to 1 \\ dt=4y^3dy & 0\to 0 \end{array}\right] =$$

$$= \frac{1}{4}\int_0^1 \frac{dt}{\sqrt{1+t^2}} = \underline{\frac{1}{4}\ln(1+\sqrt{2})}.$$

b) $f(x,y) = y/(4-x^2-y^2)^{3/2} = \frac{d}{dy}\frac{1}{\sqrt{4-x^2-y^2}}$;



$$\iint_D f(x,y)\,dx\,dy = \int_0^1 \left(\int_0^y \frac{y\,dx}{(4-x^2-y^2)^{3/2}}\right)dy$$

$$= \int_0^1 dx \int_x^1 \frac{y}{(4-x^2-y^2)^{3/2}}\,dy =$$

$$= \int_0^1 \left(\left[\frac{1}{\sqrt{4-x^2-y^2}}\right]_{y=x}^{y=1}\right)dx =$$

$$= \int_0^1 \left(\frac{1}{\sqrt{3-x^2}} - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2-x^2}}\right)dx =$$

$$= \left[\arcsin\frac{x}{\sqrt{3}} - \frac{1}{\sqrt{2}}\arcsin\frac{x}{\sqrt{2}}\right]_0^1 =$$

$$= \arcsin\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{2}}\arcsin\frac{1}{\sqrt{2}} =$$

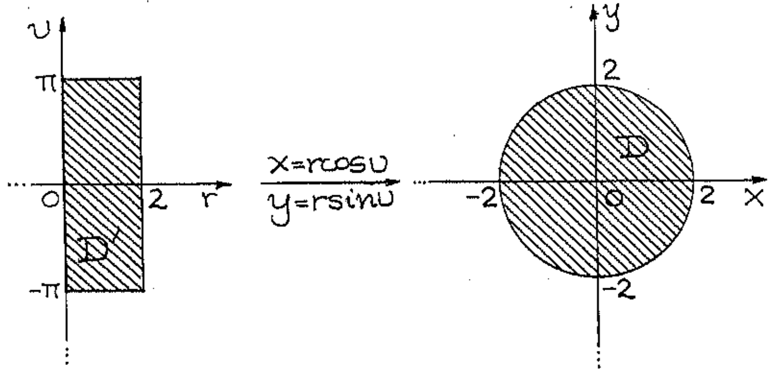
$$= \arcsin\frac{1}{\sqrt{3}} - \frac{\pi}{4\sqrt{2}}.$$

Problem 6.8 (Sid. 15)

Lösning

a) $\underline{f(x,y) = e^{x^2+y^2};\ D = \{(x,y): x^2+y^2 \le 4\}.}$

$$\begin{cases} x = r\cos v \\ y = r\sin v \end{cases} \Rightarrow dx\,dy \overset{!}{=} \frac{d(x,y)}{d(r,v)}\,dr\,dv = r\,dr\,dv;\quad D': \begin{cases} 0 \le r \le 2 \\ -\pi \le v \le \pi \end{cases}$$

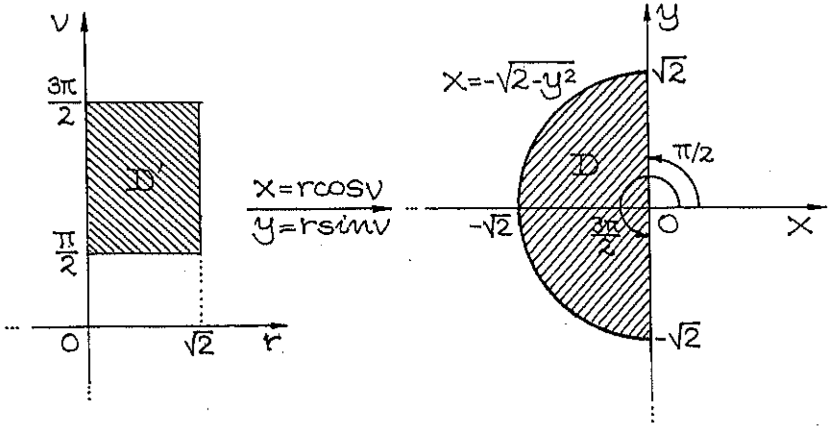


$$\iint_D e^{x^2+y^2}\,dx\,dy = \iint_{D'} e^{r^2} r\,dr = \int_0^2 re^{r^2}dr \int_{-\pi}^{\pi} dv =$$

$$= \left[\frac{1}{2}e^{r^2}\right]_0^2 \cdot \left[v\right]_{-\pi}^{\pi} = \frac{1}{2}(e^4-1)\cdot 2\pi =$$

$$= \pi(e^4-1).$$

b) $D=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le 2,\ x\le 0\}$; $f(x,y)=\dfrac{x}{1+(x^2+y^2)^{3/2}}$.

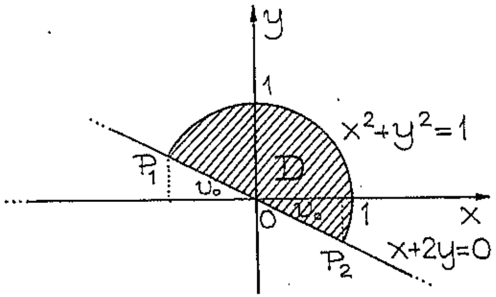


$$\begin{cases} x=r\cos v \\ y=r\sin v \end{cases} \Rightarrow dxdy \longrightarrow \frac{d(x,y)}{d(r,v)}drdv = r\,dr\,dv;\quad D': \begin{cases} 0\le r\le\sqrt{2} \\ \frac{\pi}{2}\le v\le\frac{3\pi}{2} \end{cases}$$

$$\iint_D f(x,y)\,dxdy = \iint_{D'} \frac{r^2\cos v}{1+r^3}\,drdv = \int_0^{\sqrt{2}}\frac{r^2}{1+r^3}dr\int_{\pi/2}^{3\pi/2}\cos v\,dv =$$

$$= \left[\tfrac{1}{3}\ln(1+r^3)\right]_0^{\sqrt{2}}\cdot\left[\sin v\right]_{\pi/2}^{3\pi/2} = \tfrac{1}{3}\ln(1+\sqrt{8})\cdot 2 = \underline{\tfrac{2}{3}\ln(1+\sqrt{8})}.$$

c) $$\begin{cases} x^2+y^2=1 \\ x+2y=0 \end{cases} \Leftrightarrow \begin{cases} 5y^2=1 \\ x=-2y \end{cases} \Leftrightarrow \begin{cases} y=\pm\frac{1}{\sqrt{5}} \\ x=-2y \end{cases} \Leftrightarrow \begin{cases} P_1 : (-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}) \\ P_2 : (\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}) \end{cases};$$



$$\begin{cases} x=r\cos v \\ y=r\sin v \end{cases} \Rightarrow \begin{cases} P_1 = (\arccos(-\frac{2}{\sqrt{5}}),\ \arcsin\frac{1}{\sqrt{5}}) \\ P_2 = (\arccos\frac{2}{\sqrt{5}},\ \arcsin(-\frac{1}{\sqrt{5}})) \end{cases};$$

$$\iint_D (x+2y)\,dxdy = \left[\begin{array}{c|c} x=r\cos v & 0\le r\le 1 \\ y=r\sin v & v_0\le v\le v_0+\pi \end{array}\right] =$$

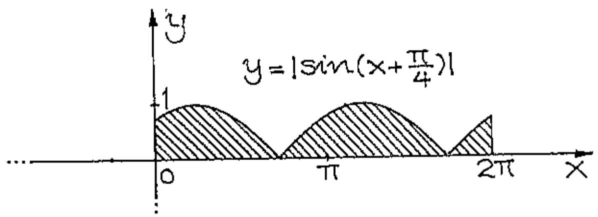
$$= \int_0^1 r^2dr\int_{v_0}^{v_0+\pi}(\cos v+2\sin v)\,dv = \tfrac{1}{3}\left[\sin v-2\cos v\right]_{v_0}^{v_0+\pi} =$$

$$= \tfrac{1}{3}(\sin(v_0+\pi)-\sin v_0 - 2(\cos(v_0+\pi)-\cos v_0)) =$$

$$= \tfrac{1}{3}(4\cos v_0 - 2\sin v_0) = \tfrac{1}{3}(4\cdot\tfrac{2}{\sqrt{5}} - 2\cdot\tfrac{1}{\sqrt{5}}) = \tfrac{1}{3}\,\tfrac{6}{\sqrt{5}} = \tfrac{2}{\sqrt{5}}.$$

d) $$\iint_D |x+y|\,dxdy = \left[\begin{array}{c|c} x=r\cos v & 0\le r\le 1 \\ y=r\sin v & 0\le v\le 2\pi \end{array}\right] = \int_0^1 r^2dr\int_0^{2\pi}|\cos v+\sin v|\,dv$$

$$= \left[\tfrac{1}{3}r^3\right]_0^1\cdot\int_0^{2\pi}\sqrt{2}\,|\sin(v+\tfrac{\pi}{4})|\,dv = \tfrac{1}{3}\cdot\sqrt{2}\cdot 4 = \tfrac{4\sqrt{2}}{3}$$ (se figur).



Det skuggade området har arean 4 (ae).

Problem 6.9 (Sid. 15)

Lösning

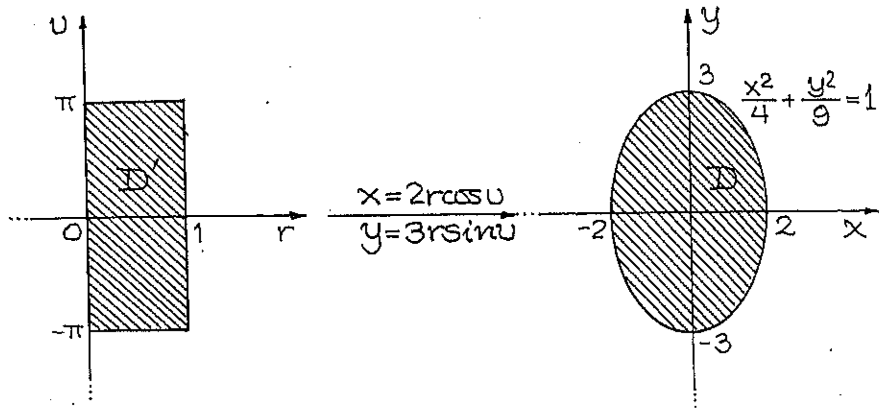
a) $$\begin{cases} x=r\cos v \\ y=r\sin v \end{cases} \Rightarrow (x^2+y^2\le 2x) \Rightarrow \underbrace{r\le 2\cos v,\ -\tfrac{\pi}{2}\le v\le\tfrac{\pi}{2}}_{D'}.$$

$$\iint (x^2+y^2)\,dxdy = \iint_{D'} r^3\,dr\,dv = \int_{-\pi/2}^{\pi/2} dv \int_0^{2\cos v} r^3\,dr =$$
$$= \int_{-\pi/2}^{\pi/2} \left(\left[\frac{r^4}{4}\right]_0^{2\cos v}\right) dv =$$
$$= 4\int_{-\pi/2}^{\pi/2} \cos^4 v\,dv =$$
$$= 8\int_0^{\pi/2} \cos^4 v\,dv =$$
$$= \int_0^{\pi/2} (3+4\cos 2v+\cos 4v)\,dv = \frac{3\pi}{2}.$$

b) $$\iint_D f(x,y)\,dxdy = \iint \sqrt{x^2+y^2}\,dxdy = \iint_{D'} r\cdot r\,dr\,dv =$$
$$= \int_{-\pi/2}^{\pi/2} dv \int_0^{2\cos v} r^2\,dr = \int_{-\pi/2}^{\pi/2} \left(\left[\frac{r^3}{3}\right]_0^{2\cos v}\right) dv =$$
$$= \frac{8}{3}\int_{-\pi/2}^{\pi/2} \cos^3 v\,dv = \frac{16}{3}\int_0^{\pi/2} (1-\sin^2 v)\cos v\,dv = \begin{bmatrix} u = \sin v \\ du = \cos v\,dv \end{bmatrix}$$
$$= \frac{16}{3}\int_0^1 (1-t^2)\,dt = \frac{16}{3}\cdot\frac{2}{3} = \frac{32}{9}.$$

Övning 6.10 (Sid. 16)

Lösning: a) $f(x,y) = x^2+y^2$; $D: \frac{x^2}{4}+\frac{y^2}{9} \le 1.$



$$\begin{cases} x = 2r\cos\varphi \\ y = 3r\sin\varphi \end{cases} \Rightarrow x^2+y^2 = \frac{1}{2}r^2(13+5\cos 2v) \wedge \frac{d(x,y)}{d(r,\varphi)} = 6r.$$

$$\iint_D (x^2+y^2)\,dxdy = \iint_{D'} 3r^3(13+5\cos 2\varphi)\,dr\,d\varphi =$$
$$= 3\int_0^1 r^3\,dr \int_0^{2\pi} (13+5\cos 2\varphi)\,d\varphi = \left[\frac{3}{4}r^4\right]_0^1 \cdot \left[13\varphi+\frac{5}{2}\sin 2\varphi\right]_0^{2\pi} = \frac{39\pi}{2}.$$

b) $f(x,y) = x^2$; $D = \{(x,y):\ 1 \le x^2+9y^2 \le 9,\ x-3y \ge 0\}.$

$$\begin{cases} x = u \\ y = v/3 \end{cases} \Rightarrow f(u, \tfrac{v}{3}) = u^2;\ dxdy \to \frac{d(x,y)}{d(u,v)} = \frac{1}{3}\,du\,dv.$$

$D' := \{(u,v):\ 1 \le u^2+v^2 \le 9,\ u-v \ge 0\}.$

$$\iint_D x^2\,dxdy = \iint_{D'} u^2\frac{1}{3}\,du\,dv \overset{!}{=} \left[\begin{array}{l|l} u = r\cos\varphi & 1 \le r \le 3 \\ v = r\sin\varphi & -3\pi/4 \le \varphi \le \pi/4 \end{array}\right] =$$
$$= \frac{1}{3}\int_1^3 r^3\,dr \int_{-3\pi/4}^{\pi/4} \cos^2\varphi\,d\varphi =$$
$$= \frac{1}{3}\left[\frac{r^4}{4}\right]_1^3 \cdot \left[\frac{1}{2}\left(\varphi+\frac{1}{2}\sin 2\varphi\right)\right]_{-3\pi/4}^{\pi/4} =$$
$$= \frac{1}{3}\cdot\frac{3^4-1}{4}\cdot\frac{1}{2}(\pi+0) = \frac{10\pi}{3}.$$

I $\overset{!}{=}$ underförstås följande:

$$\begin{cases} u = r\cos\varphi \\ v = r\sin\varphi \end{cases} \Rightarrow \begin{cases} 1 \le u^2+v^2 \le 9 \Leftrightarrow 1 \le r^2 \le 9 \Leftrightarrow 1 \le r \le 3 \\ v \le u \Leftrightarrow \sin\varphi \le \cos\varphi \Leftrightarrow -\frac{3\pi}{4} \le \varphi \le \frac{\pi}{4} \end{cases}$$

c) $f(x,y) = x$; $D = \{(x,y) \in \mathbb{R}^2:\ x^2+2xy+4y^2 \le 1,\ x,y \ge 0\}.$

(1) $x^2+2xy+4y^2 = (x+y)^2+3y^2 = (x+y)^2+(\sqrt{3}y)^2.$

(2) $$\begin{cases} u = x+y \\ v = \sqrt{3}y \end{cases} \Rightarrow \frac{d(u,v)}{d(x,y)} = \sqrt{3} = \left(\frac{d(x,y)}{d(u,v)}\right)^{-1} \Leftrightarrow \frac{d(x,y)}{d(u,v)} = \frac{1}{\sqrt{3}}.$$

(3) $\begin{cases} x+y=u \\ \sqrt{3}y=v \end{cases} \Leftrightarrow \begin{cases} x=u-v/\sqrt{3} \\ y=v/\sqrt{3} \end{cases} \Rightarrow \left\{\begin{matrix} x\geq 0 \Rightarrow v\leq\sqrt{3}u \\ y\geq 0 \Rightarrow v\geq 0 \end{matrix}\right\} \Rightarrow$

$$\Rightarrow \underline{D'=\{(u,v):\ u^2+v^2\leq 1,\ v\leq\sqrt{3}u,\ v\geq 0\}.}$$

(4) $\begin{cases} u=r\cos\varphi \\ v=r\sin\varphi \end{cases} \Rightarrow \begin{cases} v\leq\sqrt{3}u \Rightarrow \sin\varphi\leq\sqrt{3}\cos\varphi \Rightarrow \tan\varphi\leq\sqrt{3} \\ v\geq 0 \Rightarrow \sin\varphi\geq 0 \Leftrightarrow \varphi\geq 0 \end{cases}$

$$\Rightarrow \underline{D''=\{(r,\varphi):\ 0\leq r\leq 1,\ 0\leq\varphi\leq\pi/3\}.}$$

$$\iint_D x\,dxdy=\iint_{D'}\left(u-\frac{v}{\sqrt{3}}\right)\left|\frac{d(x,y)}{d(u,v)}\right|dudv=$$
$$=\iint_{D'}\left(u-\frac{v}{\sqrt{3}}\right)\frac{1}{\sqrt{3}}dudv=$$
$$=\iint_{D''}\frac{r}{\sqrt{3}}\left(\cos\varphi-\frac{\sin\varphi}{\sqrt{3}}\right)r\,drd\varphi=$$
$$=\frac{1}{3}\int_0^1 r^2\,dr\int_0^{\pi/3}(\sqrt{3}\cos\varphi-\sin\varphi)\,d\varphi=$$
$$=\frac{1}{3}\left[\frac{r^3}{3}\right]_0^1\cdot\left[\sqrt{3}\sin\varphi+\cos\varphi\right]_0^{\pi/3}=$$
$$=\frac{1}{3}\cdot\frac{1}{3}\left(\frac{3}{2}+\frac{1}{2}-1\right)=\frac{1}{9}.$$

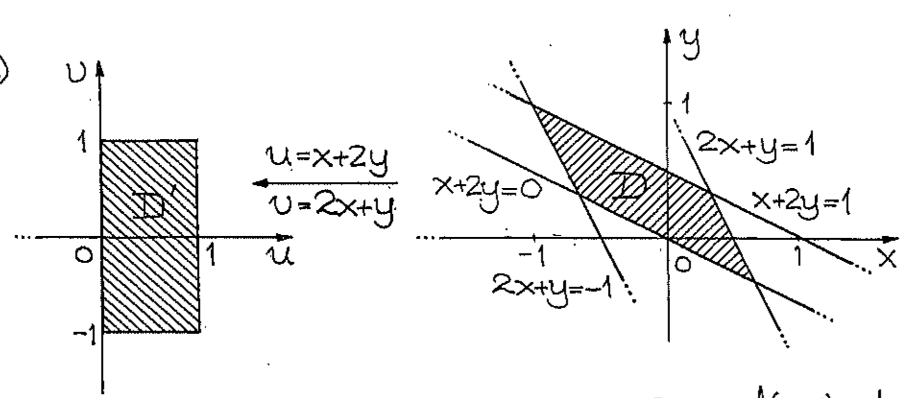
Problem 6.11 (Sid. 16)

Lösning

a) $\underline{f(x,y)=(x+2y)\cos(2x+y)}$; $\underline{D:\ 0\leq x-2y\leq 1,\ -1\leq 2x+y\leq 1.}$

(1) $\begin{cases} u=x+2y \\ v=2x+y \end{cases} \Rightarrow \left|\frac{d(u,v)}{d(x,y)}\right| = \left|\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}\right| = 3 \Rightarrow \left|\frac{d(x,y)}{d(u,v)}\right| = \frac{1}{3}$

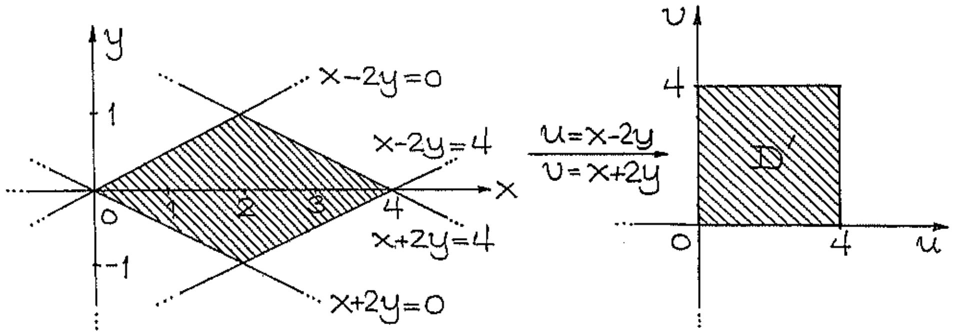
(2)



(3) $\iint_D (x+2y)\cos(2x+y)\,dxdy \left[\begin{matrix} u=x+2y \\ v=2x+y \end{matrix}\ \vdots\ \begin{matrix} \left|\frac{d(x,y)}{d(u,v)}\right|=\frac{1}{3} \\ D'=[0,1]\times[-1,1] \end{matrix}\right]$

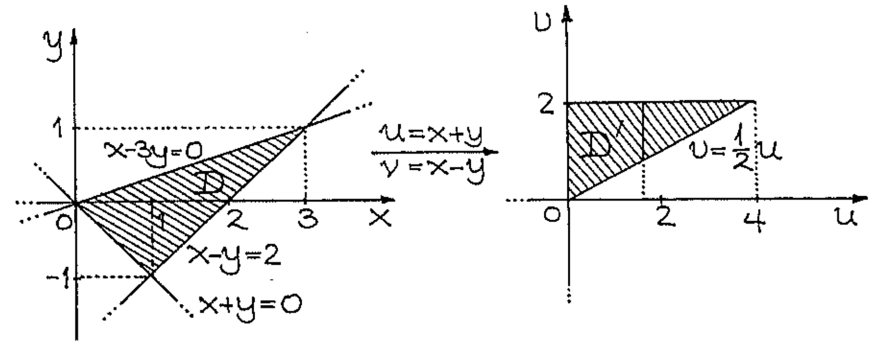
$$=\frac{1}{3}\iint_{D'} u\cdot\cos v\,dudv=\frac{1}{3}\int_0^1 u\,du\int_{-1}^1\cos v\,dv=$$
$$=\frac{1}{3}\left[\frac{u^2}{2}\right]_0^1\left[\sin v\right]_{-1}^1=\frac{1}{3}\cdot\frac{1}{2}(\sin 1-\sin(-1))=\frac{\sin 1}{3}.$$

b) $\underline{f(x,y)=(x+2y)e^{x-2y}}$; $\underline{D:\ 0\leq x-2y\leq 4,\ 0\leq x+2y\leq 4.}$



$$\iint_D (x+2y)e^{x-2y}\,dxdy=\iint_{D'} v e^u\left|\frac{d(x,y)}{d(u,v)}\right|dudv=$$
$$=\frac{1}{4}\iint_{D'} v e^u\,dudv=\frac{1}{4}\int_0^4 v\,dv\int_0^4 e^u\,du=2\cdot(e^4-1).$$

c)



$$D=\{(x,y):\ x-y\le 2,\ x+y\ge 0,\ x-3y\ge 0\}=$$
$$=\{(x,y):\ |x-1|-1\le y\le x/3\}.$$

$$\begin{cases} u=x+y \\ v=x-y \end{cases} \Rightarrow \underline{D'=\{(u,v):\ v\le 2,\ u\ge 0,\ v\ge \tfrac{1}{2}u\}.}$$

$$\frac{d(u,v)}{d(x,y)}=-2 \Rightarrow \left|\frac{d(x,y)}{d(u,v)}\right|=\left|\frac{d(u,v)}{d(x,y)}\right|^{-1}=\frac{1}{2};$$

$$\iint_D \frac{dxdy}{(1+x^2-y^2)^2}=\frac{1}{2}\iint_{D'}\frac{dudv}{(1+uv)^2}=\frac{1}{2}\int_0^4 du\int_{u/2}^2\frac{1}{(1+uv)^2}dv=$$
$$=\frac{1}{2}\int_0^4\left(\left[-\frac{1}{u}\,\frac{1}{1+uv}\right]_{v=u/2}^2\right)du=$$
$$=\frac{1}{2}\int_0^4\frac{1}{u}\left(\frac{2}{u^2+2}-\frac{1}{2u+1}\right)du=$$
$$\overset{!}{=}\frac{1}{2}\left[\ln\frac{2u+1}{\sqrt{u^2+2}}\right]_0^4=\frac{1}{2}\ln 3.$$

Anm. $\alpha\ne\beta \Rightarrow \dfrac{1}{(x-\alpha)(x-\beta)}=\dfrac{1}{\alpha-\beta}\left(\dfrac{1}{x-\alpha}-\dfrac{1}{x-\beta}\right)\Rightarrow(\text{jfr } \overset{!}{=})$

$$\Rightarrow \frac{1}{u}\left(\frac{2}{u^2+2}-\frac{1}{2u+1}\right)=\frac{2u}{u^2(u^2+2)}-\frac{2}{2u(2u+1)}=\frac{2u}{2}\left(\frac{1}{u^2}-\frac{1}{u^2+2}\right)-$$
$$-2\left(\frac{1}{2u}-\frac{1}{2u+1}\right)=\frac{2}{2u+1}-\frac{u}{u^2+2}.$$

Problem 6.12 (Sid. 16)

Lösning

$\underline{f(x,y)=y^2\sin y^2;\ D=\{(x,y)\in\mathbb{R}^2_+:\ 1\le xy\le 2,\ x\le y\le 2x\}.}$

(1) $0<x\le y\le 2x \Leftrightarrow 1\le \dfrac{y}{x}\le 2.$

(2) $u=xy \wedge v=y/x \Rightarrow \dfrac{d(u,v)}{d(x,y)}=2\dfrac{y}{x}=2v \Leftrightarrow \dfrac{d(x,y)}{d(u,v)}=\dfrac{1}{2v}$

(3) $\underline{D'=\{(u,v):\ 1\le u\le 2,\ 1\le v\le 2\}.}$

(4) $\displaystyle\iint_D y^2\sin y^2\,dxdy=\iint_{D'}(uv)\sin(u\cdot v)\cdot\frac{1}{2v}\,dudv=$

$$=\frac{1}{2}\int_1^2 du\int_1^2 u\cdot\sin(uv)dv=\frac{1}{2}\int_1^2\left(\left[-\cos(uv)\right]_{v=1}^{v=2}\right)du=$$
$$=\frac{1}{2}\int_1^2(\cos u-\cos 2u)du=\frac{1}{2}\left[\sin u-\frac{1}{2}\sin 2u\right]_1^2=$$
$$=\frac{1}{2}\left(\sin 2-\sin 1+\frac{\sin 2-\sin 4}{2}\right)=\underline{\frac{1}{3}(3\sin 2-2\sin 1-\sin 4).}$$

Problem 6.13 (Sid. 16)

Lösning

$$J(a)=\iint_{D(a)}\sin(x^2+y^2)dxdy=\left[\begin{array}{c|c} x=r\cos v & 0\le r\le\sqrt{a} \\ y=r\sin v & 0\le v\le\pi/2\end{array}\right]=$$
$$=\int_0^{\sqrt{a}} r\sin r^2dr\int_0^{\pi/2}dv=\frac{\pi}{4}\int_0^{\sqrt{a}}\sin r^2\,2rdr\left[\begin{array}{c} t=r^2 \\ dt=2rdr\end{array}\right]=$$
$$=\int_0^a\frac{\pi}{4}\sin t\,dt=\frac{\pi}{4}\left[-\cos t\right]_0^a=\frac{\pi}{2}\,\frac{1-\cos a}{2}=\frac{\pi}{2}\sin^2\left(\frac{a}{2}\right)\Rightarrow$$

$\Rightarrow \underline{J_{max}=\frac{\pi}{2}}$, för $a=(1+2k)\pi$.

Problem 6.14 (Sid. 16)

Lösning

$f(x,y)=x^2+y^2$; $E=\{(x,y): \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\}$. $\mu(E)=\pi ab$.

$$\iint_E (x^2+y^2)\,dx\,dy = \left[\begin{array}{l|l} x=ar\cos v & 0\le r\le 1 \\ y=br\sin v & 0\le v\le 2\pi \end{array}\right] =$$

$$=\int_0^1 abr^3dr\int_0^{2\pi}(a^2\cos^2 v+b^2\sin^2 v)\,dv =$$

$$= ab\left[\tfrac{1}{4}r^4\right]_0^1\cdot(a^2+b^2)\pi = \pi ab\cdot\frac{a^2+b^2}{4} =$$

$$=\frac{1}{2}\pi ab\cdot\frac{a^2+b^2}{2} \geq \frac{1}{2}\pi ab\cdot\left(\frac{a+b}{2}\right)^2 \overset{!}{\geq}$$

$$\geq \frac{1}{2}\pi ab\cdot ab = \frac{(\pi ab)^2}{2\pi} = \frac{\mu(E)^2}{2\pi} = \frac{A^2}{2\pi}.$$

Anm. För två positiva heltal a och b är
$\frac{a^2+b^2}{2}\geq\left(\frac{a+b}{2}\right)^2$ och $\frac{a+b}{2}\geq\sqrt{ab}$.

Trippelintegraler

Problem 6.15 (Sid. 16)

Lösning: $f(x,y,z)=\sqrt[3]{x}+y+\sqrt{z}$; $D: \begin{cases} 0\le x\le 1 \\ 0\le y\le 2 \\ 0\le z\le 4 \end{cases}$;

$f_{max}=f(1,2,4)=1+2+2=5 \Rightarrow \int_D f \le 5\cdot 8 = 40 < 50$.

8 är rätblockets volym; $I=50$ är alltså fel.

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_D (x^{1/3}+y+z^{1/2})\,dx\,dy\,dz =$$

$$=\int_0^1 dx\int_0^2 dy\int_0^4 (x^{1/3}+y+z^{1/2})\,dz =$$

$$=\int_0^1 dx\left(\int_0^2\left(\left[x^{1/3}z+yz+\tfrac{2}{3}z^{3/2}\right]_{z=0}^4\right)dy\right)=$$

$$=\int_0^1 dx\int_0^2 (4x^{1/3}+4y+\tfrac{16}{3})\,dy$$

$$=\int_0^1\left(\left[4x^{1/3}y+2y^2+\tfrac{16}{3}y\right]_{y=0}^2\right)dx=$$

$$=\int_0^1 (8x^{1/3}+8+\tfrac{32}{3})\,dx = 8\int_0^1 (x^{1/3}+\tfrac{7}{3})\,dx=$$

$$=8\left[\tfrac{3}{4}x^{4/3}+\tfrac{7}{3}x\right]_0^1 = 8\cdot\left(\tfrac{3}{4}+\tfrac{7}{3}\right)=\frac{37\cdot 8}{12}=\frac{74}{3}.$$

Problem 6.16 (Sid. 16)

Lösning

$f(x,y,z)=xyz$; $K: x^2+y^2\le 9,\ 0\le z\le 2$; $D: x^2+y^2\le 9$.

$$\iiint_K f(x,y,z)\,dx\,dy\,dz = \left(\int_0^2 z\,dz\right)\iint_D xy\,dx\,dy =$$

$$=2\int_{-3}^3 dx\, x\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} y\,dy = 0.$$

Problem 6.17 (Sid. 16)

Lösning

Den koniska ytans ekvation ges av $z=k\sqrt{x^2+y^2}$.

$P_0:(2,0,1)$ ligger på dess mantelyta, varför

$1=k\sqrt{0+2^2}=2k \Leftrightarrow \underline{k=1/2}.$

$\underline{f(x,y,z)=x^2+y^2,}$ $\underline{K:(1/2)\sqrt{x^2+y^2}\le z\le 1}$ $(D: x^2+y^2\le 4).$

Med $u=u(x,y)=\frac{1}{2}\sqrt{x^2+y^2}$ fås

$$\iiint_K f(x,y,z)\,dxdydz=\iint_D dxdy(x^2+y^2)\int_u^1 dz=$$

$$=\iint_D (x^2+y^2)\left(1-\tfrac{1}{2}\sqrt{x^2+y^2}\right)dxdy=\begin{bmatrix}x=r\cos v\\ y=r\sin v\end{bmatrix}=$$

$$=\int_0^2 r^2\left(1-\tfrac{1}{2}r\right)r\,dr\int_0^{2\pi}dv=\pi\int_0^2(2r^3-r^4)dr=$$

$$=\pi\left[\tfrac{1}{2}r^4-\tfrac{1}{5}r^5\right]_0^2=\pi\left(8-\tfrac{32}{5}\right)=\underline{\frac{8\pi}{5}}.$$

Problem 6.18 (Sid. 16)

Lösning $\quad\underline{K: 0\le\sqrt{1-x^2-y^2},\ D: x^2+y^2\le 1.}$

Jag inför rymdpolära (sfäriska) koordinater:

$$\underline{x=r\sin\theta\cos\varphi,\ y=r\sin\theta\sin\varphi,\ z=r\cos\theta};$$

$$D': \underline{0\le r\le 1,\ 0\le\theta\le\pi/2,\ 0\le\varphi\le 2\pi}$$

$$\underline{J(r,\theta,\varphi)=r^2\sin\theta.}$$

$$\iiint_K 3z\,dxdydz=\iiint_{D'}3r\cos\theta\cdot r^2\sin\theta\,drd\theta d\varphi=$$

$$=\int_0^1 3r\,dr\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta\int_0^{2\pi}d\varphi=$$

$$=\left[\frac{3r^4}{4}\right]_0^1\left[\tfrac{1}{2}\sin^2\theta\right]_0^{\pi/2}\left[\varphi\right]_0^{2\pi}=\underline{\frac{3\pi}{4}}.$$

forts

Anm. $x^2+y^2\le 1 \Rightarrow 0\le z\le 1 \Rightarrow 0\le f(x,y,z)\le 3 \Rightarrow$

$$\Rightarrow \iiint_K 0\,dV\le\iiint_K f(x)\,dV\le\iiint_K 3\,dV=3\cdot\frac{2\pi}{3}=2\pi.$$

Det är vad hon skulle ha i bakhuvudet innan svaret skulle avges.

Problem 6.19 (Sid. 16)

Lösning

Läs författarnas beskrivning på sidorna 29-30.

Problem 6.20 (Sid. 16)

Lösning

(1) $0\le z\le y\le x^2\le 1 \Leftrightarrow \begin{Bmatrix}0\le z\le y\\ y\le x^2\le 1\\ 0\le y\le 1\end{Bmatrix} \Leftrightarrow D: \begin{cases}0\le z\le y\\ \sqrt{y}\le x\le 1\\ 0\le y\le 1\end{cases};$

(2) $$\iiint_D f(x,y,z)\,dxdydz=\int_0^1 dy\int_{\sqrt{y}}^1 dx\frac{1}{x^2+1}\int_0^y z\,dz=$$

$$=\int_0^1 dy\int_{\sqrt{y}}^1\left(\left[\frac{z^2}{2}\right]_0^y\right)\frac{1}{x^2+1}dx=\frac{1}{2}\int_0^1 dy\,y^2\int_{\sqrt{y}}^1\frac{dx}{x^2+1}$$

$$=\frac{1}{2}\int_0^1\left(\left[\arctan x\right]_{\sqrt{y}}^1\right)y^2dy=$$

$$=\frac{1}{2}\int_0^1\left(\arctan\sqrt{y}-\frac{\pi}{4}\right)y^2dy=\left[\begin{array}{c|c}y=t^2 & 1\to 1\\ dy=2t\,dt & 0\to 0\end{array}\right]$$

$$=\frac{1}{2}\int_0^1\left(\arctan t-\frac{\pi}{4}\right)2t^5dt=$$

$$=\int_0^1\left(t^5\arctan t-\frac{\pi}{4}t^5\right)dt=\frac{\pi}{24}-\int_0^1 t^5\,\mathrm{atn}\,t\,dt=$$

$$=\frac{\pi}{24}-\left[\frac{t^6}{6}\operatorname{atn}t\right]_0^1+\frac{1}{6}\int_0^1\frac{t^6}{t^2+1}dt=$$

$$=\frac{\pi}{24}-\frac{\pi}{24}+\frac{1}{6}\int_0^1(t^4-t^2+1-\frac{1}{t^2+1})dt=$$

$$=\frac{1}{6}\left[\frac{t^5}{5}-\frac{t^3}{3}+t-\operatorname{atn}t\right]_0^1=\frac{1}{6}(\frac{1}{5}-\frac{1}{3}+1-\frac{\pi}{4})=$$

$$=\frac{52-15\pi}{360}.$$

Problem 6.21 (Sid. 16)

Lösning

$f(x,y,z)=(1+x+y+z)^{-3}$; $D: x+y+z\leq 1,\ x,y,z\geq 0$ (*)

(1) $x+y+z\leq 1 \Leftrightarrow 0\leq z\leq 1-x-y \Rightarrow 0\leq y\leq 1-x \Rightarrow 0\leq x\leq 1.$

$$\Rightarrow D:\ 0\leq x\leq 1,\ 0\leq y\leq 1-x,\ 0\leq z\leq 1-x-y.$$

(2) $$\iiint_D f(x,y,z)dxdydz=\int_0^1 dx\int_0^{1-x}dy\int_0^{1-x-y}\frac{dz}{(1+x+y+z)^3}=$$

$$=\int_0^1 dx\int_0^{1-x}(\left[-\frac{1}{2}\frac{1}{(1+x+y+z)^2}\right]_{z=0}^{1-x-y})dy=$$

$$=\int_0^1 dx\frac{1}{2}\int_0^{1-x}(\frac{1}{(1+x+y)^2}-\frac{1}{4})dy=$$

$$=\frac{1}{2}\int_0^1(\left[-\frac{1}{1+x+y}-\frac{y}{4}\right]_{y=0}^{1-x})dx=$$

$$=\frac{1}{2}\int_0^1(\frac{1}{x+1}+\frac{x-1}{4}-\frac{1}{2})dx=\frac{1}{2}\left[\ln(x+1)+\frac{(x-1)^2}{8}-\frac{x}{2}\right]_0^1=$$

$$=\frac{1}{2}(\ln 2+0-\frac{1}{2}-\frac{1}{8})=\frac{1}{2}(\ln 2-\frac{5}{8})=\frac{1}{2}\ln 2-\frac{5}{16}.$$

Problem 6.22 (Sid. 17)

Lösning: $f(x,y,z)=ye^z$, $D: x-y\leq z\leq x+y,\ |x|+|y|\leq 1.$

(1) $|x|+|y|\leq 1 \Leftrightarrow |y|\leq 1-|x| \Leftrightarrow -(1-|x|)\leq y\leq 1-|x| \wedge |x|\leq 1$

$$\Leftrightarrow \Delta:\ -(1-|x|)\leq y\leq 1-|x|,\ -1\leq x\leq 1.$$

(2) $$\iiint_D f(x)dV=\iint_\Delta dxdy\,y\int_{x-y}^{x+y}e^z dz=\iint_\Delta(e^y-e^{-y})e^x y\,dxdy$$

$$=\iint_\Delta 2e^x y\cdot\sinh y\,dxdy=2\int_{-1}^1 dx\,e^x\int_{|x|-1}^{1-|x|}y\sinh y\,dy=$$

$$=4\int_{-1}^1 dx\,e^x\int_0^{1-|x|}y\sinh y\,dy=4\int_{-1}^1 S(x)dx;$$

(3) $$S(x)=\int_0^{1-|x|}y\sinh y\,dy=[y\cosh y]_0^{1-|x|}-\int_0^{1-|x|}\cosh y\,dy=$$

$$=(1-|x|)\cosh(1-|x|)-\sinh(1-|x|).$$

(4) $$\iiint_D f(x)dV=4\int_{-1}^1 e^x((1-|x|)\cosh(1-|x|)-\sin(1-|x|))dx=$$

$$=4\int_{-1}^0 e^x((1+x)\cosh(1+x)-\sinh(1+x))dx+$$

$$+4\int_0^1 e^x((1-x)\cosh(1-x)-\sinh(1-x))dx=$$

$$=2\int_{-1}^0 e^x((1+x)(e^{x+1}+e^{-x+1})-(e^{x+1}-e^{-x-1}))dx+$$

$$+2\int_0^1 e^x((1-x)(e^{-x+1}+e^{x-1})-(e^{1-x}-e^{x-1}))dx=$$

$$=2\int_{-1}^0((1+x)(e^{2x+1}+e^{-1})-e^{2x+1}+e^{-1})dx+$$

$$+2\int_0^1((1-x)(e+e^{2x-1})-e+e^{2x-1})dx=$$

$$=\left[(1+x)(e^{2x+1}+\frac{2}{e}x)-\frac{1}{2}(e^{2x+1}+\frac{x^2}{e})-e^{2x+1}+\frac{2}{e}x\right]_{-1}^0+$$

$$+\left[(1-x)(e^{2x-1}+2ex)+\frac{1}{2}(e^{2x-1}+2ex^2)+e^{2x-1}-2ex\right]_0^1=$$

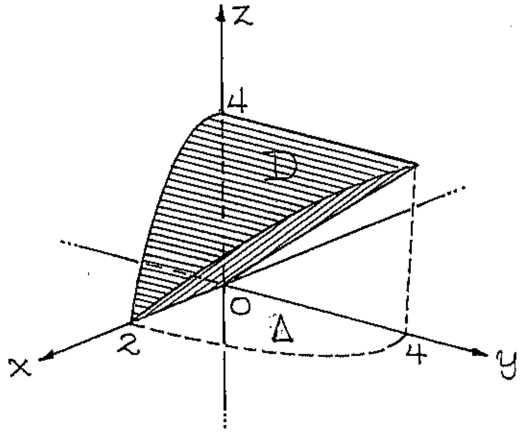
$$=e-\frac{e}{2}-e-\frac{1}{2e}-\frac{1}{e}+\frac{2}{e}+e+\frac{e}{2}-2e+e-\frac{1}{e}-\frac{1}{2e}-\frac{1}{e}=\frac{1}{e}.$$

Problem 6.23 (Sid. 16)

Lösning

$f(x) = 2x$; $D: x \geq 0,\ 0 \leq y \leq z \leq 4 - x^2$

D:s projektion på xy-planet är $\Delta: \begin{cases} 0 \leq y \leq 4 - x^2 \\ 0 \leq x \leq 2 \end{cases}$.



$$\iiint_D 2x\,dV = \iint_\Delta \Big(\int_y^{4-x^2} dz\Big) 2x\,dx\,dy =$$

$$= \iint_\Delta (4 - x^2 - y)\, 2x\,dx\,dy =$$

$$= \int_0^2 dx\, 2x \int_0^{4-x^2} (4 - x^2 - y)\,dy =$$

$$= \int_0^2 \Big(\Big[(4-x^2)y - \tfrac{1}{2}y^2 \Big]_{y=0}^{4-x^2} \Big) 2x\,dx =$$

$$= \int_0^2 \tfrac{1}{2}(4 - x^2)^2 \cdot 2x\,dx =$$

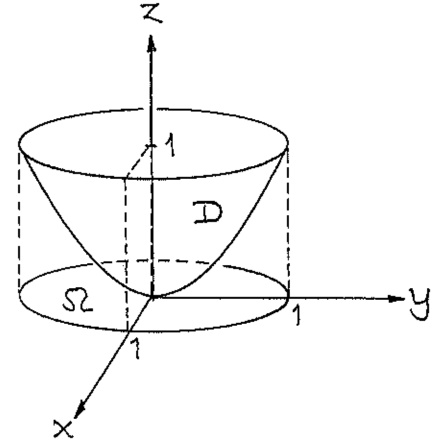
$$= \int_0^2 (16x - 8x^3 + x^5)\,dx =$$

$$= \Big[8x^2 - 2x^4 + \frac{x^6}{6} \Big]_0^2 = 32 - 32 + \frac{64}{6} = \frac{32}{3}.$$

Problem 6.24 (Sid. 17)

Lösning

$f(x,y,z) = z\sqrt{x^2+y^2}$, $D: x^2 + y^2 \leq z \leq 1.$



D:s projektion i xy-planet är $\Omega: x^2 + y^2 \leq 1.$

$$\iiint_D f(x,y,z)\,dx\,dy\,dz =$$

$$= \iint_\Omega \Big(\int_{x^2+y^2}^1 z\,dz \Big) \sqrt{x^2+y^2}\,dx\,dy =$$

$$= \frac{1}{2} \iint_\Omega \big(1 - (x^2+y^2)^2\big) \sqrt{x^2+y^2}\,dx\,dy = \left[\begin{array}{l|l} x = r\cos v & 0 \leq r \leq 1 \\ y = r\sin v & 0 \leq v \leq 2\pi \end{array} \right]$$

$$= \frac{1}{2} \int_0^1 (1 - r^4)\, r \cdot r\,dr \int_0^{2\pi} dv =$$

$$= \pi \int_0^1 (r^2 - r^6)\,dr = \frac{4\pi}{21}.$$

Problem 6.25 (Sid. 17)

Lösning

$$f(x,y,z)=x^2+y^2-z^2,\quad D:\ x^2+y^2+z^2\le 1.$$

$$\iiint_D (x^2+y^2-z^2)\,dxdydz \left[\underbrace{\begin{array}{l|l} x=r\sin\theta\cos\varphi & 0\le r\le 1\\ y=r\sin\theta\sin\varphi & 0\le\theta\le\pi \\ z=r\cos\theta & 0\le\varphi\le 2\pi\end{array}}_{D'}\right]=$$

$$=\iiint_{D'} r^2(\sin^2\theta-\cos^2\theta)\,r^2\sin\theta\,dr\,d\theta\,d\varphi=$$

$$=\underbrace{\int_0^1 r^4\,dr}_{=1/5}\int_0^\pi(1-2\cos^2\theta)\sin\theta\,d\theta\underbrace{\int_0^{2\pi}d\varphi}_{=2\pi}=$$

$$=\frac{2\pi}{5}\int_0^\pi(1-2\cos^2\theta)\sin\theta\,d\theta\left[\begin{array}{l|l} t=\cos\theta & \theta=\pi\Rightarrow t=-1\\ dt=-\sin\theta\,d\theta & \theta=0\Rightarrow t=1\end{array}\right]=$$

$$=\frac{2\pi}{5}\int_1^{-1}(1-2t^2)(-dt)=\frac{2\pi}{5}\int_{-1}^1(1-2t^2)\,dt=\frac{4\pi}{5}\int_0^1(1-2t^2)\,dt=$$

$$=\frac{4\pi}{5}\left[t-\frac{2}{3}t^3\right]_0^1=\frac{4\pi}{5}\cdot\frac{1}{3}=\frac{4\pi}{15}=-\iiint_D(z^2-x^2-y^2)\,dxdydz.$$

b) $f(x,y,z)=xe^{x^2+y^2+z^2}$, $D:\ x^2+y^2+z^2\le 1,\ x\ge 0.$

Jag inför rymdpolära koordinater och får:

$$\begin{cases} x=r\sin\theta\cos\varphi\\ y=r\sin\theta\sin\varphi\\ z=r\cos\theta\end{cases}\Rightarrow\begin{cases} x^2+y^2+z^2=r^2\\ \dfrac{d(x,y,z)}{d(r,\theta,\varphi)}=r^2\sin\theta\end{cases};\quad D':\begin{cases}0\le r\le 1\\ 0\le\theta\le\pi\\ |\varphi|\le\pi/2\end{cases}$$

$$\iiint_D f(x)\,dV=\iiint_{D'} r\sin\theta\cos\varphi\, e^{r^2}\,r^2\sin\theta\,dr\,d\theta\,d\varphi=$$

$$=\int_0^1 r^3e^{r^2}dr\int_0^\pi\sin^2\theta\,d\theta\int_{-\pi/2}^{\pi/2}\cos\varphi\,d\varphi=$$

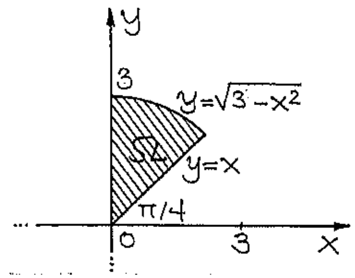
$$=\int_0^1 r^3e^{r^2}dr\cdot\frac{\pi}{2}\cdot 2=\pi\int_0^1 r^2e^{r^2}r\,dr=\left[\begin{array}{l|l} t=r^2 & 1\to 1\\ dt=2r\,dr & 0\to 0\end{array}\right]=$$

$$=\frac{\pi}{2}\int_0^1 te^t\,dt=\frac{\pi}{2}\left[(t-1)e^t\right]_0^1=\frac{\pi}{2}.$$

c) $f(x,y,z)=x$, $D:\ x^2+y^2+z^2\le 3,\ 0\le y\le x.$

D:s ortogonala projektion i xy-planet är

$$\Omega:\ x\le y\le\sqrt{3-x^2},\ x\ge 0\quad\text{(se figur).}$$



$$\begin{cases} x=r\sin\theta\cos\varphi\\ y=r\sin\theta\sin\varphi\\ z=r\cos\theta\end{cases}\Rightarrow\ D':\ 0\le r\le\sqrt3,\ 0\le\theta\le\pi,\ \pi/4\le\varphi\le\pi/2$$

$$\iiint_D x\,dxdydz=\iiint_{D'} r\sin\theta\cos\varphi\, r^2\sin\theta\,dr\,d\theta\,d\varphi=$$

$$=\int_0^{\sqrt3}r^3\,dr\int_0^\pi\sin^2\theta\,d\theta\int_{\pi/4}^{\pi/2}\cos\varphi\,d\varphi=$$

$$=\left[\frac{r^4}{4}\right]_0^{\sqrt3}\cdot\left[\frac12\left(\theta-\frac12\sin 2\theta\right)\right]_0^\pi\cdot\left[\sin\varphi\right]_{\pi/4}^{\pi/2}=$$

$$=\frac{3^2}{4}\cdot\frac{\pi}{2}\cdot\left(1-\frac{\sqrt2}{2}\right)=\frac{9\pi}{16}(2-\sqrt2).$$

Problem 6.26 (Sid. 17)

Lösning: $f(x,y,z)=y-x-z;\quad D:\begin{cases}0\le x+y+z\le 1\\ 0\le x+2y+3z\le 1;\\ 0\le x+4y+9z\le 1\end{cases}$

(1) $\begin{cases} u = x+y+z \\ v = x+2y+3z \\ w = x+4y+9z \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \Leftrightarrow AX = Y \Leftrightarrow X = A^{-1}Y$

$$\Leftrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} \Leftrightarrow \begin{cases} x = 3u+5v/2+w/2 \\ y = -3u+4v-w \\ z = u-3v/2+w/2 \end{cases} \Rightarrow$$

$$\Rightarrow g(u,v,w) = 8v-7u-2w = g(x,y,z)$$

$$dxdydz \longmapsto \left|\frac{d(x,y,z)}{d(u,v,w)}\right| dudvdw = \frac{1}{2} dudvdw.$$

(2) $\iiint_D (y-x-z)\,dxdydz = \frac{1}{2}\iiint_{D'} (8v-7u-2w)\,dudvdw =$

$$= \frac{1}{2}\int_0^1 du \int_0^1 dv \int_0^1 (8v-7u-2w)\,dw =$$

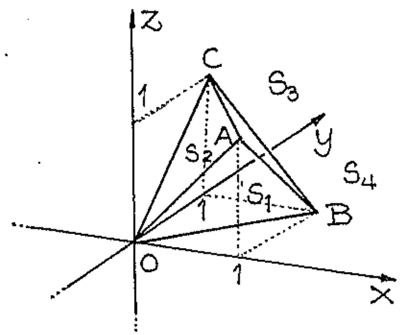
$$= \frac{1}{2}\int_0^1 du \int_0^1 \left(\left[8wv-7uw-w^2\right]_{w=0}^{1}\right) dv =$$

$$= \frac{1}{2}\int_0^1 du \int_0^1 (8v-7u-1)\,dv =$$

$$= \frac{1}{2}\int_0^1 \left(\left[4v^2-7uv-v\right]_{v=0}^{1}\right) du =$$

$$= \frac{1}{2}\int_0^1 (3-7u)\,du = \frac{1}{2}\left[3u-\frac{7}{2}u^2\right]_0^1 = \underline{-\frac{1}{4}}.$$

b) $\underline{f(x,y,z) = x}$, D är tetraedern nedan.



$\underline{A:(1,0,1)}$, $\underline{B:(1,1,0)}$, $\underline{C:(0,1,1)}$

$\overline{OA} = (1,0,1)$, $\overline{OB} = (1,1,0)$, $\overline{OC} = (0,1,1)$, $\overline{AB} = (0,1,-1)$, $\overline{AC} = (-1,1,0)$

Jag bestämer ekvationerna för S_1, S_2, S_3 och S_4.

$$S_1: \begin{vmatrix} x & 1 & 1 \\ y & 0 & 1 \\ z & 1 & 0 \end{vmatrix} = 0 \Leftrightarrow y+z-x = 0 \Leftrightarrow \underline{S_1: x-y-z=0.}$$

$$S_2: \begin{vmatrix} x & 1 & 0 \\ y & 0 & 1 \\ z & 1 & 1 \end{vmatrix} = 0 \Leftrightarrow z-y-x = 0 \Leftrightarrow \underline{S_2: x+y-z=0.}$$

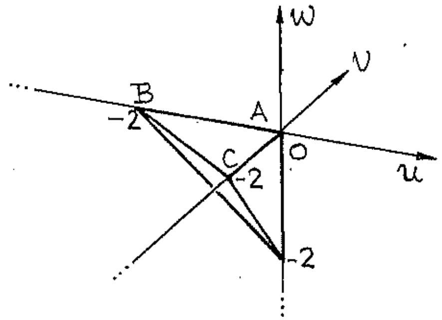
$$S_3: \begin{vmatrix} x & 1 & 0 \\ y & 1 & 1 \\ z & 0 & 1 \end{vmatrix} = 0 \Leftrightarrow x+z-y = 0 \Leftrightarrow \underline{S_3: x-y+z=0.}$$

$$S_4: \begin{vmatrix} x-1 & 0 & -1 \\ y-0 & 1 & 1 \\ z-1 & -1 & 0 \end{vmatrix} = 0 \Leftrightarrow y+z-1+x-1 = 0 \Leftrightarrow \underline{S_4: x+y+z=2.}$$

$$\begin{cases} x-y+z = u \\ -x+y-z = v \\ -x-y+z = w \end{cases} \Leftrightarrow \begin{cases} x = (-v-w)/2 \\ y = (-u-w)/2 \\ z = (-u-v)/2 \end{cases} \Rightarrow x+y+z = \frac{1}{2}(-2u-2v-2w) = 2$$

$$\Leftrightarrow \underline{u+v+w = -2}$$

$$\begin{cases} D: x-y-z \leq 0,\ -x-y+z \leq 0,\ -x+y-z \leq 0,\ x+y+z \leq 2. \\ D': u+v+w \geq -2,\ u,v,w \leq 0. \end{cases}$$



$\underline{\Omega = \Delta ABC}$

$$f(x,y,z)=x=-\tfrac{1}{2}(v+w)=g(u,v,w).$$

$$\frac{d(u,v,w)}{d(x,y,z)}=\begin{vmatrix}1&-1&-1\\-1&1&-1\\-1&-1&1\end{vmatrix}=-1-1-1-1-1+1=-4\Leftrightarrow\left|\frac{d(x,y,z)}{d(u,v,w)}\right|=\frac{1}{4}.$$

$$\iiint_D x\,dxdydz=-\frac{1}{8}\iiint_{D'}(v+w)\,dudvdw=$$
$$=-\frac{1}{8}\int_{-2}^{0}du\int_{-2-u}^{0}dv\int_{-2-u-v}^{0}(v+w)\,dw=$$
$$=-\frac{1}{8}\int_{-2}^{0}du\int_{-2-u}^{0}\Big(\Big[\tfrac{1}{2}(w+v)^2\Big]_{w=-2-u-v}^{0}\Big)dv=$$
$$=-\frac{1}{8}\int_{-2}^{0}du\int_{-2-u}^{0}\tfrac{1}{2}(v^2-(u+2)^2)\,dv=$$
$$=-\frac{1}{16}\int_{-2}^{0}\Big(\Big[\tfrac{1}{3}v^3-(u+2)^2v\Big]_{v=-2-u}^{0}\Big)du=$$
$$=-\frac{1}{16}\int_{-2}^{0}\Big(\tfrac{1}{3}(u+2)^3-(u+2)^3\Big)du=$$
$$=\frac{1}{24}\int_{-2}^{0}(u+2)^3\,du=\frac{1}{24}\cdot\frac{2^4}{4}=\underline{\underline{\frac{1}{6}}}.$$

Problem 6.27 (Sid. 17)

Lösning

$$\forall n\geq 2:\ \int\!\!\int\cdots\int_D(x_1+x_2+\dots+x_n)\,dx_1dx_2\cdots dx_n=$$
$$=\int_0^1dx_1\int_0^1dx_2\cdots\int_0^1(x_1+x_2+\dots+x_n)\,dx_n=$$
$$=\int_0^1dx_1\int_0^1dx_2\cdots\int_0^1(x_1+x_2+\dots\tfrac{1}{2})\,dx_{n-1}=$$
$$=\int_0^1dx_1\int_0^1dx_2\cdots\int_0^1(x_1+x_2+\dots+\tfrac{1}{2}+\tfrac{1}{2})\,dx_{n-2}=\dots=$$
$$=\frac{1}{2}+\frac{1}{2}+\dots+\frac{1}{2}=n\cdot\frac{1}{2}=\frac{n}{2}\quad\text{(efter } n \text{ integrationer)}$$

Integralltillämpningar

Problem 6.28 (Sid. 17)

Lösning:

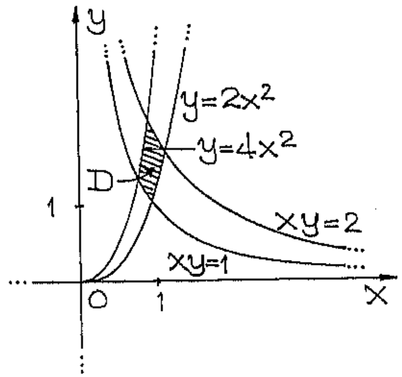
a) $D:\ |x+2y|+|3x-y|\leq 1$

$$\begin{cases}u=x+2y\\ v=3x-y\end{cases}\Rightarrow\left|\frac{d(u,v)}{d(x,y)}\right|=\left|\begin{vmatrix}1&2\\3&-1\end{vmatrix}\right|=7\Leftrightarrow\left|\frac{d(x,y)}{d(u,v)}\right|=\frac{1}{7}$$

$D':\ |u|+|v|\leq 1$ (Se Pr. 1.2 a))

$$\mu(D)=\iint_D dxdy=\iint_{D'}\left|\frac{d(x,y)}{d(u,v)}\right|dudv=\frac{1}{7}\mu(D')=\frac{2}{7}\,ae.$$

b)



$D:\ 2x^2\leq y\leq 4x^2,\ 1\leq xy\leq 2.$

$D:\ 2\leq\frac{y}{x^2}\leq 4,\ 1\leq xy\leq 2.;\quad u=\frac{y}{x^2},\ v=xy;\quad D':\begin{cases}2\leq u\leq 4\\ 1\leq v\leq 2\end{cases}$

$$\frac{d(u,v)}{d(x,y)}=\begin{vmatrix}-2y/x^3&1/x^2\\ y&x\end{vmatrix}=-2y/x^2-y/x^2=-3y/x^2=-3u;$$

$$\Leftrightarrow\left|\frac{d(x,y)}{d(u,v)}\right|=\frac{1}{3u}\Rightarrow\mu(D)=\iint_D dxdy=\iint_{D'}\frac{1}{3u}\,dudv=$$

$$= \frac{1}{3}\int_2^4 \frac{du}{u}\cdot\int_1^2 dv = \frac{1}{3}\ln\frac{4}{2}\cdot 1 = \underline{\frac{1}{3}\ln 2 \text{ ae}}.$$

Problem 6.29 (Sid. 19)

Lösning

a) $$\underline{D = \{(x,y,z) : \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1\}.}$$

$$\begin{cases} x = au \\ y = bv \\ z = cw \end{cases} \Rightarrow \begin{cases} D' : u^2+v^2+w^2 \le 1 \\ \dfrac{d(x,y,z)}{d(u,v,w)} = abc \end{cases} \Rightarrow \mu(D) = \iiint_D dxdydz =$$

$$= \iiint_{D'} \left|\frac{d(x,y,z)}{d(u,v,w)}\right| dudvdw = abc\iiint_{D'} dudvdw = \frac{4}{3}\pi abc.$$

Anm. D är en ellipsoid; D' är enhetsklotet.

b) $$\underline{D = \{(x,y,z) : 3x^2+2y^2+z^2+2xz-2yz \le 1\}.}$$

(1) $3x^2+2y^2+z^2+2xz-2yz =$

$$= x^2+(x^2+y^2+z^2-2xy+2xz-2yz)+(x^2+2xy+y^2) =$$

$$= x^2+(x+y)^2+(x-y+z)^2;$$

(2) $$\begin{cases} u = x \\ v = x+y \\ w = x-y+z \end{cases} \Rightarrow \frac{d(u,v,w)}{d(x,y,z)} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = 1 = \frac{d(x,y,z)}{d(u,v,w)}.$$

(3) $$\underline{D' = \{(u,v,w) : u^2+v^2+w^2 \le 1\}.}$$

(4) $$\mu(D) = \iiint_D dxdydz = \iiint_{D'} dudvdw = \mu(D') = \frac{4\pi}{3}.$$

D fås av D' genom en isometri (rotation).

Problem 6.30 (Sid. 17)

Lösning

(1) $$\begin{cases} u = 2x+y+z \\ v = x+2y+z \\ w = x+z+2z \end{cases} \Leftrightarrow \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \Leftrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} x = (3u-v-w)/4 \\ y = (-u+3v-w)/4 \\ z = (-u-v+3w)/4 \end{cases} \Leftrightarrow x+y+z = \frac{1}{4}(u+v+w).$$

(2) $\underline{D:\ 2x+y+z \ge 0,\ x+2y+z \ge 0,\ x+y+2z \ge 0,\ x+y+z \le 4.}$

$$\underline{D' :\ u \ge 0,\ v \ge 0,\ w \ge 0,\ u+v+w \le 16.}$$

(3) $$\frac{d(u,v,w)}{d(x,y,z)} = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 4 = \left(\frac{d(x,y,z)}{d(u,v,w)}\right)^{-1} \Leftrightarrow \frac{d(x,y,z)}{d(u,v,w)} = \frac{1}{4}.$$

(4) D' är en tetraeder i den första oktanten i uvw-planet med spetsarna i punkterna $(0,0,0)$, $(16,0,0)$, $(0,16,0)$ och $(0,0,16)$. Dess volym är $\mu(D') = \frac{1}{6}\cdot 16^3 = \frac{2^{12}}{6} = \frac{2^{11}}{3} = \frac{2048}{3}.$

(5) $$\mu(D) = \iiint_D dxdydz = \iiint_{D'} \frac{1}{4} dudvdw = \frac{1}{4}\mu(D') = \frac{512}{3}.$$

Problem 6.31 (Sid. 17)

Lösning

a) $x^2+4y^2 \le 36 \Leftrightarrow \frac{x^2}{2^2}+\frac{y^2}{3^2} \le 1$ (elliptisk cylinder).

$$D=\{(x,y,z):\ x+y-10\leq z\leq 10-x-y,\ \ 9x^2+4y^2\leq 36\}$$

$$D_o=\{(x,y): \frac{x^2}{2^2}+\frac{y^2}{3^2}\leq 1\}.$$

$$\mu(D)=\iiint_D dxdydz=\iint_{D_o}\Big(\int_{x+y-10}^{10-x-y}dz\Big)dxdy=$$

$$=\iint_{D_o}2(10-x-y)dxdy\left[\begin{array}{l|l} x=2r\cos v & 0\leq r\leq 1\\ y=3r\sin v & \underbrace{0\leq v\leq 2\pi}_{\Delta}\end{array}\right]=$$

$$=\iint_{\Delta}2(10-2r\cos v-3r\sin v)\cdot 6r\,drdv=$$

$$=12\int_0^1 dr\,r\int_0^{2\pi}(10-2r\cos v-3r\sin v)dv=$$

$$=12\int_0^1 r dr\cdot 10\cdot 2\pi=120\pi\left[r^2\right]_0^1=120\pi.$$

b) Låt oss projicera kroppen i xy-planet.

z-koordinaterna lika $\Leftrightarrow x^2+y^2=1-2x-2y \Leftrightarrow$

$\Leftrightarrow x^2+2x+1+(y^2+2y+1)=3 \Leftrightarrow (x+1)^2+(y+1)^2=3.$

Den sökta projektionen är

$$\Delta=\{(x,y,0): (\tfrac{x+1}{\sqrt3})^2+(\tfrac{y+1}{\sqrt3})^2\leq 1\}.$$

$$\underline{K=\{(x,y,z):\ x^2+y^2\leq z\leq 1-2x-2y,\ (x+1)^2+(y+1)^2\leq 3\}.}$$

$$\mu(K)=\iiint_K dxdydz=\iint_\Delta\Big(\int_{x^2+y^2}^{1-2x-2y}dz\Big)dxdy=$$

$$=\iint_\Delta(3-(x+1)^2-(y+1)^2)dxdy\left[\begin{array}{l|l} x+1=\sqrt3 r\cos v & 0\leq r\leq 1\\ y+1=\sqrt3 r\sin v & \underbrace{0\leq v\leq 2\pi}_{\Delta'}\end{array}\right]$$

$$=\iint_{\Delta'}(3-3r^2)3r\,drdv=9\int_0^1(r-r^3)dr\int_0^{2\pi}dv=$$

$$=9\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1\cdot[v]_0^{2\pi}=9\cdot\frac14\cdot 2\pi=\frac{9\pi}{2}\text{ ve.}$$

Problem 6.32 (Sid. 17)

Lösning

$$\underline{D=\{(x,y,z)\in\mathbb{R}^3:\ x+y^2+z\leq 1;\ x,y,z\geq 0\}.}\quad (*)$$

a) $x+y^2+z\leq 1 \overset{(*)}{\Rightarrow} 0\leq z\leq 1-x-y^2 \overset{(*)}{\Rightarrow} 0\leq x\leq 1-y^2 \overset{(*)}{\Rightarrow} 0\leq y\leq 1;$

$$\underline{D=\{(x,y,z): 0\leq z\leq 1-x-y^2,\ 0\leq x\leq 1-y^2,\ 0\leq y\leq 1\}}$$

b) $\mu(D)=\iiint_D dxdydz=\int_0^1 dy\int_0^{1-y^2}dx\int_0^{1-x-y^2}dz=$

$$=\int_0^1 dy\int_0^{1-y^2}(1-x-y^2)dx=\int_0^1\Big(\Big[(1-y^2)x-\frac12x^2\Big]_0^{1-y^2}\Big)dy=$$

$$=\int_0^1\frac12(1-y^2)^2dy=\frac12\int_0^1(1-2y^2+y^4)dy=$$

$$=\frac12\Big[y-\frac23y^3+\frac15y^5\Big]_0^1=\frac12\Big(1-\frac23+\frac15\Big)=\frac12\cdot\frac{15-10+3}{15}=\frac{4}{15}.$$

Problem 6.33 (Sid. 17)

Lösning

$$\underline{D=\{(x,y,z):\ y\geq x^2,\ z\geq y^2,\ x\geq z^2\}.}$$

$$(1)\quad \begin{cases} z^2\leq x\\ x^2\leq y\\ y^2\leq z\end{cases}\Leftrightarrow\begin{cases} z^2\leq x\\ x\leq\sqrt y\\ y\leq\sqrt z\end{cases}\Leftrightarrow\begin{cases} z\leq\sqrt x\\ z^2\leq x\leq\sqrt y\leq\sqrt[4]{z}\leq\sqrt[8]{x}\end{cases}\Rightarrow x^2\leq y\leq\sqrt[4]{x}.$$

(2) $y^2 \le z \wedge z^2 \le x \Leftrightarrow y^2 \le z \wedge z \le \sqrt{x} \Leftrightarrow \underline{y^2 \le z \le \sqrt{x}}$;

(3) $x \le \sqrt{y} \le \sqrt[8]{x} \Leftrightarrow \underline{0 \le x \le 1}$.

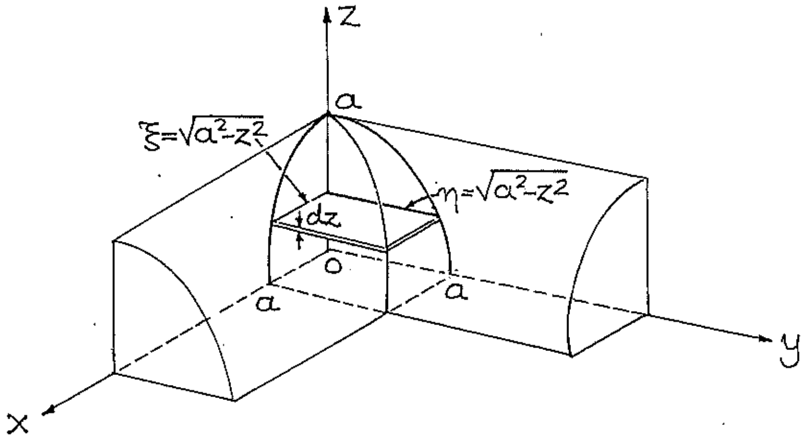
$$\underline{D = \{(x,y,z):\ y^2 \le z \le \sqrt{x},\ x^2 \le y \le \sqrt[4]{x},\ 0 \le x \le 1\}}.$$

(4) $$\mu(D) = \iiint_D dx\,dy\,dz = \int_0^1 dx \int_{x^2}^{\sqrt[4]{x}} dy \int_{y^2}^{\sqrt{x}} dz =$$
$$= \int_0^1 dx \int_{x^2}^{\sqrt[4]{x}} (\sqrt{x} - y^2)\,dy =$$
$$= \int_0^1 \left(\left[\sqrt{x}\,y - \frac{y^3}{3}\right]_{x^2}^{\sqrt[4]{x}}\right) dx =$$
$$= \int_0^1 \left(\frac{2}{3}x^{3/4} - x^{5/2} + \frac{1}{3}x^6\right) dx =$$
$$= \left[\frac{8}{21}x^{7/4} - \frac{2}{7}x^{7/2} + \frac{1}{21}x^7\right]_0^1 =$$
$$= \frac{8}{21} + \frac{1}{21} - \frac{2}{7} = \frac{1}{7}.$$

Problem 6.34 (Sid. 17)

Lösning

a)



I figuren syns en åttondedel av volymen ifråga.

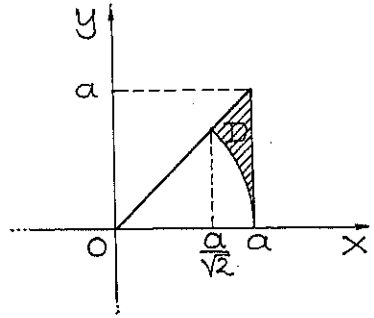
$$\begin{cases} x^2 + z^2 = a^2 \Leftrightarrow x = \sqrt{a^2 - z^2} \\ y^2 + z^2 = a^2 \Leftrightarrow y = \sqrt{a^2 - z^2} \end{cases} \Rightarrow dV = \xi \cdot \eta\, dz = (a^2 - z^2)\,dz \Rightarrow$$

$$\Rightarrow \frac{1}{8}V = \int_0^a (a^2 - z^2)\,dz = \left[a^2 z - \frac{z^3}{3}\right]_0^a = \frac{2}{3}a^3 \Leftrightarrow V = \frac{16}{3}a^3.$$

b) Cylindern $x^2 + y^2 = a^2$ "kapar" 4 st "strimlor av kroppen ovan (de 4 vertikala kanterna).

Jag kommer att bestämma volymen av en fjärdedel av "strimlan" vars projektion på xy-planet syns i figuren på nästföljande figur:

$$\underline{D = \{(x,y,0):\ x^2 + y^2 \ge a^2,\ x \le y\}}.$$



$$K = \{(x,y,z):\ 0 \le z \le \sqrt{a^2 - x^2},\ x^2 + y^2 \ge a^2,\ 0 \le y \le x\} =$$
$$= \{(x,y,z):\ 0 \le z \le \sqrt{a^2 - x^2},\ \sqrt{a^2 - x^2} \le y \le x,\ \frac{a}{\sqrt{2}} \le x \le a\}$$

$$\frac{V}{16} = \int_{a/\sqrt{2}}^{a} dx \int_{\sqrt{a^2 - x^2}}^{x} dy \int_0^{\sqrt{a^2 - x^2}} dz = \int_{a/\sqrt{2}}^{a} dx \int_{\sqrt{a^2 - x^2}}^{x} \sqrt{a^2 - x^2}\,dy =$$

$$= \int_{a/\sqrt{2}}^{a} (x\sqrt{a^2-x^2} - a^2 + x^2)\,dx = \int_{a/\sqrt{2}}^{a} \sqrt{a^2-x^2}\,x\,dx -$$

$$- \int_{a/\sqrt{2}}^{a} (a^2-x^2)\,dx = \left[-\frac{1}{3}(a^2-x^2)^{3/2} - a^2x + \frac{x^3}{3}\right]_{a/\sqrt{2}}^{a} =$$

$$= \frac{a^3}{6\sqrt{2}} - \frac{2a^3}{3} + \frac{\sqrt{2}a^3}{2} - \frac{\sqrt{2}a^3}{12} = \frac{3\sqrt{2}-4}{6}a^3 \Rightarrow V = \frac{8}{3}(3\sqrt{2}-4)a^3.$$

Den sökta volymen blir alltså

$$V_0 = \frac{16}{3}a^3 - \frac{8}{3}(3\sqrt{2}-4)a^3 = \underline{(16-8\sqrt{2})a^3}.$$

Problem 6.35 (Sid. 17)

Lösning

$$\underline{K = \{(x,y,z):\ x^2+y^2 \leq z \leq 5\}}$$

$$\underline{D = \{(x,y,0): \sqrt{x^2+y^2} \leq \sqrt{5}\}.}$$

$$\mu(K) = \iiint_K dx\,dy\,dz = \iint_D \left(\int_{x^2+y^2}^{5} dz\right) dx\,dy =$$

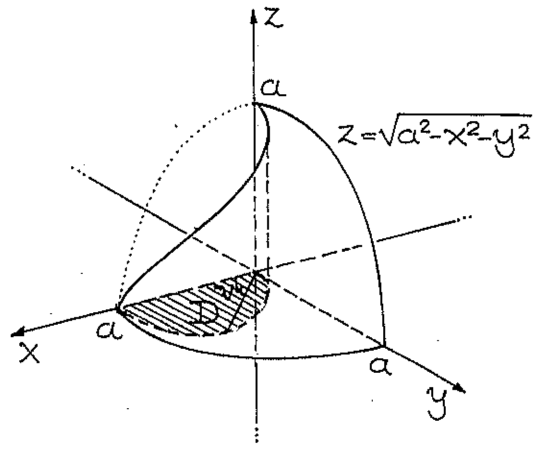
$$= \iint_D (5-x^2-y^2)\,dx\,dy \left[\begin{array}{l|l} x = r\cos v & 0 \leq r \leq \sqrt{5} \\ y = r\sin v & 0 \leq v \leq 2\pi \end{array}\right] =$$

$$= \int_0^{\sqrt{5}} (5-r^2)r\,dr \int_0^{2\pi} dv = 2\pi \int_0^{\sqrt{5}} (5r-r^3)\,dr =$$

$$= \pi\left[5r^2 - \frac{1}{2}r^4\right]_0^{\sqrt{5}} = \pi\left(25 - \frac{25}{2}\right) = \frac{25\pi}{2} \approx 39{,}3 \text{ ml}.$$

Problem 6.36 (Sid. 18)

Lösning: Jag räknar i den första oktanten och multiplicerar därefter med 8 (se figur).



$$\underline{D = \{(x,y,0):\ x^2+y^2 \leq ax\}.}$$

$$\underline{K = \{(x,y,z):\ x^2+y^2+z^2 \leq a^2,\ x^2+y^2 \leq ax,\ a>0\}.}$$

$$\frac{1}{8}V = \iint_D \left(\int_0^{\sqrt{a^2-x^2-y^2}} dz\right) dx\,dy = \iint_D \sqrt{a^2-x^2-y^2}\,dx\,dy =$$

$$= \left[\begin{array}{l|l} x = r\cos v & 0 \leq r \leq a\cos v \\ y = r\sin v & 0 \leq v \leq \pi/2 \end{array}\right] =$$

$$= \int_0^{\frac{\pi}{2}} dv \int_0^{a\cos v} \sqrt{a^2-r^2}\,r\,dr = \int_0^{\pi/2} \left(\left[-\frac{1}{3}(a^2-r^2)^{3/2}\right]_0^{a\cos v}\right) dv =$$

$$= \int_0^{\pi/2} \frac{1}{3}(1-\sin^3 v)a^3\,dv = \frac{\pi a^3}{6} - \frac{a^3}{3}\int_0^{\pi/2} (1-\cos^2 v)\sin v\,dv$$

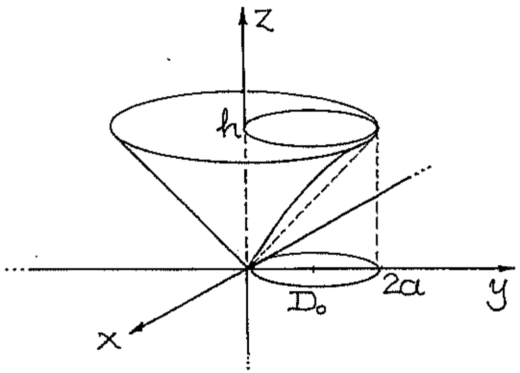
$$= \frac{\pi a^3}{6} + \frac{a^3}{6}\left[\cos v - \frac{1}{3}\cos^3 v\right]_0^{\pi/2} = \frac{\pi a^3}{6} - \frac{2a^3}{9} \Leftrightarrow$$

$$\Leftrightarrow V = \frac{4\pi}{3}a^3 - \frac{16a^3}{9}.$$

Anm. $x^2+y^2 \leq ax \Leftrightarrow r^2\cos^2 v + r^2\sin^2 v \leq ar\cdot\cos v \ldots$

Problem 6.37 (Sid. 18)

Lösning



Konens mantelyta har ekvationen

$$z = k\cdot\sqrt{x^2+y^2},$$

där k är är en konstant; $(0,2a,h)$ ligger på ytan, dvs $h = k\cdot 2a \Leftrightarrow k = h/2a,\ a>0$.

Den bit av konen som urborras bestäms av

$$\underline{D = \{(x,y,z):\ k\sqrt{x^2+y^2} \le z \le h,\ x^2+y^2 \le 2ay\}.}$$

$$\underline{D_0 = \{(x,y,0):\ x^2+y^2 \le 2ay,\ a>0\}.}$$

$$\mu(D) = \iiint_D dx\,dy\,dz = \iint_{D_0}\left(\int_{k\sqrt{x^2+y^2}}^{h} dz\right)dx\,dy =$$

$$= \iint_{D_0}(h - k\sqrt{x^2+y^2})\,dx\,dy \left[\begin{array}{l|l} x = r\cos v & 0 \le r \le 2a\sin v \\ y = r\sin v & 0 \le v \le \pi \end{array}\right] =$$

$$= \int_0^{\pi} dv \int_0^{2a\sin v}(h - kr)\,r\,dr =$$

$$= \int_0^{\pi}\left(\left[\tfrac{1}{2}hr^2 - \tfrac{1}{3}kr^3\right]_0^{2a\sin v}\right)dv =$$

$$= \int_0^{\pi}(2ha^2\sin^2 v - \tfrac{8}{3}ka^3\sin^3 v)\,dv =$$

$$= \int_0^{\pi}(ha^2(1-\cos 2v) - \tfrac{8}{3}ka^3(1-\cos^2 v)\sin v)\,dv =$$

$$= \left[ha^2(v + \tfrac{1}{2}\sin 2v) + \tfrac{8}{3}ka^3(\cos v - \tfrac{1}{3}\cos^3 v)\right]_0^{\pi} =$$

$$= (\pi - \tfrac{16}{9})a^2h.$$

Konens volym är $V_0 = \frac{4}{3}\pi a^2 h$, så det som blir kvar efter urborrningen har volymen $\frac{3\pi+16}{9}a^2h$.

Problem 6.38 (Sid. 17)

Lösning

a) $\underline{K:\ x^2+y^2+z^2 = R^2,\ z \ge 0;\ \rho(x) = \rho_0.}$

Rymdpolära koordinater införs:

$$\underline{K':\ 0 \le r \le R,\ 0 \le \theta \le \pi/2,\ 0 \le \varphi \le 2\pi.}$$

$$\underline{dm = \rho\,dV = \rho_0 r^2 \sin\theta\,dr\,d\theta\,d\varphi.}$$

$$m = \iiint_{K'} \rho\,dV = \rho_0\int_0^R r^2\,dr \int_0^{\pi/2}\sin\theta\,d\theta \int_0^{2\pi} d\varphi =$$

$$= \rho_0\left[\frac{r^3}{3}\right]_0^R \cdot \left[-\cos\theta\right]_0^{\pi/2} \cdot \left[\varphi\right]_0^{2\pi} =$$

$$= \rho\cdot\frac{R^3}{3}\cdot 1\cdot 2\pi = \underline{\frac{2\pi}{3}R^3\rho_0.}$$

b) $K: x^2+y^2+z^2 \le R^2,\ z \ge 0;\ \rho(\mathbf{x}) = |\mathbf{x}| = r.$

$$m = \iiint_K \rho\, dV = \iiint_{K'} kr \cdot r^2 \sin\theta\, dr\, d\theta\, d\varphi =$$
$$= k\int_0^R r^3 dr \int_0^{\pi/2} \sin\theta\, d\theta \int_0^{2\pi} d\varphi =$$
$$= k \cdot \frac{R^4}{4} \cdot 1 \cdot 2\pi = \frac{\pi}{2} k R^4.$$

Om volymelementet i rymdpolära koordinater kan du läsa på sidan 293 i läroboken.

Problem 6.39 (Sid. 18)

Lösning

Cylinderkoordinaterna (ρ, φ, z) införs på sidan 312 i läroboken. (Man kan räkna utan dessa koordinater dock).

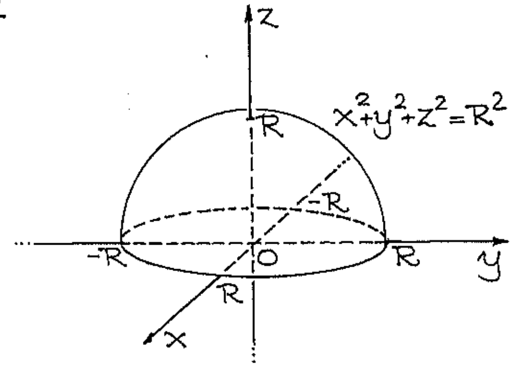
$$dm = \tilde{\rho}\, dV = (10-z)^2(4-\rho)\, 2\pi\rho\, d\rho\, dz;\ \underbrace{0 \le \rho \le 3,\ 0 \le z \le 8}_{D}$$

$$m = \iiint_D \tilde{\rho}\, dV = 2\pi \int_0^3 (4\rho - \rho^2)\, d\rho \int_0^8 (10-z)^2 dz =$$
$$= 2\pi \left[2\rho^2 - \tfrac{1}{3}\rho^3\right]_0^3 \cdot \left[-\tfrac{1}{3}(10-z)^3\right]_0^8 =$$
$$= 2\pi(18-9) \cdot \tfrac{1}{3}(1000-8) =$$
$$= 2\pi \cdot 9 \cdot \tfrac{1}{3} \cdot 988 = 5928\pi \text{ kg}.$$

Svar: 18,5 ton ungefär.

Problem 6.40 (Sid. 18)

Lösning



a) Pga homogeniteten och rotationssymetrin kring z-axeln faller tyngdpunkten på denna axel, dvs. $x_T = y_T = 0$. Halvklotets massa är $m = \frac{2}{3}\pi R^3 \rho_0$.

$K: x^2+y^2+z^2 \le R^2,\ z \ge 0.$

Med rymdpolära koordinater fås

$K': 0 \le r \le R,\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \varphi \le 2\pi.$

$$m z_T = \iiint_K z \cdot \rho \cdot dV = \iiint_{K'} r\cos\theta \cdot \rho_0 \cdot r^2 \sin\theta\, dr\, d\theta\, d\varphi =$$
$$= \rho_0 \int_0^R r^3 dr \int_0^{\pi/2} \sin\theta \cos\theta\, d\theta \int_0^{2\pi} d\varphi =$$
$$= \rho_0 \cdot \frac{R^4}{4} \cdot \frac{1}{2} \cdot 2\pi = \frac{\pi}{4} R^4 \rho_0 \Leftrightarrow$$
$$\Leftrightarrow \frac{2}{3}\pi R^3 \rho_0 z_T = \frac{\pi}{4} R^4 \rho_0 \Leftrightarrow z_T = \frac{3R}{8}, \text{ R är radien.}$$

Svar: Tyngdpunkten faller på symmetriaxeln och på avståndet $\frac{3}{8}R$ från den plana sidan.

b) $m = \iiint_K \rho\, dV = \iiint_{K'} kr \cdot r^2 \sin\theta\, dr d\theta d\varphi =$

$= k\int_0^R r^3 dr \int_0^{\pi/2} \sin\theta\, d\theta \int_0^{2\pi} d\varphi = k\frac{R^4}{4} \cdot 1 \cdot 2\pi = k\frac{\pi}{2}R^4;$

$mz_T = \iiint_K z\rho\, dV = \iiint_{K'} r\cos\theta \cdot kr \cdot r^2 \sin\theta\, dr d\theta d\varphi =$

$= k\int_0^R r^4 dr \int_0^{\pi/2} \sin\theta\cos\theta\, d\theta \int_0^{2\pi} d\varphi = k\frac{\pi}{5}R^5 \Leftrightarrow$

$\Leftrightarrow k\frac{\pi}{2}R^4 z_T = k\frac{\pi}{5}R^5 \Leftrightarrow z_T = \frac{2R}{5}.$

Svar: Tyngdpunkten ligger på symmetri-axeln $\frac{2}{5}R$ från den plana ytan.

Anm. Svara inte i termer av koordinater.

Problem 6.41 (Sid. 18)

Lösning

$O = (0,0,0)$, $\overrightarrow{OA} = (a,0,0)$, $\overrightarrow{OB} = (0,b,0)$, $\overrightarrow{OH} = (0,0,h)$

$\overrightarrow{OT} = \frac{1}{4}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OH}) = \frac{1}{4}(a,b,h) \Rightarrow z_T = \frac{h}{4}.$

Anm. Tyngdpunkten för en tetraeder med hörnen i A, B, C och D är (se linjär algebra):

$$\overrightarrow{OT} = \frac{1}{4}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD}).$$

Det går att använda analytiska metoder...

Generaliserade integraler

Problem 6.42 (Sid. 18)

Lösning

a) $f(x,y) = e^{-x-y}$; $D_n = \{(x,y) : 0 \le x \le n, 0 \le y \le n\}$.

$\iint_{D_n} f(x,y)\,dxdy = \int_0^n e^{-x} dx \int_0^n e^{-y} dy = \left(\int_0^n e^{-x} dx\right)^2 =$

$= \left(\left[-e^{-x}\right]_0^n\right)^2 = (1-e^{-n})^2$, $n = 1,2,3,\ldots$

b) $f(x,y) = e^{-x-y}$; $D_n = \{(x,y) : 0 \le y \le n-x, 0 \le x \le n\}$.

$\iint_{D_n} f(x,y)\,dxdy = \int_0^n dx\, e^{-x} \int_0^{n-x} e^{-y} dy =$

$= \int_0^n \left(\left[-e^{-y}\right]_0^{n-x}\right) e^{-x} dx =$

$= \int_0^n (e^{-x} - e^{-n})\, dx =$

$= \left[-e^{-x} - xe^{-x}\right]_0^n =$

$= 1-(n+1)e^{-n}$, $n = 1,2,3,\ldots$

c) D_n, $n \ge 1$, i a) och b) är uttömmande sviter för den första kvadranten $(x,y \ge 0)$.

$\iint_D f(x,y)\,dxdy = \lim_{n\to\infty} \iint_{D_n} f(x,y)\,dxdy = \ldots =$

$= \lim_{n\to\infty} (1-e^{-n})^2 = 1 = \lim_{n\to\infty} (1-(n+1)e^{-n}).$

Problem 6.43 (Sid. 18)

Lösning

$$f(x)=\frac{xy}{(1+x^2+y^2)^2};\quad D: x,y\geq 0.$$

$D_n=\{(x,y): x^2+y^2\leq n^2,\ x,y\geq 0\}$, $n=1,2,3,\dots$ är en uttömmande svit för D, den första kvadranten. Planpolära koordinater ger

$$\iint_{D_n} f(x,y)\,dx\,dy=\int_0^n\frac{r^3}{(1+r^2)^2}dr\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta=$$

$$=\int_0^n\frac{r^3}{(1+r^2)^2}dr\cdot\left[\frac{1}{2}\sin^2\theta\right]_0^{\pi/2}=$$

$$=\frac{1}{2}\int_0^n\left(\frac{r}{(1+r^2)^2}-\frac{r}{(1+r^2)^3}\right)dr=$$

$$=\frac{1}{2}\left[-\frac{1}{2}\frac{1}{1+r^2}+\frac{1}{4}\frac{1}{(1+r^2)^2}\right]_0^n=$$

$$=\frac{1}{8}\left(1+\frac{1}{(n^2+1)^2}-\frac{2}{n^2+1}\right)\xrightarrow[n\to\infty]{}\frac{1}{8}.$$

b) $f(x)=x^2e^{-\sqrt{x^2+y^2}};\quad D=\mathbb{R}^2=\mathbb{R}\times\mathbb{R}.$

$D_n=\{(x,y): x^2+y^2\leq n^2\}$, $n=1,2,3,\dots$, är en uttömmande svit för $D=\mathbb{R}^2$. Planpolära koordinater ger

$$\iint_{D_n} f(x,y)\,dx\,dy=\int_0^n r^3e^{-r}dr\int_0^{2\pi}\cos^2 v\,dv=\int_0^n r^3e^{-r}\cdot\pi=$$

$$=\pi\left[-r^3e^{-r}\right]_0^n+3\pi\int_0^n r^2e^{-r}dr=$$

$$=-\pi n^3e^{-n}+3\pi\left[-r^2e^{-r}\right]_0^n+6\pi\int_0^n re^{-r}dr=$$

$$=-\pi n^3e^{-n}-3\pi n^2e^{-n}+6\pi\left[-(r+1)e^{-r}\right]_0^n=$$

$$=6\pi-(6+6n-3n^2+n^3)e^{-n}\xrightarrow[n\to\infty]{}6\pi.$$

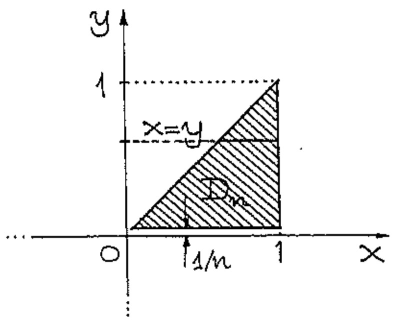
Problem 6.44 (Sid. 18)

Lösning

a) $f(x,y)=x/y;\quad D=\{(x,y): 0<y\leq x\leq 1\}.$

$D_n=\{(x,y): \frac{1}{n}\leq y\leq x\leq 1\}$, $n=1,2,3,\dots$, är en uttömmande svit för triangeln D (se figur).

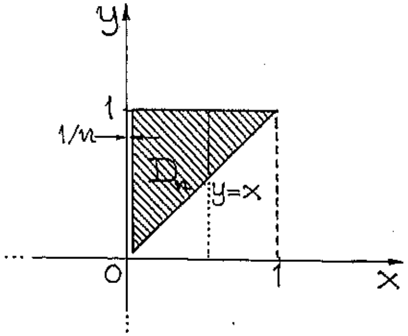
$$\iint_{D_n} f(x,y)\,dx\,dy=\int_{1/n}^n dy\,\frac{1}{y}\int_y^1 x\,dx=\int_{1/n}^n\left(\left[\frac{x^2}{2}\right]_y^1\right)\frac{1}{y}dy=$$



$$=\frac{1}{2}\int_{1/n}^1\left(\frac{1}{y}-y\right)dy=\frac{1}{2}\left[\ln y-\frac{1}{2}y^2\right]_{1/n}^1=\frac{1}{2}\left(\ln 1-\frac{1}{2}+\ln(n)+\frac{1}{2}n^{-2}\right)\xrightarrow[n\to\infty]{}\infty,$$ dvs integralen är divergent.

b) $f(x,y)=x/y;\quad D:=\{(x,y): 0<x\leq y\leq 1\}.$

$D_n=\{(x,y): \frac{1}{n}\le x\le y\le 1\}$, $n=1,2,3,\dots$, är en uttömmande svit för triangeln D (se figur);



$$\iint_{D_n}(x/y)\,dx\,dy=\int_{1/n}^{1}dx\,x\int_x^1\frac{1}{y}\,dy=-\int_{1/n}^{1}x\ln x\,dx=$$

$$=-\left[\tfrac{1}{2}x^2\ln x\right]_{1/n}^{1}+\frac{1}{2}\int_{1/n}^{1}x\,dx=$$

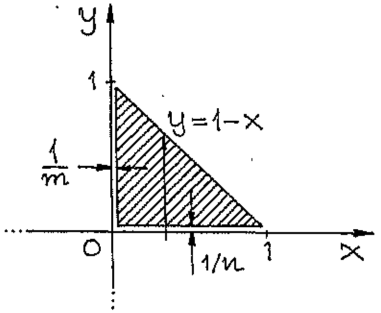
$$=-\frac{\ln n}{2n^2}+\frac{1}{2}\left[\frac{x^2}{2}\right]_{1/n}^{1}=$$

$$=\frac{1}{4}\left(1-\frac{1}{n^2}\right)\xrightarrow[n\to\infty]{}\frac{1}{4}.$$

Problem 6.45 (Sid. 18)

Lösning

$f(x,y)=1/\sqrt{xy}$; $D=\{(x,y)\in\mathbb{R}_+^2: x+y<1\}$.



$D_{m,n}=\{(x,y): x+y\le 1,\ x\ge\frac{1}{m},\ y\ge\frac{1}{n}\}$, $m,n=1,2,\dots$, är uttömmande svit för D (se figur ovan).

$$\iint_{D_{m,n}}f(x,y)\,dx\,dy=\int_{1/m}^{1}dx\,\frac{1}{\sqrt{x}}\int_{1/n}^{1-x}\frac{dy}{\sqrt{y}}=$$

$$=\int_{1/m}^{1}\left(\left[2\sqrt{y}\right]_{1/n}^{1-x}\right)\frac{1}{\sqrt{x}}\,dx=$$

$$=2\int_{1/m}^{1}\left(\sqrt{1-x}-\sqrt{1/n}\right)\frac{1}{\sqrt{x}}\,dx=$$

$$=2\int_{1/m}^{1}\sqrt{\frac{1}{x}-1}\,dx-\frac{2}{\sqrt{n}}\int_{1/m}^{1}\frac{dx}{\sqrt{x}}=$$

$$=2\int_{1/m}^{1}\sqrt{\frac{1}{x}-1}\,dx-\frac{4}{\sqrt{n}}\left(1-\frac{1}{\sqrt{m}}\right)=$$

$$=\frac{4}{\sqrt{n}}\left(\frac{1}{\sqrt{m}}-1\right)+2\int_{1/m}^{1}\sqrt{x^{-1}-1}\,dx\left[\begin{array}{l}t^2=x^{-1}-1\\ dx=\frac{-2t}{(t^2+1)^2}dt\end{array}\right]$$

$$=\frac{4}{\sqrt{n}}\left(\frac{1}{\sqrt{m}}-1\right)+4\int_0^{\sqrt{m-1}}\frac{t^2}{(t^2+1)^2}\,dt=$$

$$=(m,n\longrightarrow\infty)=\int_0^\infty 4\,\frac{t^2}{(t^2+1)^2}\,dt=\int_0^\infty 4\left(\frac{1}{t^2+1}-\frac{1}{(t^2+1)^2}\right)dt=$$

$$=4\int_0^\infty\frac{dt}{t^2+1}-4\int_0^\infty\frac{dt}{(t^2+1)^2}=$$

$$=4\cdot\frac{\pi}{2}-4\int_0^\infty\frac{dt}{(t^2+1)^2}\,[t=\tan v]=$$

$$=2\pi-4\int_0^{\pi/2}\cos^2 v\,dv=2\pi-\pi=\pi.$$

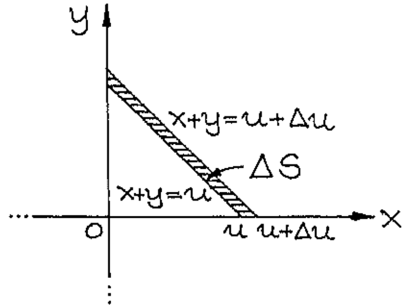
$$=2\pi-4\cdot\frac{\pi}{4}=2\pi-\pi=\underline{\pi.}$$

Innan man läser detta avsnitt bör man läsa Alexanderssons GNP.

Problem 6.46 (Sid. 18)

Lösning: $f(x,y)=1/(1+(x+y)^4)$, $D=[0,\infty[^2$.

Jag väljer integrationselement som i figuren:



$$\Delta S=\frac{1}{2}(u+\Delta u)^2-\frac{1}{2}u^2=u\,\Delta u+\frac{1}{2}(\Delta u)^2\approx u\,\Delta u \Leftrightarrow$$

$\Leftrightarrow dS=u\,du$ (Jfr integration via nivåkurvor).

$$\iint_D f(x,y)=\lim_{u\to\infty}\int_0^u \frac{v}{1+v^4}\,dv \overset{!}{=} \lim_{u\to\infty}\frac{1}{2}\arctan u^2=\frac{\pi}{4}.$$

Vid $\overset{!}{=}$ underförstås substitutionen $v=u^2$.

Problem 6.47 (Sid. 18)

Lösning

$$f(x)=e^{-x^2-xy-y^2},\quad D=\mathbb{R}^2=\mathbb{R}\times\mathbb{R}.$$

(1) $D_n=\{(x,y): x^2+xy+y^2\leq n^2\}$, $n=1,2,3,\ldots$, är en uttömmande svit för $\mathbb{R}^2$.

(2) $x^2+y^2+xy=(x+\frac{1}{2}y)^2+(\frac{\sqrt{3}}{2}y)^2$.

(3) $\begin{cases} u=x+\frac{1}{2}y \\ v=\frac{\sqrt{3}}{2}y \end{cases} \Rightarrow \frac{d(u,v)}{d(x,y)}=\frac{\sqrt{3}}{2}=\left(\frac{d(x,y)}{d(u,v)}\right)^{-1} \Leftrightarrow \frac{d(x,y)}{d(u,v)}=\frac{2}{\sqrt{3}}$;

D_n avbildas på $D_n'=\{(u,v): u^2+v^2\leq n^2\}$ så att även D_n', $n=1,2,3,\ldots$, är en uttömmande svit för $\mathbb{R}^2$.

(4) $\iint_D f(x,y)\,dxdy=\lim_{n\to\infty}\frac{2}{\sqrt{3}}\iint_{D_n'} e^{-u^2-v^2}\,dudv=$

$$=\lim_{n\to\infty}\frac{1}{\sqrt{3}}\int_0^n e^{-r^2}2r\,dr\int_0^{2\pi}d\varphi=\lim_{n\to\infty}\frac{1}{\sqrt{3}}(1-e^{-n^2})\cdot 2\pi=\frac{2\pi}{\sqrt{3}}.$$

Problem 6.48 (Sid. 18)

Lösning

Jag använder samma integrationselement som i 6.46 ovan.

$$\iint_D f(x,y)\,dxdy=\lim_{\varepsilon\to 0^+}\int_0^{1-\varepsilon}\frac{u}{(1-u)^\alpha}\,du=$$

$$=\lim_{\varepsilon\to 0^+}\left(\left[\frac{1}{\alpha-1}\frac{u}{(1-u)^{\alpha-1}}\right]_0^{1-\varepsilon}-\frac{1}{\alpha-1}\int_0^{1-\varepsilon}\frac{du}{(1-u)^{\alpha-1}}\right)=$$

$$=\lim_{\varepsilon\to 0^+}\left(\frac{(1-\varepsilon)\varepsilon^{1-\alpha}}{\alpha-1}-\frac{1}{(\alpha-1)(\alpha-2)}\left[\frac{1}{(1-u)^{\alpha-2}}\right]_0^{1-\varepsilon}\right)=$$

$$=\lim_{\varepsilon\to 0^+}\left(\frac{(1-\varepsilon)\varepsilon^{1-\alpha}}{\alpha-1}-\frac{\varepsilon^{2-\alpha}-1}{(\alpha-1)(\alpha-2)}\right)=(\alpha<1)=$$

$$=\frac{1}{(1-\alpha)(2-\alpha)}.$$

Svar: Integralens värde är $\frac{1}{(1-\alpha)(2-\alpha)}$ för $\alpha<1$; för $\alpha\geq 1$ divergerar den.

Problem 6.49 (Sid. 18)

Lösning

$D_n = \{(x,y,z) : x^2+y^2+z^2 \leq n^2\}$, $n=1,2,3,\ldots$, är en uttömmande svit för $\mathbb{R}^3$.

Rymdpolära koordinater (r,θ,φ) införs. Vi får

$$D_n': \ 0 \leq r \leq n,\ 0 \leq \theta \leq \pi,\ 0 \leq \varphi \leq 2\pi.$$

$$\iiint_{D_n} e^{-|x|} \cdot \frac{1}{|x|}\, dx\,dy\,dz = \iiint_{D_n'} e^{-r} \cdot \frac{1}{r} r^2 \sin\theta\, dr\, d\theta\, d\varphi =$$

$$= \int_0^n r e^{-r}\, dr \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\varphi = \left[-(r+1)e^{-r}\right]_0^n \cdot 2 \cdot 2\pi =$$

$$= 4\pi(1-(n+1)e^{-n}) \xrightarrow[n\to\infty]{} 4\pi.$$

Resultat: $\iiint_{\mathbb{R}^3} e^{-|x|} \frac{1}{|x|}\, dx\,dy\,dz = 4\pi.$

Problem 6.50 (Sid. 18)

Lösning

$$f(x) = |x|^{-1};\quad D: \sqrt{x^2+y^2} \leq z \leq 1$$

(1) Rymdpolära koordinater införs: $\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$;

$$z \leq 1 \Rightarrow r\cos\theta \leq 1 \Leftrightarrow 0 \leq r \leq \frac{1}{\cos\theta}$$

$$\sqrt{x^2+y^2} \leq z \Leftrightarrow r\sin\theta \leq r\cos\theta \Leftrightarrow \tan\theta \leq 1 \Leftrightarrow 0 \leq \theta \leq \frac{\pi}{4}.$$

Rotationssymmetri kring z-axeln $\Rightarrow 0 \leq \varphi \leq 2\pi$.

(2) $$\iiint_D \frac{1}{|x|}\, dx\,dy\,dz = \int_0^{\pi/4} d\theta \sin\theta \int_0^{1/\cos\theta} r\,dr \int_0^{2\pi} d\varphi =$$

$$= \int_0^{\pi/4} \left(\left[\tfrac{1}{2} r^2\right]_0^{1/\cos\theta}\right) \sin\theta\, d\theta \cdot 2\pi =$$

$$= \pi \int_0^{\pi/4} \frac{\sin\theta}{\cos^2\theta}\, d\theta =$$

$$= \pi \left[\frac{1}{\cos\theta}\right]_0^{\pi/4} = \pi(\sqrt{2}-1).$$

Problem 6.51 (Sid. 18)

Lösning

a) $f(x) = x/(1+|x|^2)$; $D_n: x^2+y^2 \leq n^2$, $n=1,2,\ldots$

$$\iint_{D_n} \frac{x}{1+x^2+y^2}\, dx\,dy \left[\begin{array}{c|c} x = r\cos v & 0 \leq r \leq n \\ y = r\sin v & 0 \leq v \leq 2\pi \end{array}\right] =$$

$$= \int_0^n \frac{r^2}{1+r^2}\, dr \int_0^{2\pi} \cos v\, dv =$$

$$= \int_0^n \frac{r^2}{r^2+1}\, dr \cdot 0 = 0, \text{ för alla } n.$$

Anm. $f(x,y) = \dfrac{x}{1+x^2+y^2} \Rightarrow f(-x,y) = -f(x,y) \Rightarrow$

$$\Rightarrow \iint_{D_n} f(x,y)\, dx\,dy = \int_{-n}^{n} dy \int_{-\sqrt{n^2-y^2}}^{\sqrt{n^2-y^2}} \frac{x}{1+x^2+y^2}\, dx = 0.$$

b) $\iint_{D_n} \frac{x}{1+(x^2+y^2)^2}\, dx\,dy = 0$, enl. anmärkningen ovan.

c) $f(x,y) = (x-2y)e^{-2x-y}$, $D: x \geq 0, y \geq 0$.

Som uttömmande svit för D kan användas

$D_{mn}=\{(x,y): 0\leq x\leq m,\ 0\leq y\leq n\}$, $m,n=0,1,\ldots$

$$\iint_{D_{mn}} f(x,y)\,dx\,dy=\int_0^n dy\, e^{-y}\int_0^m (x-2y)e^{-2x}\,dx=$$

$$=\int_0^n \left(\left[\tfrac{1}{2}(2y-x)e^{-2x}-\tfrac{1}{4}e^{-2x}\right]_{x=0}^{m}\right)e^{-y}\,dy=$$

$$=\int_0^n \left(\tfrac{1}{2}(2y-m)e^{-2m}-\tfrac{1}{4}e^{-2m}-y+\tfrac{1}{4}\right)e^{-y}\,dy \xrightarrow[m\to\infty]{}$$

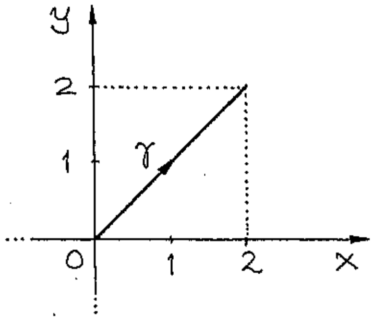
$$\xrightarrow[m\to\infty]{}\int_0^n \left(\tfrac{1}{4}-y\right)e^{-y}\,dy=\left[\left(y+\tfrac{3}{4}\right)e^{-y}\right]_0^n=-\tfrac{3}{4}-\left(n+\tfrac{3}{4}\right)e^{-n}\xrightarrow[n\to\infty]{}-\tfrac{3}{4}.$$

Vektoranalys i planet

Problem 9.1 (Sid. 19)

Lösning

a)



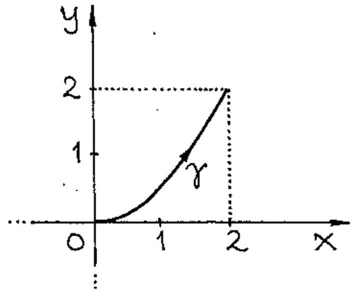
$\gamma: (x,y)=(t,t),\ 0\leq t\leq 2.$

$$\omega=(x^2+xy)\,dx+(y^2-xy)\,dy=\begin{bmatrix}x=t\Rightarrow dx=dt\\ y=t\Rightarrow dy=dt\end{bmatrix}=2t^2\,dt;$$

$$\int_\gamma \omega=\int_0^2 2t^2\,dt=\left[\tfrac{2}{3}t^3\right]_0^2=\tfrac{16}{3}.$$

Anm. Att beräkna en linjeintegral är detsamma som att integrera en differentialform.

b)



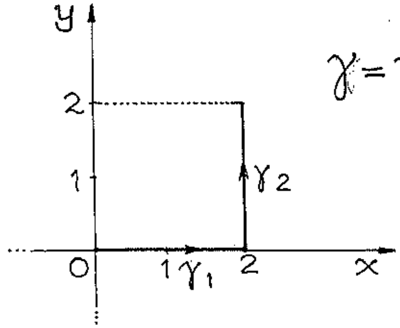
$\gamma: (x,y)=(2t,2t^2),\ 0\leq t\leq 1.$

$$\begin{cases}x=2t\Rightarrow dx=2dt\\ y=2t^2\Rightarrow dy=4t\,dt\end{cases}\Rightarrow \omega=(4t^2+4t^3)\cdot 2dt+(4t^4-4t^3)\cdot 4dt=$$

$$=(8t^2+8t^3+16t^5-16t^4)\,dt\Rightarrow \int_\gamma\omega=\int_0^1 8(t^2+t^3+2t^5-2t^4)\,dt=$$

$$=8\left[\frac{t^3}{3}+\frac{t^4}{4}+\frac{1}{3}t^6-\frac{2}{5}t^5\right]_0^1=8\left(\frac{2}{3}+\frac{1}{4}-\frac{2}{5}\right)=8\,\frac{40+15-24}{3\cdot4\cdot5}=\frac{62}{15}.$$

c)



$\gamma=\gamma_1+\gamma_2$

$\gamma_1: (x,y)=(t,0),\ 0\leq t\leq 2;$ $\gamma_2: (x,y)=(2,t),\ 0\leq t\leq 2.$

$$\int_\gamma\omega=\int_{\gamma_1}\omega+\int_{\gamma_2}\omega=\int_0^2 t^2\,dt+\int_0^2 (t^2-2t)\,dt=\frac{8}{3}-4=-\frac{4}{3}.$$

Problem 9.2 (Sid. 19)

Lösning

Det gäller att integrera formeln

$$\omega = y^2 dx + x^2 dy$$

över cirkelkurvan C: $(x-a)^2+(y-b)^2=r^2$.

a) C: $\gamma(t)=(a+r\cos t, b+r\sin t)$, $0\le t\le 2\pi$.

$$\omega(\gamma)=(b+r\sin t)^2(-r\sin t)dt+(a+r\cos t)^2(r\cos t)dt=$$
$$=(b^2+r^2\sin^2 t+2br\sin t)(-r\sin t)\,dt+$$
$$+(a^2+r^2\cos^2 t+2ar\cos t)(r\cos t)\,dt=$$
$$=r(a^2\cos t-b^2\sin t)\,dt+$$
$$+r^2(2b\sin^2 t-2a\cos^2 t)\,dt+$$
$$+r^3(\cos^3 t-\sin^3 t)\,dt \Rightarrow$$

$$\oint_\gamma \omega=\int_0^{2\pi}\omega(\gamma)=r\int_0^{2\pi}(a^2\cos t-b^2\sin t)\,dt+$$
$$+r^2\int_0^{2\pi}(2b\sin^2 t-2a\cos^2 t)\,dt+$$
$$+r^3\int_0^{2\pi}(\cos^3 t-\sin^3 t)\,dt=$$
$$=0+2\pi(b-a)r^2+0=\underline{\underline{2\pi(b-a)r^2}}.$$

Anmärkningar

(1) $\int_0^{2\pi}\cos kt\,dt=\int_0^{2\pi}\sin kt\,dt=0$, k konstant $\neq 0$.

(2) $\int_0^{2\pi}\cos^2 kt\,dt=\frac{1}{2}\int_0^{2\pi}(1+\cos 2kt)\,dt=$
$$=\frac{1}{2}\left(\int_0^{2\pi}dt+\int_0^{2\pi}\cos 2kt\,dt\right)=(1)=$$
$$=\frac{1}{2}\cdot 2\pi=\pi.$$

(3) $\int_0^{2\pi}\sin^2 kt\,dt=\int_0^{2\pi}(1-\cos^2 t)\,dt=2\pi-\pi=\pi.$

(4) $\int_0^{2\pi}\cos^3 t\,dt=\int_0^{2\pi}\cos^2 t\cos t\,dt=\int_0^{2\pi}(1-\sin^2 t)\cos t\,dt=$
$$=\left[\sin t-\frac{1}{3}\sin^3 t\right]_0^{2\pi}=0=\int_0^{2\pi}\sin^3 t\,dt.$$

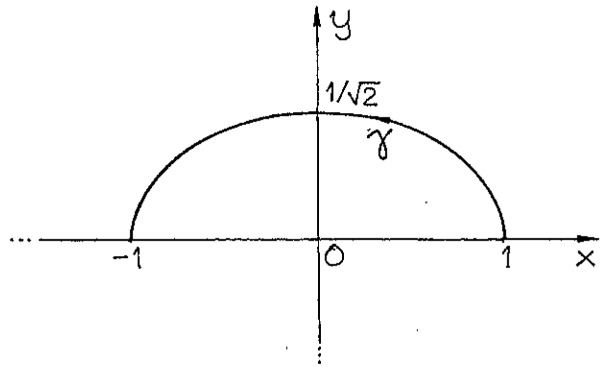
b) $\oint_\gamma \omega=\iint_D d\omega=\iint_D 2(x-y)\,dx\,dy\left[\begin{array}{l|l} x=a+\rho\cos t & 0\le\rho\le r \\ y=b+\rho\sin t & 0\le t\le 2\pi\end{array}\right]$
$$=\int_0^r 2\rho\,d\rho\int_0^{2\pi}(a-b+\rho(\cos t-\sin t))\,dt=(1)=\underline{\underline{2\pi r^2(a-b)}}.$$

Problem 9.3 (Sid. 19)

Lösning

$\underline{\omega=(x-y)dx+(x+y)dy}$; $\underline{\gamma: x^2+2y^2=1,\ (1,0)\to(-1,0)}$

Metod 1 (som linjeintegral)



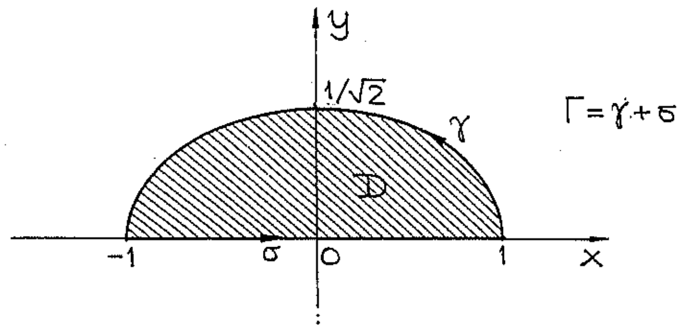
$C: (x,y)=(\cos t, \frac{1}{\sqrt{2}}\sin t), \quad 0\le t\le \pi.$

$$\omega = (\cos t - \frac{\sin t}{\sqrt{2}})(-\sin t)dt + (\cos t + \frac{\sin t}{\sqrt{2}})\frac{\cos t}{\sqrt{2}}dt =$$
$$= (-\sin t\cos t + \frac{1}{\sqrt{2}}\sin^2 t + \frac{1}{\sqrt{2}}\cos^2 t + \frac{1}{2}\sin t\cos t)dt =$$
$$= (\frac{1}{\sqrt{2}}(\sin^2 t + \cos^2 t) - \frac{1}{2}\sin t\cos t)dt = (\text{trig. ettan}) =$$
$$= (\frac{1}{\sqrt{2}} - \frac{1}{2}\sin t\cos t)dt;$$

$$\int_C \omega = \int_0^{\pi}(\frac{1}{\sqrt{2}} - \frac{1}{2}\sin t\cos t)dt = \frac{\pi}{\sqrt{2}} - 0 = \underline{\frac{\pi}{\sqrt{2}}}.$$

Metod 2 (med Greens formel)

Jag stänger kurvbågen med storaxeln:



$$d\omega = (\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial y}(x-y))dxdy = 2dxdy;$$

$$\oint_\Gamma \omega = \iint_D d\omega = 2\iint_D dxdy \overset{!}{=} 2\cdot\frac{1}{2}\cdot 1\cdot\frac{1}{\sqrt{2}}\pi = \frac{\pi}{\sqrt{2}} \Leftrightarrow$$

$$\Leftrightarrow \int_\gamma \omega + \int_\sigma \omega = \frac{\pi}{\sqrt{2}} \Leftrightarrow \int_\gamma \omega = \frac{\pi}{\sqrt{2}} - \int_{-1}^{1} x\,dx = \underline{\frac{\pi}{\sqrt{2}}}.$$

Anm. $\omega = Pdx + Qdy \Rightarrow d\omega = (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy.$

Problem 9.4 (Sid. 19)

Lösning

$y^2 = x^3 \Rightarrow x = t^2 \wedge y = t^3 \Rightarrow \underline{(x,y) = (t^2, t^3),\ 0\le t\le 1.}$

$$dW = \mathcal{F}\cdot d\mathbf{r} = (3x^3y, -2xy^2)\cdot(dx, dy) =$$
$$= (3\cdot t^6\cdot t^3, -2t^2\cdot t^6)\cdot(2t, 3t^2)dt =$$
$$= (3t^9, -2t^8)\cdot(2t, 3t^2)dt = 0.$$

Svar: Kraftfältet uträttar inget arbete på partikeln.

Problem 9.5 (Sid. 19)

Lösning

a) $A = (A_x, A_y) = (2x-2y, -2x+6y) \Rightarrow \begin{cases} A_x = 2x-2y \\ A_y = -2x+6y \end{cases} \Rightarrow$

$$\Rightarrow \frac{\partial}{\partial y}A_x = -2 = \frac{\partial}{\partial x}A_y \Rightarrow A \text{ potentialfält.}$$

$$d\Phi = \frac{\partial \Phi}{\partial x}dx + \frac{\partial \Phi}{\partial y}dy = A_x dx + A_y dy = (2x-2y)dx +$$
$$+(-2x+6y)dy = 2xdx - 2(ydx + xdy) + 6ydy =$$
$$= d(x^2) - 2d(yx) + d(3y^2) =$$
$$= d(x^2 - 2xy + 3y^2) \Leftrightarrow$$

$$\Leftrightarrow \underline{\Phi(x,y) = x^2 - 2xy + 3y^2 + C,\ C \text{ konstant}.}$$

Anm Läs om differentialer på sidan 116.

Annan lösning

$$\operatorname{grad}\Phi(\mathbf{x})=\left(\frac{\partial\Phi}{\partial x},\frac{\partial\Phi}{\partial y}\right)=(A_x,A_y)=\mathbf{A}\Leftrightarrow\begin{cases}\frac{\partial\Phi}{\partial x}=2x-2y & (1)\\ \frac{\partial\Phi}{\partial y}=-2x+6y & (2)\end{cases}$$

$$(1)\Rightarrow\Phi(\mathbf{x})=x^2-2xy+g(y)\Rightarrow\frac{\partial\Phi}{\partial y}=-2x+g'(y)\overset{(2)}{=}-2x+6y$$

$$\Leftrightarrow g'(y)=6y\Leftrightarrow g(y)=3y^2+C\Rightarrow\underline{\Phi(\mathbf{x})=x^2-2xy+3y^2+C.}$$

b) $$\left.\begin{matrix}A_x=y^2-x^2\\ A_y=2xy\end{matrix}\right\}\Rightarrow\frac{\partial}{\partial x}A_y=2y=\frac{\partial}{\partial y}A_x\Rightarrow\underline{\mathbf{A}\text{ potentialfält.}}$$

$$d\Phi=\frac{\partial\Phi}{\partial x}dx+\frac{\partial\Phi}{\partial y}dy=A_xdx+A_ydy=(y^2-x^2)dx+2xydy=$$

$$=y^2dx+2xydy+(-x^2)dx=y^2dx+xdy^2-d\left(\frac{x^3}{3}\right)=$$

$$=d(xy^2)-d\left(-\frac{x^3}{3}\right)=d\left(xy^2-\frac{1}{3}x^3\right)\Leftrightarrow\underline{\Phi(\mathbf{x})=xy^2-\frac{x^3}{3}+C.}$$

c) $$\mathbf{A}=(A_x,A_y)=(2xy,y^2-x^2)\Leftrightarrow A_x=2xy\wedge A_y=y^2-x^2$$

$$\Rightarrow\frac{\partial}{\partial y}A_x-\frac{\partial}{\partial x}A_y=2x-(-2x)=4x\neq0\Rightarrow\underline{\mathbf{A}\text{ är inget potentialfält.}}$$

d) $$\mathbf{A}=(A_x,A_y)=\left(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2}\right)\Leftrightarrow\left\{\begin{matrix}A_x=\frac{x}{x^2+y^2}\\ A_y=\frac{y}{x^2+y^2}\end{matrix}\right\}\Rightarrow$$

$$\Rightarrow\frac{\partial}{\partial x}A_y=-\frac{2xy}{(x^2+y^2)^2}=\frac{\partial}{\partial y}A_x\Rightarrow\underline{\mathbf{A}\text{ potentialfält.}}$$

$$d\Phi=\frac{\partial\Phi}{\partial x}dx+\frac{\partial\Phi}{\partial y}dy=A_xdx+A_ydy=\frac{xdx+ydy}{x^2+y^2}=$$

$$=\frac{1}{2}\frac{2xdx+2ydy}{x^2+y^2}=\frac{1}{2}\frac{d(x^2+y^2)}{x^2+y^2}=\frac{1}{2}d(\ln(x^2+y^2))\Leftrightarrow$$

$$\Leftrightarrow\underline{\Phi(x)=\frac{1}{2}\ln(x^2+y^2)+C.}$$

Problem 9.6 (Sid. 19)

Lösning

a) $\underline{C:(x,y)=(\cos t,\sin t),\ 0\leq t\leq 2\pi}$; $(\underline{C}: x^2+y^2=1)$.

$$\oint_C A_xdx+A_ydy=\oint_C\frac{-ydx+xdy}{x^2+y^2}=\int_0^{2\pi}\frac{\sin^2t+\cos^2t}{\cos^2t+\sin^2t}dt=$$

$=\int_0^{2\pi}dt=2\pi\neq0$; hos ett potentialfält blir cirkulationen 0; vårt fält är således inget potentialfält i ett område som omfattar origo.

b) $\underline{D=\mathbb{R}_+\times\mathbb{R}=\{(x,y): x>0\}.}$

$$d\Phi=\frac{\partial\Phi}{\partial x}dx+\frac{\partial\Phi}{\partial y}dy=A_xdx+A_ydy=\frac{-y}{x^2+y^2}dx+\frac{x}{x^2+y^2}dy=$$

$$=\frac{xdy-ydx}{x^2+y^2}=\frac{1}{1+(y/x)^2}\frac{xdy-ydx}{x^2}=\frac{1}{1+(y/x)^2}d\left(\frac{y}{x}\right)=$$

$$=d\left(\arctan\frac{y}{x}\right)\Leftrightarrow\underline{\Phi(\mathbf{x})=\arctan\frac{y}{x}+C.}$$

c) $\underline{D=\mathbb{R}\times\mathbb{R}_+=\{(x,y): y>0\}.}$

$$d\Phi=\frac{ydx-xdy}{x^2+y^2}=-\frac{1}{1+(x/y)^2}\frac{ydx-xdy}{y^2}=-\frac{1}{1+(x/y)^2}d\left(\frac{x}{y}\right)=$$

$$=-d\left(\arctan\frac{x}{y}\right)\Leftrightarrow\underline{\Phi(\mathbf{x})=-\arctan\frac{x}{y}+C.}$$

Anm. $$\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}\Rightarrow\begin{cases}\tan\theta=\frac{y}{x}\\ \cot\theta=\frac{x}{y}\end{cases}\Rightarrow\begin{cases}\theta=\arctan\frac{y}{x}\\ \theta=\operatorname{arccot}\frac{x}{y}\end{cases};$$

Problem 9.7 (Sid. 19)

Lösning

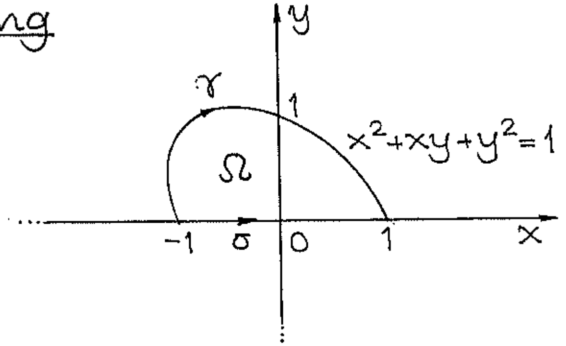
$$d\Phi=\frac{\partial\Phi}{\partial x}dx+\frac{\partial\Phi}{\partial y}dy=A_x dx+A_y dy=\frac{2x^2y\,dy-2xy^2dx}{x^4+y^4}=$$

$$=\frac{1}{1+y^4/x^4}\cdot\frac{2x^2y\,dy-2xy^2dx}{x^4}=\frac{1}{1+(y^2/x^2)^2}\,d\left(\frac{y^2}{x^2}\right)=$$

$$=d\left(\arctan\left(\frac{y}{x}\right)^2\right)\Leftrightarrow \underline{\Phi(x)=\arctan\left(\frac{y}{x}\right)^2+C.}$$

Problem 9.8 (Sid. 19)

Lösning



Jag tillämpar Greens formel på ovanstående kontur för fältet $\mathbb{F}=(x^2,y^2)$.

$$\oint_\Gamma x^2dx+y^2dy=\left(\int_\gamma-\int_\sigma\right)x^2dx+y^2dy=\iint_\Omega 0\,dxdy$$

$$\Leftrightarrow\int_\gamma x^2dx+y^2dy=\int_\sigma x^2dx=\int_{-1}^{1}x^2dx=\underline{\frac{2}{3}}.$$

Greens formel tillämpas på slutna, enkla konturer; σ stänger γ och avskärmar Ω.

Problem 9.9 (Sid. 19)

Lösning

$$dS=\frac{1}{2}(x\,dy-y\,dx)=\frac{1}{2}\begin{vmatrix}x & y\\ dx & dy\end{vmatrix}=\frac{1}{2}\begin{vmatrix}x & y\\ \dot{x} & \dot{y}\end{vmatrix}dt\,;\ \begin{cases}\dot{x}=\frac{dx}{dt}\\ \dot{y}=\frac{dy}{dt}\end{cases}$$

a) $|x|^{2/3}+|y|^{2/3}=1=\cos^2t+\sin^2t\Leftrightarrow\begin{cases}x=\cos^3t\\ y=\sin^3t\end{cases},\ 0\le t\le 2\pi;$

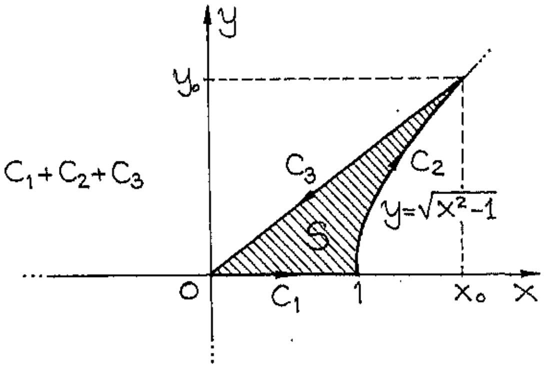
$$\begin{cases}\frac{dx}{dt}=-3\cos^2t\sin t\\ \frac{dy}{dt}=3\sin^2t\cos t\end{cases}\Rightarrow dS=\frac{1}{2}\begin{vmatrix}\cos^3t & \sin^3t\\ -3\cos^2t\sin t & 3\sin^2t\cos t\end{vmatrix}dt=$$

$$=\frac{3}{2}\cos^2t\cdot\sin^2t\begin{vmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{vmatrix}dt=\frac{3}{8}(2\sin t\cdot\cos t)^2dt=$$

$$=\frac{3}{8}\sin^2 2t\,dt=\frac{3}{8}\,\frac{1-\cos 4t}{2}\,dt=\frac{3}{16}(1-\cos 4t)\,dt\Rightarrow$$

$$\Rightarrow S=\frac{3}{16}\int_0^{2\pi}(1-\cos 4t)\,dt=\frac{3}{16}\cdot 2\pi=\underline{\frac{3\pi}{8}}.$$

b)



$C=C_1+C_2+C_3$

$\underline{dS=\frac{1}{2}(x\,dy-y\,dx)}.$

Den hyperboliska ettan är: $\cosh^2t-\sinh^2t=1$.

På C_1 och C_3 har vi $dS=0$ och på C_2 är $dS=dt$

$$S=\frac{1}{2}\left(\int_{C_1}+\int_{C_2}+\int_{C_3}\right)(x\,dy-y\,dx)=\frac{1}{2}\int_0^{t_0}dt=\frac{1}{2}t_0 \Leftrightarrow$$

$\Leftrightarrow S=\frac{1}{2}\operatorname{Arsinh} x_0$ (areasinushyperbolicus).

Problem 9.10 (Sid. 20)

Lösning: $C: (x,y)=(\cos t, \sin t),\ 0\leq t\leq 2\pi$.

(1) $\omega = A_x\,dx + A_y\,dy = \dfrac{-y^3dx+xy^2dy}{(x^2+y^2)^2}$;

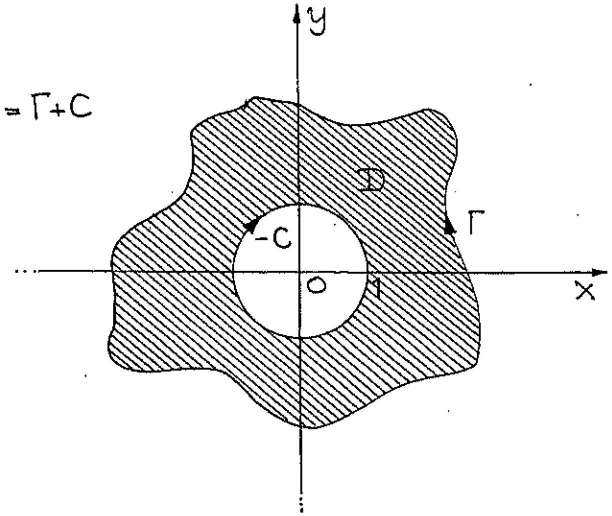
$\omega(C)=(\sin^4 t+\cos^2 t\sin^2 t)\,dt=\sin^2 t(\cos^2 t+\sin^2 t)\,dt=$

$=\sin^2 t\,dt=\frac{1}{2}(1-\cos 2t)\,dt$;

$\oint_C \omega=\int_0^{2\pi}\frac{1}{2}(1-\cos 2t)\,dt=\frac{1}{2}\cdot 2\pi=\pi.$

(2) $\dfrac{\partial}{\partial y}A_x=\dfrac{y^4-x^2y^2}{(x^2+y^2)^3}=\dfrac{\partial}{\partial x}A_y$, utanför origo

$\partial D=\Gamma+C$



För att tillämpa Greens formel på D ska dess rand genomlöpas positivt, dvs D ska vara till vänster om sin rand. Origo ligger utanför D, dvs $d\omega=\left(\frac{\partial}{\partial x}A_y-\frac{\partial}{\partial y}A_x\right)dx\,dy=0$ och vi får

$$\oint_{\partial D}\omega=\iint_D d\omega=0 \Leftrightarrow \int_\Gamma\omega+\int_{-C}\omega=\int_\Gamma\omega-\int_C\omega=0\Leftrightarrow$$

$\Leftrightarrow \oint_\Gamma\omega=\oint_C\omega=\pi$, VSV.

(3) Om kurvan inte omsluter origo är ω exakt och då är dubbelintegralen i Greens form 0.

Problem 9.11 (Sid. 20)

Lösning

a) $\begin{cases} z'_x+zf=0 \\ z'_y+zg=0 \end{cases} \Rightarrow f\,dx+g\,dy=-\dfrac{z'_x dx+z'_y dy}{z}=-\dfrac{dz}{z}$;

b) $d\Phi=\dfrac{\partial\Phi}{\partial x}dx+\dfrac{\partial\Phi}{\partial y}dy\overset{!}{=}f\,dx+g\,dy=-\dfrac{dz}{z}\Leftrightarrow\Phi=-\ln|z|+C$

$\Leftrightarrow \underline{z=Ae^{-\Phi},\ A \text{ konstant.}}$

Problem 9.12 (Sid. 20)

Lösning

(1) $\mathcal{F}=(P,Q)$ konservativ $\Leftrightarrow \dfrac{\partial Q}{\partial x}=\dfrac{\partial P}{\partial y}$.

(2) $\mathbb{F}=(h,-g) \Rightarrow \text{rot}\,\mathbb{F}=-(\frac{\partial g}{\partial x}+\frac{\partial h}{\partial y})\overset{!}{=}0 \Leftrightarrow \mathbb{F}$ konservativt $\Rightarrow$

$\Rightarrow$ det existerar ett C^2-fält $\mathcal{U}$ s.a: $\nabla \mathcal{U}=(h,-g)$.

(3) $\mathbb{G}=(-g,f) \Rightarrow \text{rot}\,\mathbb{G}=\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}\overset{!}{=}0 \Rightarrow \mathbb{G}$ konservativt $\Rightarrow$

$\Rightarrow$ det existerar ett C^2-fält V s.a. $\nabla V=(-g,f)$.

(4) $\text{grad}\,\mathcal{U}(\mathbf{x})=(h,-g) \Rightarrow \mathcal{U}(\mathbf{x})=\int_\gamma h\,dx-g\,dy$;

$\text{grad}\,V(\mathbf{x})=(-g,f) \Rightarrow V(\mathbf{x})=\int_\delta -g\,dx+f\,dy$;

(5) $K=(\mathcal{U},V) \overset{(4)}{\Rightarrow} \frac{\partial \mathcal{U}}{\partial y}=-g=\frac{\partial V}{\partial x} \Rightarrow \mathbb{K}$ konservativt $\Rightarrow$

$\Rightarrow$ det existerar C^2-fält u s.a. $\text{grad}\,u(\mathbf{x})=(\mathcal{U},V)$.

(6) $\frac{\partial u}{\partial x}=\mathcal{U} \Rightarrow \frac{\partial^2 u}{\partial x^2}=\frac{\partial \mathcal{U}}{\partial x}=h$; $\frac{\partial u}{\partial y}=V \overset{(4)}{\Rightarrow} \frac{\partial^2 u}{\partial x \partial y}=\frac{\partial V}{\partial x}=-g$ och

$\frac{\partial^2 u}{\partial y^2}=\frac{\partial V}{\partial y}=f$. (Skilj mellan u och $\mathcal{U}$)

Problem 9.13 (Sid. 20)

Lösning

$$\oint_\gamma -f\frac{\partial f}{\partial y}dx+f\frac{\partial f}{\partial x}dy = (\text{Greens formel})=$$

$$=\iint_D \left(\frac{\partial}{\partial x}\left(f\frac{\partial f}{\partial x}\right)+\frac{\partial}{\partial y}\left(f\frac{\partial f}{\partial y}\right)\right)dxdy=$$

$$=\iint_D \left(\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+f\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)\right)dxdy=$$

$$=\iint_D \left(\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2\right)dxdy \overset{!}{=} 0 \Leftrightarrow$$

$$\Leftrightarrow \left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2=0 \Leftrightarrow \frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0 \Rightarrow f(x,y)=C, \text{ konstant}.$$

$\underline{f(\mathbf{x})=0}$, för alla $\mathbf{x}\in\gamma$, dvs $f(\mathbf{x})\equiv 0$