1. Funktioner av flera variabler

Mängder i $\mathbb{R}^n$. Funktioner

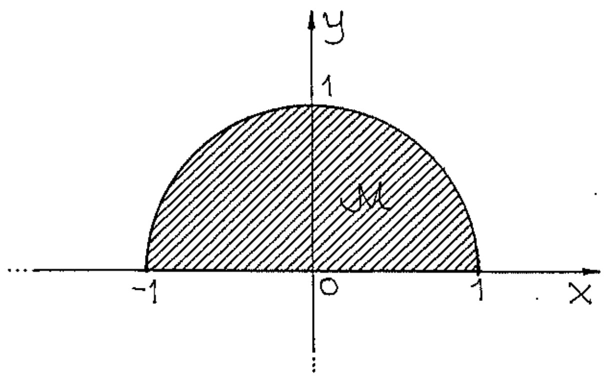
Problem 1.1 (Sid. 1)

Lösning

a) $$\underline{M = \{(x,y) : x^2+y^2 \leq 1 , y \geqslant 1\}}$$

$$M = \{(x,y) : x^2+y^2 \leq 1 \wedge y \geqslant 0\} =$$

$$= \{(x,y) : x^2+y^2 \leq 1\} \cap \{(x,y) : y \geqslant 0\} = M_1 \cap M_2 .$$

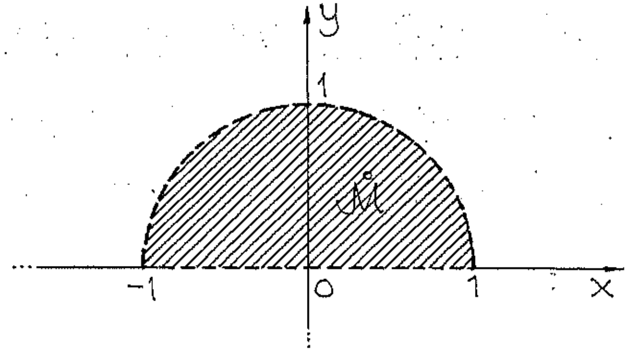


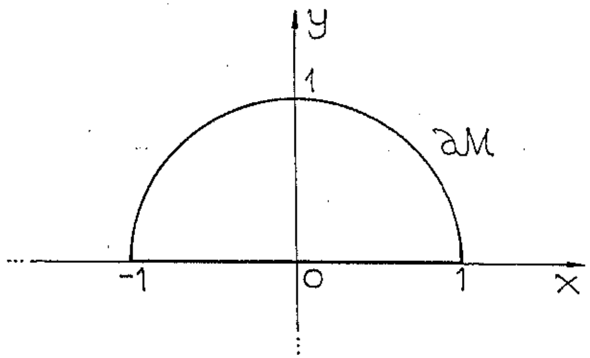
M är den del av enhetscirkeln som ligger på det övre halvplanet, halva enhetsdisken. Randen (konturen) ingår.

Anm. Det inre av en (punkt)mängd M betecknas $\mathring{M}$ och ibland $\mathrm{Int}(M)$; randen betecknas ∂M eller $\mathrm{Rand}(M)$.

I vårt fall är $\underline{\mathring{M} = \{(x,y) : 0 < y < \sqrt{1-x^2}\}}$ och $\underline{\partial M = \{(x,y) : y = \sqrt{1-x^2}, -1 \leq x \leq 1\} \cup \{(x,0) : -1 \leq x \leq 1\}}$.

Det inses lätt att $\underline{M = \mathring{M} \cup \partial M}$ (se figurer).



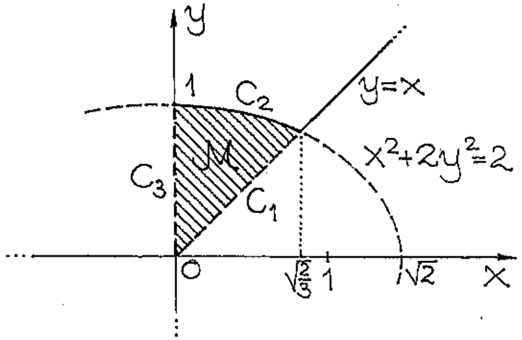


Punktmängdstopologin studeras i Appendix A; mängdläran ingår i en kurs i diskret matematik; punktmängder studeras i topologin.

b) $\underline{\mathcal{M}=\{(x,y): x^2+2y^2\leq 2,\ 0<x<y\}}$

$$\mathcal{M}=\{(x,y)\in\mathbb{R}^2:\ x^2+2y^2\leq 2 \wedge 0<x<y\}=$$
$$=\{(x,y)\in\mathbb{R}^2_+:\ x^2+2y^2\leq 2 \wedge x<y\}=$$
$$=\{(x,y)\in\mathbb{R}^2_+:\ \frac{x^2}{(\sqrt{2})^2}+\frac{y^2}{1^2}\leq 1 \wedge x<y\}.$$

$\mathcal{M}$ består av punkterna i den första kvadranten som ligger innanför och på ellipsen $\frac{x^2}{2}+y^2=1$ och samtidigt ovanför axelbisektrisen $y=x$.



$$\mathring{\mathcal{M}}=\{(x,y)\in\mathbb{R}^2_+:\ x<y\leq\sqrt{1-\frac{x^2}{2}}\};\qquad \mathbb{R}_+=\{x\in\mathbb{R}: x>0\}$$
$$\partial\mathcal{M}=C_1\cup C_2\cup C_3=\{(x,x): 0\leq x\leq\sqrt{\tfrac{2}{3}}\}\cup$$
$$\cup\{(x,y): y=\sqrt{1-\tfrac{x^2}{2}},\ 0\leq x\leq\sqrt{\tfrac{2}{3}}\}\cup$$
$$\cup\{(0,y): 0\leq y\leq 1\}$$

Anm. Endast bågen C_2 ingår i $\mathcal{M}$.

Problem 1.2 (Sid.1)

Lösning

Absolutbeloppet definieras av (a konstant)

$$|u-a|=\begin{cases} u-a, & u\geq a\\ -(u-a), & u<a\end{cases}$$

dvs avståndet från u till a på talaxeln.

a) $\underline{\mathcal{M}=\{(x,y): |x|+|y|<1\}}.$

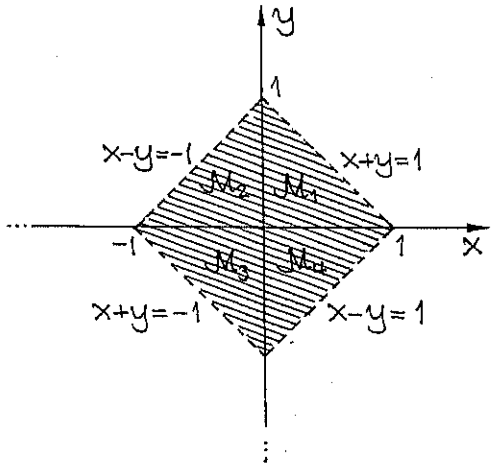
$$(1)\ \begin{cases} x\geq 0\Rightarrow |x|=x\\ y\geq 0\Rightarrow |y|=y\end{cases}\Rightarrow |x|+|y|=x+y\leq 1\Rightarrow \mathcal{M}_1:\begin{cases}x+y<1\\ x\geq 0\\ y\geq 0\end{cases};$$

$$(2)\ \begin{cases} x\leq 0\Rightarrow |x|=-x\\ y\geq 0\Rightarrow |y|=y\end{cases}\Rightarrow |x|+|y|=-x+y\leq 1\Rightarrow \mathcal{M}_2:\begin{cases}-x+y<1\\ x\leq 0\\ y\geq 0\end{cases};$$

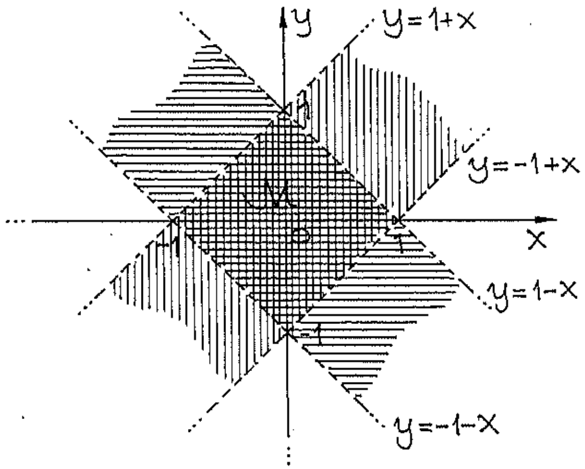
$$(3)\ \begin{cases} x\leq 0\Rightarrow |x|=-x\\ y\leq 0\Rightarrow |y|=-y\end{cases}\Rightarrow |x|+|y|=-x-y\leq 1\Rightarrow \mathcal{M}_3:\begin{cases}-x-y<1\\ x\leq 0\\ y\leq 0\end{cases}$$

$$(4)\ \begin{cases} x\geq 0\Rightarrow |x|=x\\ y\leq 0\Rightarrow |y|=-y\end{cases}\Rightarrow |x|+|y|=x-y\leq 1\Rightarrow \mathcal{M}_4:\begin{cases}x-y<1\\ x\geq 0\\ y\leq 0\end{cases}$$

$$\mathcal{M}=\mathcal{M}_1\cup\mathcal{M}_2\cup\mathcal{M}_3\cup\mathcal{M}_4=$$
$$=\underline{\{(x,y)\in\mathbb{R}^2: -1<x+y<1\}}\cap\underline{\{(x,y)\in\mathbb{R}^2: -1<x-y<1\}}.$$



M kan betraktas som unionen av 4 halva kvadrater, en i varje kvadrant, som en romb eller som skärningen (snittet) mellan två oändliga "band" i planet:

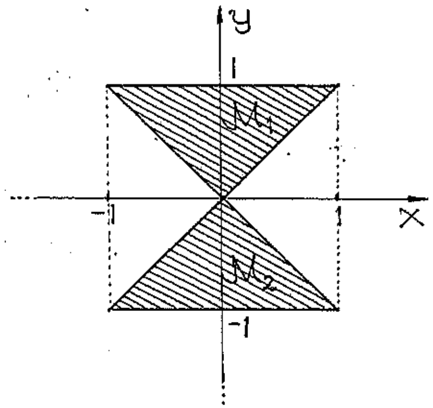


$M = \{(x,y) : -1-x < y < 1-x \ \wedge \ -1+x < y < 1+x\}$ (2 streck).

b) $\underline{M = \{(x,y) \in \mathbb{R}^2 : \max\{|x|,|y|\} \leq 1\}}$

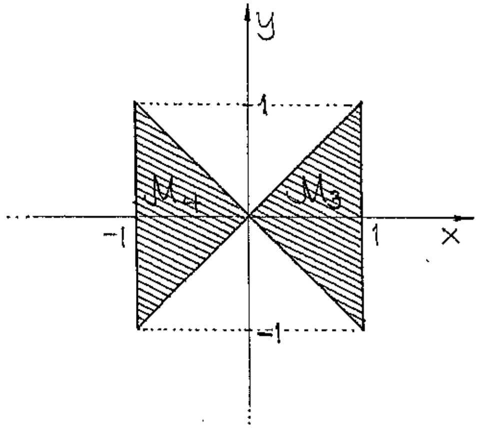
(1) $|x| \leq |y| \leq 1 \Leftrightarrow |x| \leq \pm y \leq 1 \Leftrightarrow |x| \leq y \leq 1 \vee |x| \leq -y \leq 1$

$\Leftrightarrow \underline{M_1 : |x| \leq y \leq 1} \quad \vee \quad \underline{M_2 : -1 \leq y \leq -|x|.}$



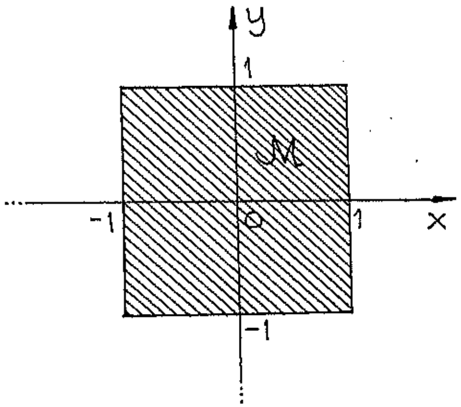
(2) $|y| \leq |x| \leq 1 \Leftrightarrow |y| \leq \pm x \leq 1 \Leftrightarrow |y| \leq x \leq 1 \vee |y| \leq -x \leq 1$

$\Leftrightarrow \underline{M_3 : |y| \leq x \leq 1} \quad \vee \quad \underline{M_4 : -1 \leq x \leq -|y|}.$



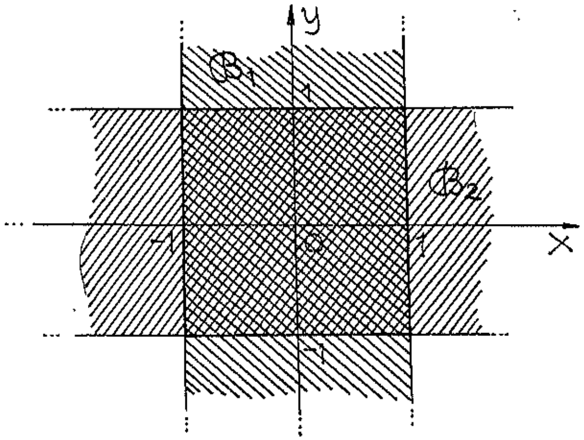
forts

$\mathcal{M}$ kan framställas som unionen mellan 4 halva kvadrater $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 \cup \mathcal{M}_4$ eller som en kvadrat, nämligen



$\underline{\mathcal{M} = \{(x,y) \in \mathbb{R}^2 : -1 \leq x, y \leq 1\} = [-1,1] \times [-1,1] = [-1,1]^2.}$

Den kan även uppfattas som skärningen mellan "banden" $B_1 : -1 \leq x \leq 1$ och $B_2 : -1 \leq y \leq 1$.



Problem 1.3 (Sid. 1)

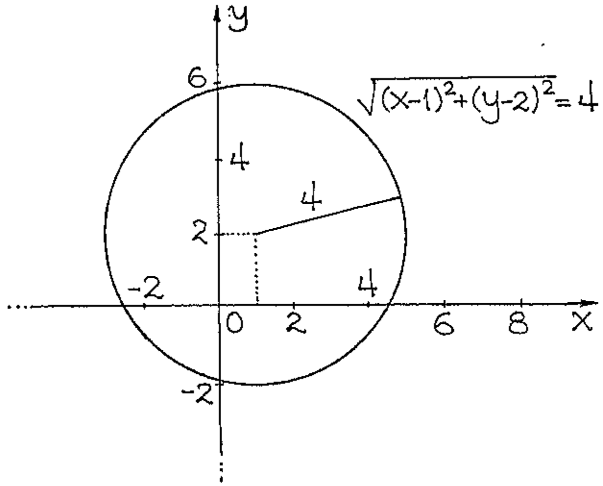
Lösning

a) $\underline{x^2 + y^2 - 2x - 4y = 11}$

$$x^2 - 2x + 1 + (y^2 - 4y + 4) = 16 \Leftrightarrow (x-1)^2 + (y-2)^2 = 4^2 \Leftrightarrow$$

$$\Leftrightarrow \underline{\sqrt{(x-1)^2 + (y-2)^2} = 4.}$$

Avståndet från punkten (x,y) till punkten $(1,2)$ är konstant; en cirkel (kurva) med radien 4 och medelpunkten i $(1,2)$.

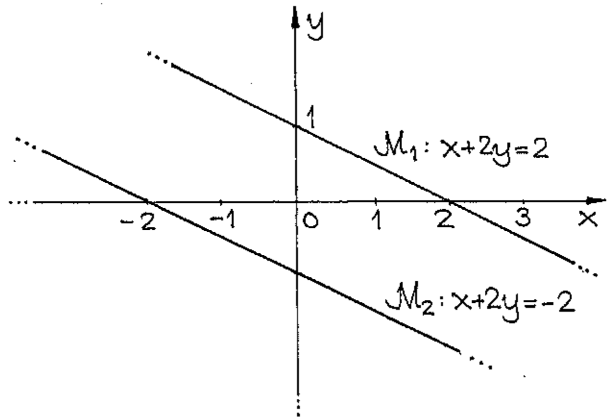


$\mathcal{M} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 - 2x - 4y = 11\}$ saknar inre punkter, den är för tunn; den består alltså av idel randpunkter. En randpunkt definieras som en punkt där varje omgivning omfattar punkter ur mängden och punkter ur dess komplement. Om detta kan man läsa på sidan 11 i läroboken.

Skilj mellan cirkel<u>kurva</u> och cirkel<u>skiva</u>:

$\mathcal{M}=\{(x,y):\ x^2+y^2-2x-4y=11\} \Rightarrow \mathring{\mathcal{M}}=\emptyset \wedge \partial\mathcal{M}=\mathcal{M}.$

b) $|x+2y|=2 \Leftrightarrow \underline{x+2y=2} \vee \underline{x+2y=-2}$; två linjer.



$\mathcal{M}=\mathcal{M}_1\cup\mathcal{M}_2=\{(x,y):\ x+2y=2\}\cup\{(x,y):\ x+2y=-2\}.$

$\partial\mathcal{M}=\mathcal{M}$ och $\mathring{\mathcal{M}}=\emptyset$.

Problem 1.4 (Sid. 1)

<u>Lösning</u>: En mängd $\mathcal{M}$ kallas öppen om $\mathcal{M}=\mathring{\mathcal{M}}$ och sluten om $\partial\mathcal{M}\subseteq\mathcal{M}$. Definitionen finns på sidan 13 i boken.

(1) $$\underline{\mathcal{M}_1=\{(x,y):\ x^2+y^2\leq 1,\ y\geq 0\}}$$

$\mathcal{M}_1$ omfattar sin rand (heldragen kontur); den är sluten således. Den är även begränsad, ty

$$\underline{\mathcal{M}_1\subseteq\mathcal{M}_0=\{(x,y):\ x^2+y^2\leq 2\}.}$$

(2) $$\underline{\mathcal{M}_2=\{(x,y):\ x^2+2y^2\leq 2,\ 0<x<y\}}$$

$\mathcal{M}_2$ är inte öppen, <u>en del</u> av randen (ellipsbågen) ingår. $\mathcal{M}_2$ är varken öppen eller sluten; den är dock begränsad, ty

$$\underline{\mathcal{M}_2\subseteq\mathcal{M}_0=\{(x,y):\ x^2+y^2\leq 4\}.}$$

(3) $$\underline{\mathcal{M}_3=\{(x,y):\ |x|+|y|<1\}}$$

$\mathcal{M}_3$ är konturlös, dvs öppen; den är begränsad dock, ty

$$\underline{\mathcal{M}_3\subseteq\mathcal{M}_0=\{(x,y):\ x^2+y^2\leq 4\}.}$$

(4) $$\underline{\mathcal{M}_4=\{(x,y):\ \max(|x|,|y|)\}}$$

$\mathcal{M}_4$ är sluten (heldragen kontur) och begränsad, ty $M_4 \subseteq M_0 = \{(x,y):\ x^2+y^2 \leq 4\}$.

(5) $M_5 = \{(x,y):\ x^2+y^2-2x-4y=11\}$.

M_5 är en sluten kurva, dvs lika med sin rand; M_5 är en sluten mängd; den är även begränsad, ty $M_5 \subseteq M_0 = \{(x,y):\ x^2+y^2 \leq 100\}$.

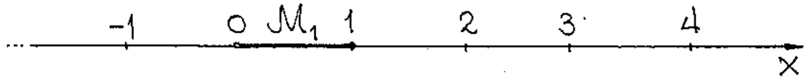
(6) $M_6 = \{(x,y):\ |x+2y| = 2\}$

$M_6 = \partial M_6$, dvs är sluten; den är dock obegränsad (se figur).

Problem 1.5 (Sid. 1)

Lösning

a) $M_1 = \{x \in \mathbb{R}:\ 0 < x \leq 1\} =]0,1]$.



$\mathring{M}_1 = \{(x,y):\ 0<x<1\} =]0,1[$; $\partial M_1 = \{0,1\}$ (2 pkr).

Anm. En lucka (ring) på talaxeln signalerar att just den punkten saknas.

Jämför med Exempel 5 på s.13 i läroboken

b) $M_2 = \{x \in \mathbb{R}:\ x^2 \geq 0\} = \mathbb{R}$

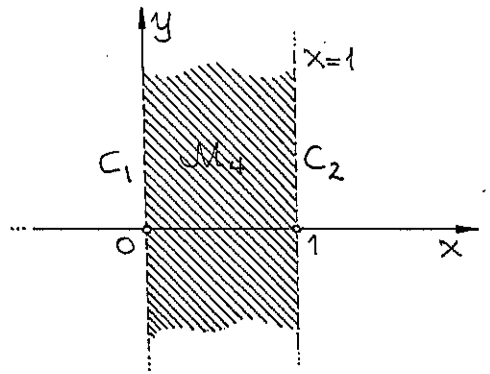
$\mathring{M}_2 = \mathbb{R}$; $\partial M_2 = \emptyset$. (Läs på sidan 13 i boken).

c) $M_3 = \{(x,y) \in \mathbb{R}:\ x^2+1 < 2x\}$

$2x > x^2+1 \Leftrightarrow x^2-2x+1<0 \Leftrightarrow (x-1)^2 < 0 \Leftrightarrow M_3 = \emptyset \Leftrightarrow$

$\Leftrightarrow \mathring{M}_3 = \emptyset = \partial M_3$.

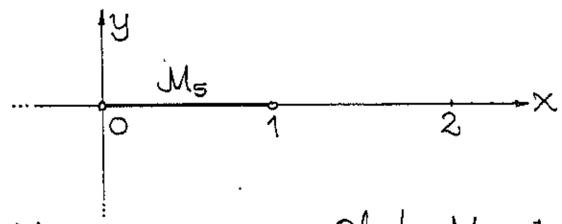
d) $M_4 = \{(x,y) \in \mathbb{R}^2:\ 0<x<1\}$.



$\mathring{M}_4 = M_4$; $\partial M_4 = C_1 \cup C_2 = \{0,1\} \times \mathbb{R}$ ("kryssprodukt").

Om produktmängd kan man läsa på s.407.

e) $M_5 = \{(x,y) \in \mathbb{R}^2:\ 0<x<1,\ y=0\}$.



$\mathring{M}_5 = \emptyset$, $\partial M_5 = [0,1] \times \{0\}$. Obs! $M_5 =]0,1[\times \{0\}$.

Problem 1.6 (Sid. 1)

Lösning

(1) $$\mathcal{M}_1 = \{x \in \mathbb{R} : 0 < x \leq 1\}$$

$\mathcal{M}_1$ är ett halvöppet intervall, dvs $\mathcal{M}_1$ är varken öppen eller sluten. Den är dock begränsad: $\mathcal{M}_1 =]0,1] \subseteq [-2,2] = \mathcal{M}_0$.

(2) $$\mathcal{M}_2 = \{x \in \mathbb{R} : x^2 \geq 0\}.$$

$\mathcal{M}_2 = \mathbb{R} =]-\infty, +\infty[$ är både sluten och öppen. $\mathcal{M}_2$ är obegränsad.

(3) $$\mathcal{M}_3 = \{x \in \mathbb{R} : 2x > x^2+1\}$$

$\mathcal{M}_3 = \emptyset = \complement\mathbb{R}$ är både sluten och öppen; $\emptyset$ är delmängd i varje mängd (enligt definition) så den är begränsad.

(4) $$\mathcal{M}_4 = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1\}$$

$\mathcal{M}_4 =]0,1[\times \mathbb{R}$ är öppen, ty produkt av två öppna; den är uppenbarligen obegränsad ty komponenten (faktorn) $\mathbb{R}$ är det.

(5) $$\mathcal{M}_5 = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1, y = 0\}$$

$\mathcal{M}_5 =]0,1[\times \{0\}$ saknar inre punkter så den är inte öppen; den är inte sluten heller ty $(0,0)$ och $(1,0)$ ligger i $\partial\mathcal{M}_5$ men inte i $\mathcal{M}_5$; $\mathcal{M}_5$ är begränsad dock.

Problem 1.7 (Sid. 1)

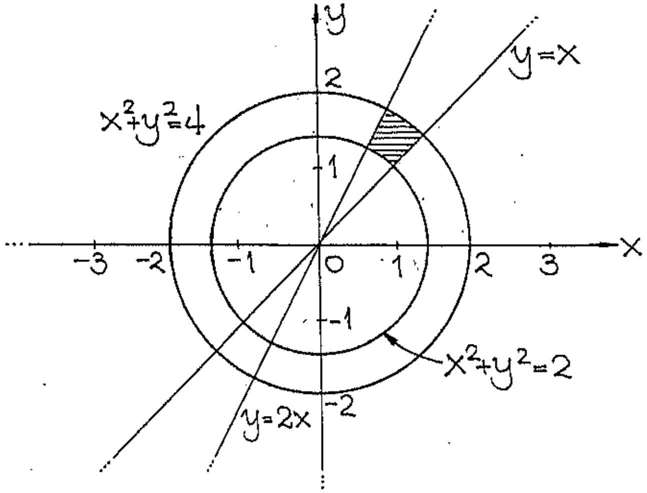
Lösning

a) $$\mathcal{M} = \{(x,y) \in \mathbb{R}^2 : 2 \leq x^2+y^2 \leq 4,\ x \leq y \leq 2x\}.$$

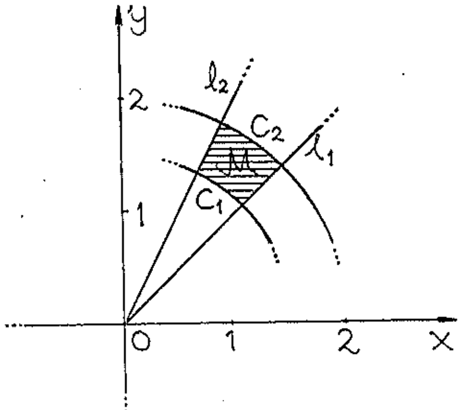
(1) $$2 \leq x^2+y^2 \leq 4 \Leftrightarrow \begin{cases} 2 \leq x^2+y^2 \\ 4 \geq x^2+y^2 \end{cases} \Leftrightarrow \begin{cases} \sqrt{2} \leq \sqrt{x^2+y^2} \\ 2 \geq \sqrt{x^2+y^2} \end{cases} \Leftrightarrow$$

$\Leftrightarrow \sqrt{2} \leq \sqrt{x^2+y^2} \leq 2 \Rightarrow$ avståndet från $(x,y) \in \mathcal{M}$ till origo $(0,0)$ är lägst $\sqrt{2}$ och högst 2; det är frågan om en origocentrisk ring med inre radien $\sqrt{2}$ och yttre radien 2. Liknande ringar studeras i samband med de komplexa talen; i det komplexa planet har man $\sqrt{2} \leq |z| \leq 2$.

(2) I samma koordinatsystem uppritas cirklarna $x^2+y^2=2$, $x^2+y^2=4$ och linjerna $y=x$ & $y=2x$:

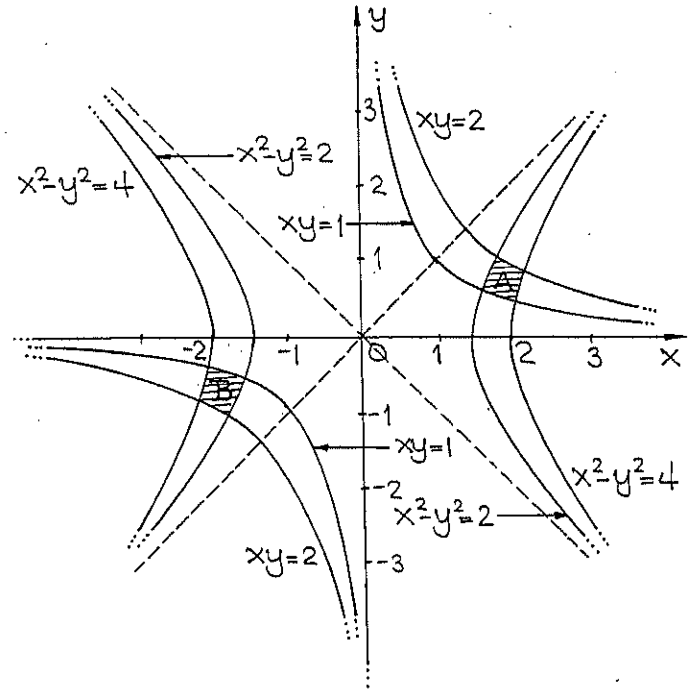


M syns skuggad i figuren; man brukar inte rita hela mönstret som ovan.



C_1: $x^2+y^2=2$, C_2: $x^2+y^2=4$, l_1: $y=x$, l_2: $y=2x$.

b)



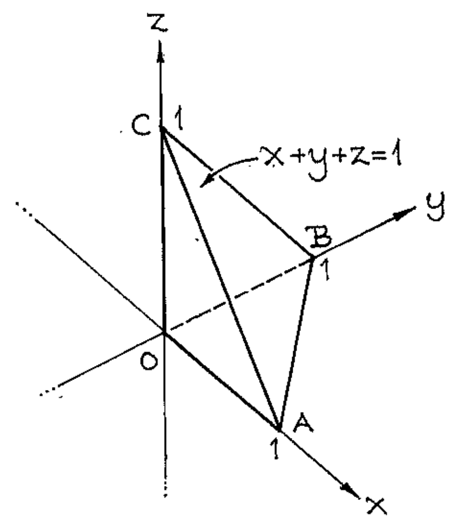
$$\left.\begin{aligned} A &= \{(x,y):\ 2 \le x^2-y^2 \le 4,\ 1 \le xy \le 2,\ x>0\} \\ B &= \{(x,y):\ 2 \le x^2-y^2 \le 4,\ 1 \le xy \le 2,\ x<0\} \end{aligned}\right\} \Rightarrow M = A \cup B.$$

Anm. Liknande mönster kan man se på sidorna 32-33 i läroboken.

Problem 1.8 (Sid. 1)

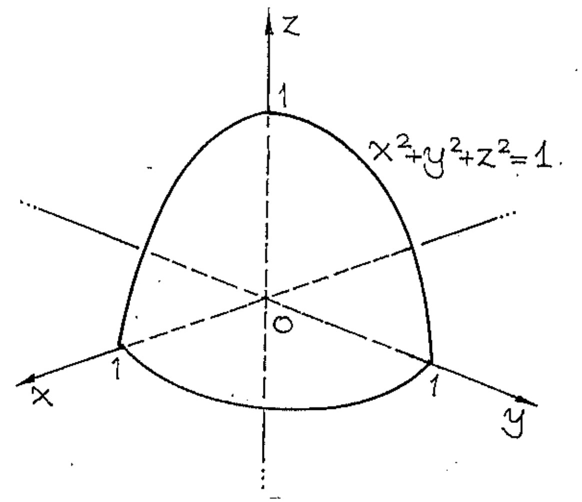
Lösning a) $M = \{(x,y,z):\ x+y+z \le 1,\ x,y,z \ge 0\}$.

M är en tetraeder i den första oktanten som i figuren på nästa sida.



Observera att tetraeder är massiv (solid), alltså inte bara skalet (randen).

b) $\underline{M_2 = \{(x,y,z):\ x^2+y^2+z^2=1,\ x,y,z \geq 0\}}$.



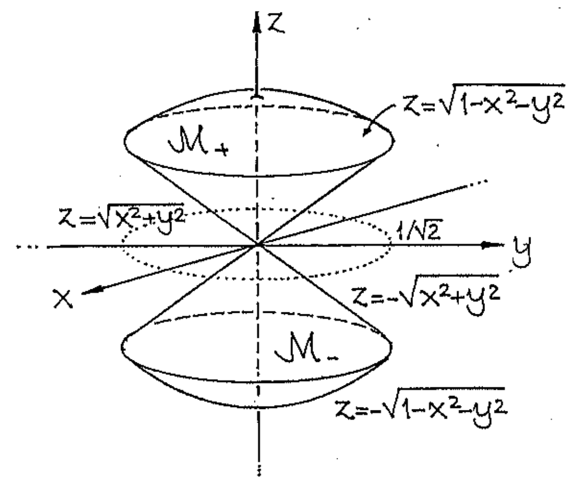
M_2 är enhetssfärens del i den 1:a oktanten.

c) $\underline{M_3 = \{(x,y,z):\ x^2+y^2 \leq z^2 \leq 1-x^2-y^2\}}$

$$x^2+y^2 \leq z^2 \leq 1-x^2-y^2 \Leftrightarrow \sqrt{x^2+y^2} \leq \sqrt{z^2} \leq \sqrt{1-x^2-y^2}$$

$$\Leftrightarrow \sqrt{x^2+y^2} \leq |z| \leq \sqrt{1-x^2-y^2} \Leftrightarrow \sqrt{x^2+y^2} \leq \pm z \leq \sqrt{1-x^2-y^2}$$

$$\Leftrightarrow \begin{cases} M_+ = \{(x,y,z):\ \sqrt{x^2+y^2} \leq z \leq \sqrt{1-x^2-y^2} \\ M_- = \{(x,y,z):\ -\sqrt{1-x^2-y^2} \leq z \leq -\sqrt{x^2+y^2} \end{cases} \Rightarrow \underline{M = M_+ \cup M_-}$$



Klotsektorerna $M_\pm$ är varandras spegelbild i xy-planet.

d) $\underline{M_4 = \{(x,y,z):\ x^2+y^2+z^2 \leq 1,\ x+y+z=1\}}$

Skärningen mellan enhetssfären (skalet) $x^2+y^2+z^2=1$ och planet $x+y+z=1$ är cirkeln genom punkterna $P_1:(1,0,0)$, $P_2:(0,1,0)$ & $P_3:(0,0,1)$.

Skärningen mellan enhetsklotet $x^2+y^2+z^2 \leq 1$ och planet $x+y+z=1$ är en cirkelskiva med denna cirkel till kontur.

Övning 1.9 (Sid. 1)

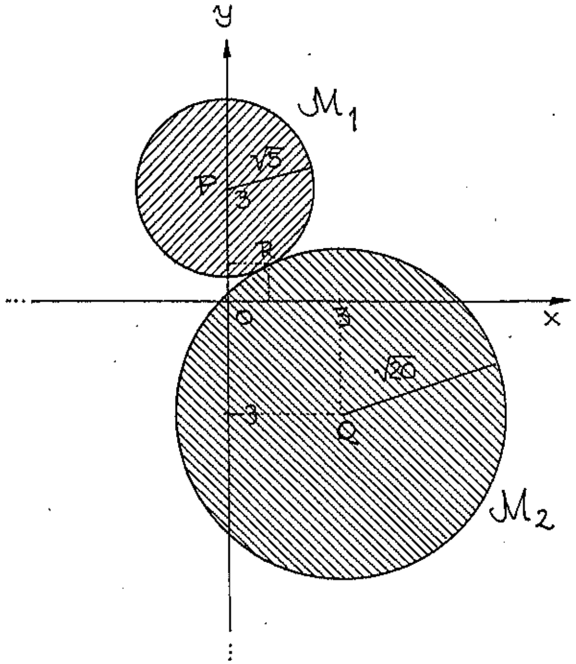
Lösning

(1) $x^2+y^2-6y+4 \leq 0 \Leftrightarrow x^2+(y-3)^2 \leq 5 \Leftrightarrow \sqrt{x^2+(y-3)^2} \leq \sqrt{5}$;

$M_1=\{(x,y): x^2+y^2-6y+4 \leq 0\}=\{(x,y): x^2+(y-3)^2 \leq 5\}$

(2) $x^2+y^2-6x+6y-2 \leq 0 \Leftrightarrow (x-3)^2+(y+3)^2 \leq 20$;

$M_2=\{(x,y): (x-3)^2+(y+3)^2 \leq 20\}$



$$\left.\begin{array}{l} P:(0,3) \\ Q:(3,-3) \end{array}\right\} \Rightarrow \overrightarrow{PQ}=(3,-6) \Rightarrow |\overrightarrow{PQ}|=\sqrt{3^2+6^2}=\sqrt{45}=$$

$=3\sqrt{5}=\sqrt{5}+2\sqrt{5}=\sqrt{5}+\sqrt{20}=|\overrightarrow{PR}|+|\overrightarrow{RQ}| \Rightarrow M_1 \cap M_2=$ $=\{(1,1)\}$, <u>cirkelskivorna tangerar varandra</u>.

<u>Anm.</u> I den linjära algebran skrivs vektorerna som kolonner; i den analytiska geometrin (koordinatgeometrin) skrivs de som rader.

Problem 1.10 (Sid. 1)

Lösning

$$M_1=\{(x,y,z): x^2+y^2+z^2-2x-2z \leq 0\}=$$
$$=\{(x,y,z): (x-1)^2+y^2+(z-1)^2 \leq 2\};$$

M_1 är ett klot med medelpunkten $P_1:(1,0,1)$ och radien $r_1=\sqrt{2}$.

$$M_2=\{(x,y,z): x^2+y^2+z^2-2y+4z \leq 0\}=$$
$$=\{(x,y,z): x^2+(y-1)^2+(z+2)^2 \leq 1\};$$

M_2 är ett klot med medelpunkten $P_2:(0,1,-2)$

och radien $r_2=1$.

$\overrightarrow{P_1P_2}=(0-1,1-0,-2-1)=(-1,1,-3) \Rightarrow |\overrightarrow{P_1P_2}|=\sqrt{11}>\sqrt{2}+1=$

$=r_1+r_2 \Rightarrow M_1 \cap M_2=\emptyset$.

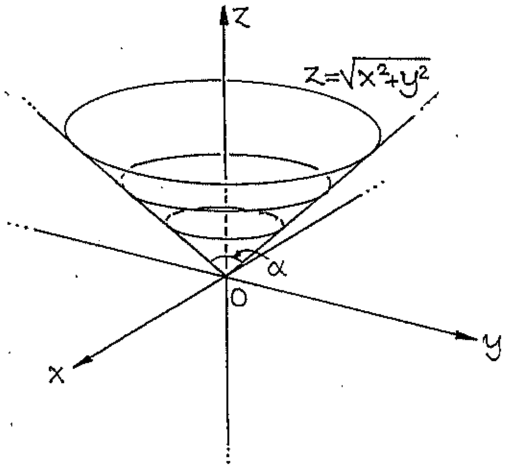
Svar: Nej, de har inte.

Problem 1.11 (Sid. 1)

Lösning

a) $z=x+2y-2 \Leftrightarrow x+2y-z=2$; ett plan genom punkterna $P_1:(2,0,0)$, $P_2:(0,1,0)$ och $P_3:(0,0,-2)$

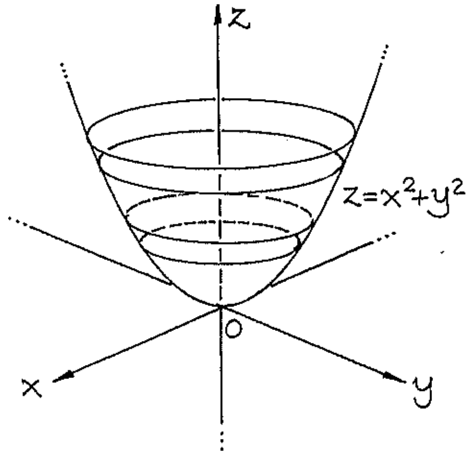
b) $z=\sqrt{x^2+y^2} \Leftrightarrow z^2=x^2+y^2 \wedge z\geq 0 \Leftrightarrow x^2+y^2-z^2=0$; en konisk yta som i figuren nedan.



$\alpha=90°$

Andragradskurvor och ytor genomgås i den linjära algebran i samband med diagonalisering av kvadratiska former. Läs även det som finns i kursboken på sidorna 29-31. Konsultera matematikhandboken "BETA".

c) $z=x^2+y^2$; en rotationsparaboloid (se figur).

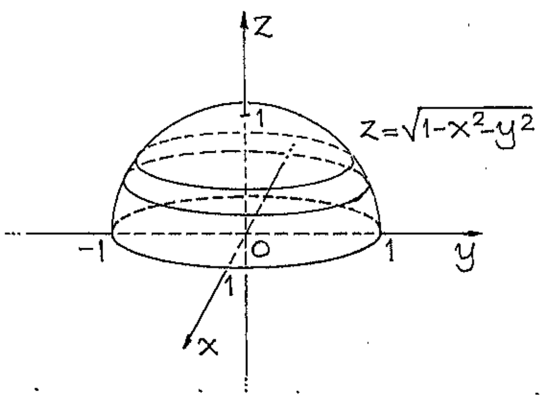


Anm. $z=f(\sqrt{x^2+y^2})$ är en kring z-axeln rotationssymmetrisk funktionsyta. (Jfr. 1.17).

d) $z=\sqrt{1-x^2-y^2}$, $x^2+y^2\leq 1$

$z^2=1-x^2-y^2 \wedge z\geq 0 \Leftrightarrow x^2+y^2+z^2=1 \wedge z\geq 0$; övre halvan av enhetssfären.

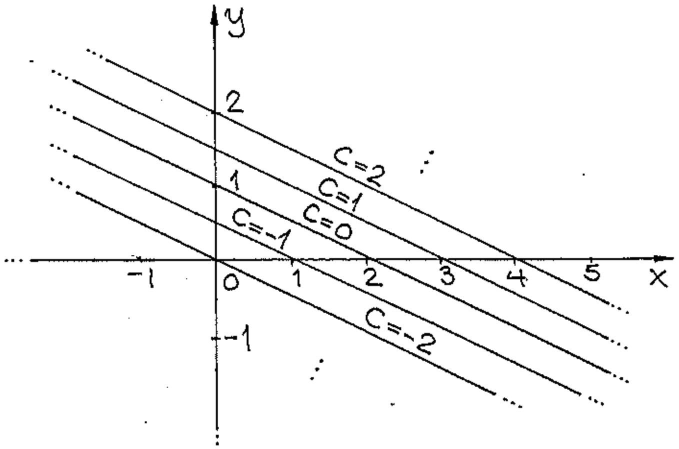
Definitionsmängden är enhetscirkeln;



Problem 1.12 (Sid. 1)

Lösning

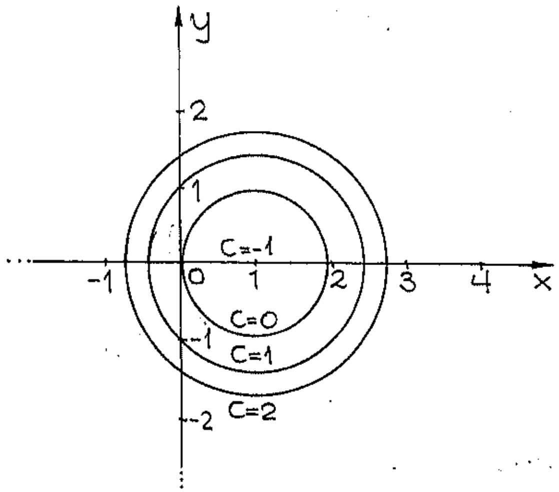
a) $\underline{f(x,y)=x+2y-2}=C \Leftrightarrow x+2y=2+C,\quad C=0,\pm 1,\pm 2.$



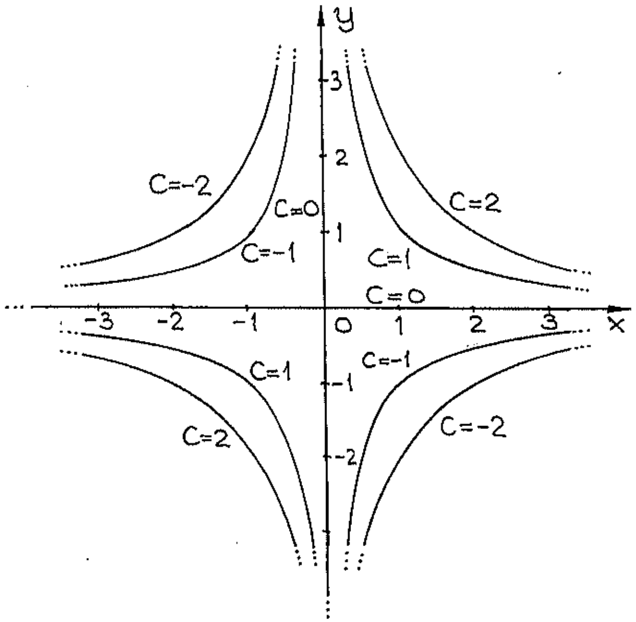
b) $\underline{f(x,y)=x^2+y^2-2x}=C \Leftrightarrow (x-1)^2+y^2=C+1,\ C=0,\pm 1,\pm 2.$

$C+1\geqslant 0 \Leftrightarrow C\geqslant -1 \Rightarrow C=-1,0,1,2.$

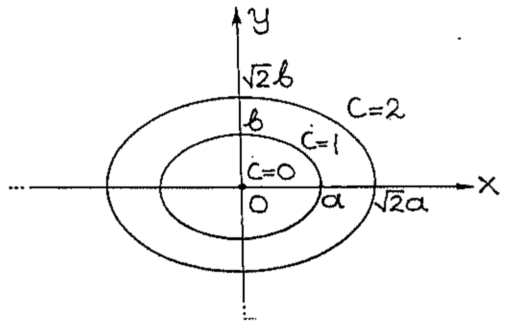
Nivåkurvorna en cirkelskara som i figuren:



c) $\underline{f(x,y)=xy}=C,\quad C=0,\pm 1,\pm 2.$



d) $\underline{f(x,y)=\frac{x^2}{a^2}-\frac{y^2}{b^2}}=C,\quad C=0,1,2.$ (Negativa C?)



Svar: a) Ett plan; $x+2y-z=2$.

b) En rotationsparaboloid med toppen $(1,0,0)$ och rotationsaxeln $\mathbf{x}=(1,0,0)+(0,0,1)\cdot t,\ t\geq 0$;

c) En parabolisk hyperboloid.

d) En elliptisk paraboloid.

Anm. Man får nivåkurvan $f(x,y)=C_0$ som skärningen mellan funktionsytan $z=f(x,y)$ och planet $z=C_0$, projicerad i xy-planet parallellt med z-axeln.

Om nivåkurvor kan du läsa i någon Calculus-text, av de som finns i handeln eller också i biblioteket.

Problem 1.13 (Sid. 1)

Lösning

a) $\begin{cases} g(t)=te^{-t^2}+1 \\ t=x+y \end{cases} \Rightarrow f(x,y)=g(x+y)=\underline{(x+y)e^{-(x+y)^2}+1}.$

b) $\begin{cases} g(t)=t^2\cos t \\ t=xy \end{cases} \Rightarrow f(x,y)=g(xy)=(xy)^2\cos xy=\underline{x^2y^2\cos xy}.$

Problem 1.14 (Sid. 1)

Lösning

a) $f(x,y)=e^{-xy}-x^2y^2=e^{-xy}-(xy)^2=g(xy) \Rightarrow \underline{g(t)=e^{-t}-t^2}.$

b) $f(x,y)=e^{-xy}-x^2y$; det finns ingen sådan g.

c) $f(x,y)=(x^2-4xy+4y^2)e^x\cdot e^{-2y}+1=(x-2y)^2e^{x-2y}+1=$

$=g(x-2y) \Rightarrow \underline{g(t)=t^2e^t+1}$; $a=1,\ b=-2$.

Problem 1.15 (Sid. 1)

Lösning

(1) $f(x,y)=x^2-y^2=(x-y)(x+y)=s\cdot t=g(s,t)$;

(2) $\begin{cases} s=x-y \\ t=x+y \end{cases} \Leftrightarrow \begin{bmatrix} s \\ t \end{bmatrix}=\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\sqrt{2}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=$

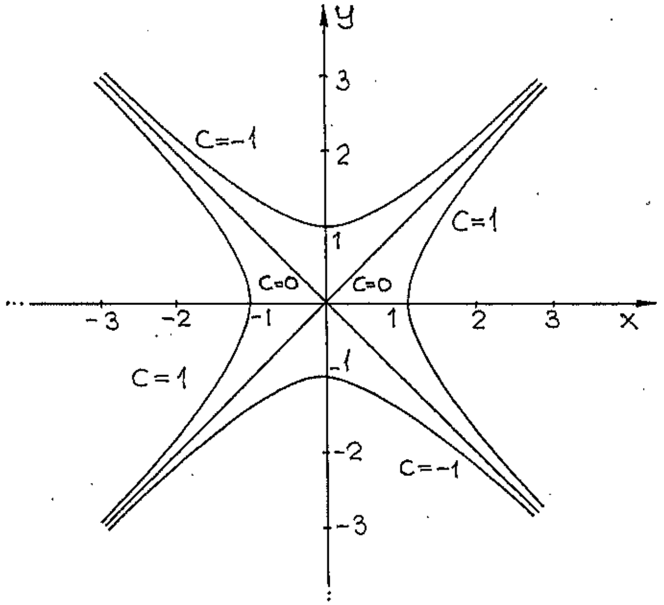
$$=\sqrt{2}\underbrace{\begin{bmatrix}\cos\frac{\pi}{4} & -\sin\frac{\pi}{4}\\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4}\end{bmatrix}}_{\text{ON-matris}}\begin{bmatrix}x\\ y\end{bmatrix};\ \begin{cases}\text{Rotation vinkeln } \frac{\pi}{4}\\ \text{och sträckning } \sqrt{2}\text{ggr.}\end{cases}$$

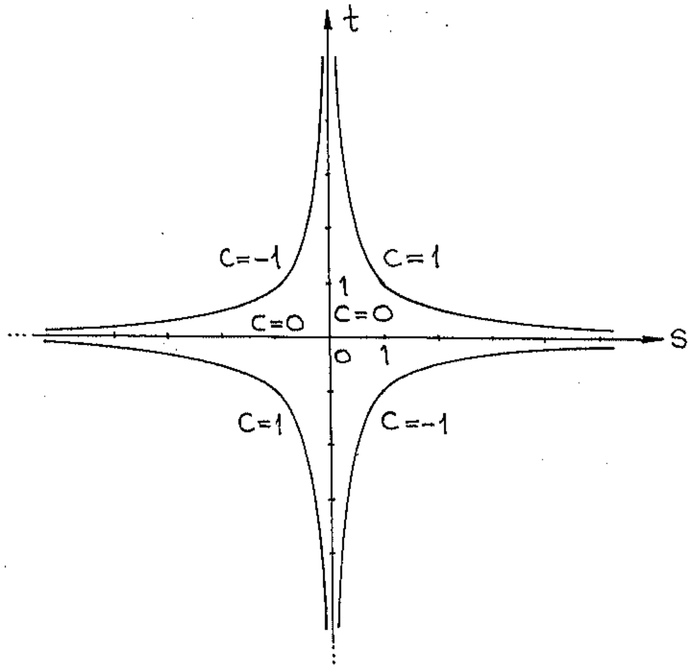
<u>Anm</u> $\zeta = s+it$, $z=x+iy$ (komplexa tal)

$$s+it=(1+i)(x+iy) = x-y+i(x+y)$$
$$=\sqrt{2}\cdot\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)(x+iy) =$$
$$=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)(x+iy)\Leftrightarrow$$

$$\begin{cases}s=\sqrt{2}(\cos\frac{\pi}{4}x-\sin\frac{\pi}{4}y)\\ t=\sqrt{2}(\sin\frac{\pi}{4}x+\cos\frac{\pi}{4}y)\end{cases}\Leftrightarrow\begin{bmatrix}s\\ t\end{bmatrix}=\sqrt{2}\begin{bmatrix}\cos\frac{\pi}{4} & -\sin\frac{\pi}{4}\\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4}\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

<u>Ovanstående tillhör komplex analys...</u>





<u>Problem 1.16</u> (Sid. 2)

<u>Lösning</u>

$$\begin{cases}s=x+y\\ t=x-y\end{cases}\Leftrightarrow\begin{cases}x=\frac{s+t}{2}\\ y=\frac{s-t}{2}\end{cases};$$

a) $f(x,y)=x=\frac{1}{2}s+\frac{1}{2}t=g(s)+h(t)\Leftrightarrow\begin{cases}g(s)=s/2\\ h(t)=t/2\end{cases}$;

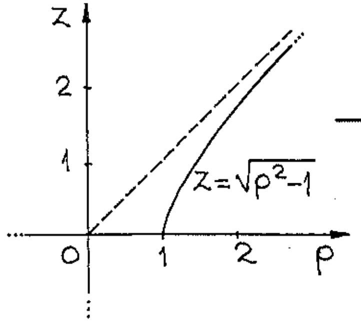
b) $f(x,y)=xy=\frac{1}{4}s^2-\frac{1}{4}t^2=g(s)+h(t)\Leftrightarrow\begin{cases}g(s)=s^2/4\\ h(t)=-t^2/4\end{cases}$;

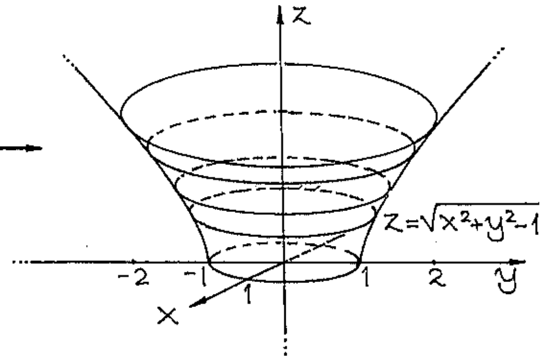
c) $f(x,y)=x^2=\frac{1}{4}s^2+\frac{1}{4}t^2+st\neq g(s)+h(t)$; det går inte.

Problem 1.17 (Sid. 2)

Lösning

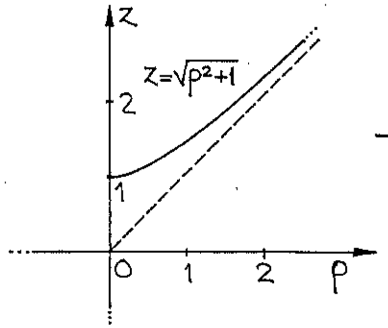
a)

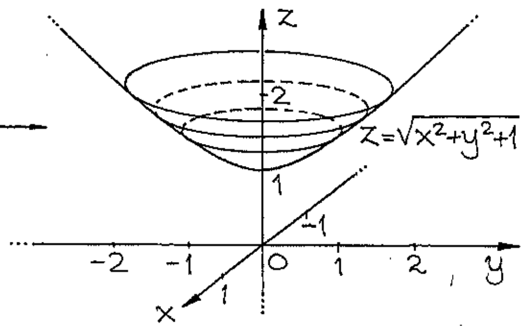




Profilkurvan $g(\rho)=\sqrt{\rho^2-1}$ roterar ett varv kring z-axeln; den uppkomna funktionsytan är en stympad (enmantad) rotationshyperboloid.

b) $g(\rho)=\sqrt{\rho^2+1} \Rightarrow z=f(x,y)=\sqrt{x^2+y^2+1}$, $(x,y)\in\mathbb{R}^2$; den övre halvan av en rotationshyperboloid.





Problem 1.18 (Sid. 2)

Lösning

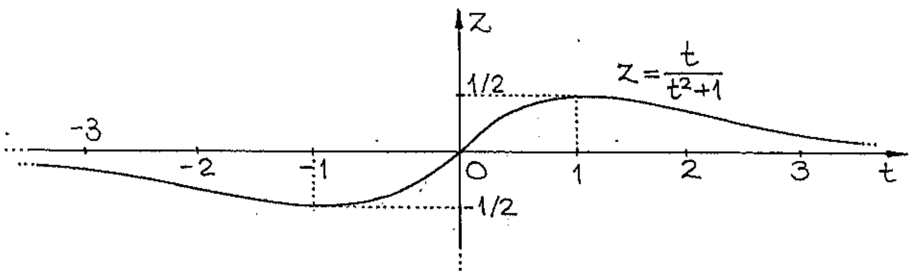
(1) $f(x,y)=\dfrac{xy}{x^2+y^2}=\dfrac{y/x}{1+(y/x)^2}=g\left(\dfrac{y}{x}\right) \Rightarrow g(t)=\dfrac{t}{1+t^2}$, $t\in\mathbb{R}$.

(2) $g(-t)=-g(t) \Rightarrow g$ udda $\Rightarrow g$:s graf är origosymmetrisk (alt. speglar i origo) i tz-systemet. Jag betraktar positiva t och speglar i origo.

$$g'(t)=\frac{1\cdot(t^2+1)-t\cdot 2t}{(t^2+1)^2}=\frac{1-t^2}{(1+t^2)^2}=\frac{1+t}{(1+t^2)^2}\cdot(1-t);$$

$$\left.\begin{cases} 0<t<1 \Rightarrow g'(t)>0 \Rightarrow g \text{ växande} \\ t>1 \Rightarrow g(t)<0 \Rightarrow g \text{ avtagande} \end{cases}\right\} \Rightarrow g_{max}=g(1)=\frac{1}{2};$$

$\lim_{t\to\infty} g(t)=\lim_{t\to\infty}\frac{1}{t}=0^+ \Rightarrow$ <u>t-axeln vågrät asymptot.</u>



(3) $f(x,0)=g(0)=0$, $f(x,x)=g(1)=\frac{1}{2}$, $f(x,7x)=g(7)=\frac{7}{50}$,
$f(0^+,y)=g(+\infty)=0$, $f(0^-,y)=g(-\infty)=0^-$, $f(x,-x)=-\frac{1}{2}$.

(4) $f(x,kx)=\dfrac{k}{1+k^2}$; <u>f konstant längs strålar från origo.</u>

Gränsvärden och kontinuitet

Problem 1.19 (Sid. 2)

Lösning: Jag betraktar vektorer som rader; i koordinatgeometrin (analytiska geometrin) är detta standardbeteckning.

a) $\begin{cases}\mathbf{x}=(x,y)\\ \mathbf{a}=(0,2)\end{cases} \Rightarrow \mathbf{x}-\mathbf{a}=(x,y-2) \Rightarrow \begin{cases}|\mathbf{x}-\mathbf{a}|\geq|x|\\ |\mathbf{x}-\mathbf{a}|\geq|y-2|\end{cases} \Rightarrow$

$|x^2+y-2| \leq x^2+|y-2| \leq |\mathbf{x}-\mathbf{a}|^2+|\mathbf{x}-\mathbf{a}| \Rightarrow g(\rho)=\rho^2+\rho.$

$\rho=|\mathbf{x}-\mathbf{a}|=\sqrt{x^2+(y-2)^2}.$

b) För små $|u|$ gäller som bekant att $|\sin u|\leq|u|$;

$|\sin(x-y)| \leq |x-y| \leq |x|+|-y| \leq |\mathbf{x}|+|\mathbf{x}| = 2|\mathbf{x}|$; $g(\rho)=2\rho$;

$\rho=|\mathbf{x}|=\sqrt{x^2+y^2}.$

c) $\begin{cases}\mathbf{x}=(x,y)\\ \mathbf{a}=(1,2)\end{cases} \Rightarrow \mathbf{x}-\mathbf{a}=(x-1,y-2) \Rightarrow |\mathbf{x}-\mathbf{a}|=\sqrt{(x-1)^2+(y-2)^2}$;

$\left|x+\frac{2}{y}-2\right| = \left|(x-1)+\frac{y-2}{y}\right| \leq |x-1|+\left|\frac{y-2}{y}\right| = |x-1|+\frac{|y-2|}{|y|}$;

$y>1 \Rightarrow \frac{1}{y}<1 \Rightarrow \left|x+\frac{2}{y}-2\right| \leq |x-1|+|y-2| \leq 2|\mathbf{x}-\mathbf{a}| \Rightarrow$

$g(\rho)=2\rho$, $\rho=\sqrt{(x-1)^2+(y-2)^2}.$

Problem 1.20 (Sid. 2)

Lösning

a) $\mathbf{x}=(x,y) \Rightarrow \sin(x^2+y^2)=\sin|\mathbf{x}|^2=\sin r^2=r^2+O(r^6)$

$=r^2(1+O(r^4)) \Rightarrow \frac{\sin(x^2+y^2)}{x^2+y^2}=1+O(r^4)\xrightarrow[r\to 0]{}1 \Rightarrow$

$\lim_{\mathbf{x}\to 0}\frac{\sin|\mathbf{x}|^2}{|\mathbf{x}|^2}=1.$

b) $|x^2y|=x^2|y|\leq|\mathbf{x}|^2|\mathbf{x}| \Rightarrow 0\leq\frac{x^2|y|}{|\mathbf{x}|^2}\leq|\mathbf{x}|\xrightarrow[\mathbf{x}\to 0]{}0 \Rightarrow$

$\lim_{\mathbf{x}\to 0}\frac{\sin(x^2+y^2)}{x^2+y^2+x^2y}=\lim_{\mathbf{x}\to 0}\frac{\sin|\mathbf{x}|^2}{|\mathbf{x}|^2}\cdot\lim_{\mathbf{x}\to 0}\frac{1}{1+x^2y/|\mathbf{x}|^2}=1.$

c) $\left|\frac{x+y+1}{\ln(x^2+2y^2)}\right|=\frac{|x+y+1|}{|\ln(x^2+2y^2)|}\leq\frac{|x|+|y|+1}{|\ln(x^2+y^2)|}\leq\frac{2|\mathbf{x}|+1}{|\ln|\mathbf{x}|^2|}\overset{!}{\leq}$

$\leq\frac{3}{2|\ln|\mathbf{x}||}\xrightarrow[\mathbf{x}\to 0]{}\frac{3}{+\infty}=0 \Rightarrow \lim_{\mathbf{x}\to 0}\frac{x+y+1}{\ln(x^2+2y^2)}=0.$

Anm. I $\overset{!}{\leq}$ underförstås "för $|\mathbf{x}|<1$"; vi är ju intresserade av (x,y) nära $(0,0)$.

Problem 1.21 (Sid. 2)

Lösning

a) $\left|\frac{x^3+y^3}{x^2+y^2}\right|=\frac{|x^3+y^3|}{x^2+y^2}\leq\frac{|x^3|+|y^3|}{x^2+y^2}\leq\frac{|x|^3+|y|^3}{x^2+y^2}\leq\frac{|\mathbf{x}|^3+|\mathbf{x}|^3}{|\mathbf{x}|^2}=$

$=\frac{2|\mathbf{x}|^3}{|\mathbf{x}|^2}=2|\mathbf{x}|\xrightarrow[\mathbf{x}\to 0]{}0 \Rightarrow \lim_{\mathbf{x}\to 0}\frac{x^3+y^3}{x^2+y^2}=0$, enligt instängningsregeln.

b) Låt oss att till vägar in mot origo välja koordinataxlarna.

$$\left.\begin{array}{l}(1)\ \lim\limits_{\mathbf{x}\to 0}\dfrac{x^2-2y^2}{2x^2+y^2}=\begin{bmatrix}x=t\\ y=0\end{bmatrix}=\lim\limits_{t\to 0}\dfrac{t^2}{2t^2}=\lim\limits_{t\to 0}\dfrac{1}{2}=\dfrac{1}{2}\\[2ex] (2)\ \lim\limits_{\mathbf{x}\to 0}\dfrac{x^2-2y^2}{2x^2+y^2}=\begin{bmatrix}x=0\\ y=s\end{bmatrix}=\lim\limits_{s\to 0}\dfrac{-2s^2}{s^2}=\lim\limits_{s\to 0}(-2)=-2\end{array}\right\}\Rightarrow$$

$\lim\limits_{\mathbf{x}\to 0}\dfrac{x^2-2y^2}{2x^2+y^2}$ <u>existerar inte.</u>

c) $$\left|\frac{x^3-2y^3}{2x^2+y^2}\right|=\frac{|x^3-2y^3|}{2x^2+y^2}\leq\frac{|x^3|+|-2y^3|}{2x^2+y^2}\leq\frac{|x|^3+2|y|^3}{x^2+y^2}\leq$$
$$\leq\frac{|\mathbf{x}|^3+2|\mathbf{x}|^3}{|\mathbf{x}|^2}=\frac{3|\mathbf{x}|^3}{|\mathbf{x}|^2}=2|\mathbf{x}|\xrightarrow[\mathbf{x}\to 0]{}0\Rightarrow\lim_{\mathbf{x}\to 0}\frac{x^3-2y^3}{2x^2+y^2}=0.$$

d) Med kursen rakt in mot origo över $y=x$ fås
$$\lim_{\mathbf{x}\to 0}\frac{x^2}{y-x^2}=\lim_{x\to 0}\frac{x^2}{x-x^2}=\lim_{x\to 0}\frac{x}{1-x}=0;\quad (*)$$
vägen över kurvan $y=x^3$ leder till
$$\lim_{\mathbf{x}\to 0}\frac{x^2}{y-x^2}=\lim_{x\to 0}\frac{x^2}{x^3-x^2}=\lim_{x\to 0}\frac{1}{x-1}=-1\overset{(*)}{\Rightarrow}$$ <u>gränsvärdet existerar inte.</u>

e) Planpolära koordinater (sida 31) införs här:
$$g(r,\varphi)=f(r\cos\varphi, r\sin\varphi)=r\cdot\frac{2\cos^3\varphi-\cos\varphi\sin^2\varphi}{1-\sin\varphi\cos\varphi};$$
$1-\sin\varphi\cos\varphi=1-\frac{1}{2}\sin 2\varphi\neq 0$, för alla $\varphi\in[0,2\pi]$;

det innebär att $h(\varphi)=\dfrac{2\cos^3\varphi-\cos\varphi\sin^2\varphi}{1-\sin\varphi\cos\varphi}$, $0\leq\varphi\leq 2\pi$, är begränsad; $m=h_{min}$ och $M=h_{max}$ ger $m|\mathbf{x}|\leq f(\mathbf{x})\leq M|\mathbf{x}|$; det innebär i sin tur att $\lim\limits_{\mathbf{x}\to 0}f(\mathbf{x})=0$ (enligt instängningsregeln).

<u>Anm.</u> $\mathbf{x}=(x,y)\Rightarrow\begin{cases}x=r\cos\varphi\\ y=r\sin\varphi\end{cases}\Rightarrow|\mathbf{x}|^2=x^2+y^2=r^2\Rightarrow$

$\Rightarrow r=|\mathbf{x}|\Rightarrow mr\leq g(r,\varphi)\leq Mr\Leftrightarrow m|\mathbf{x}|\leq f(\mathbf{x})\leq M|\mathbf{x}|$

g) $$g(x)=f(x,kx)=\frac{2x^3-xk^2x^2}{(x-kx)^2}=x\cdot\frac{2-k^2}{(1-k)^2}\xrightarrow[x\to 0]{k\neq 1}0;$$
för $k=1$, dvs $y=x$, blir det ... ingenting.
<u>Gränsvärdet existerar inte</u> helt enkelt.

<u>Problem 1.22</u> (Sid. 2)

<u>Lösning</u>

a) $$f(x,y)=\frac{xy-y}{x^2+2y^2-2x+1}=\frac{(x-1)y}{(x-1)^2+2y^2}=\begin{bmatrix}s=x-1\\ t=y\end{bmatrix}=\frac{st}{s^2+2t^2};$$
$(x,y)\longrightarrow(1,0)\Leftrightarrow(x-1,y)\longrightarrow(0,0)\Leftrightarrow(s,t)\longrightarrow(0,0)$.

Jag sätter $\mathbf{x}=(x,y)$, $\mathbf{a}=(1,0)$, $\mathbf{r}=(s,t)$, $\mathbf{0}=(0,0)$.
$$g(\mathbf{r})=\frac{st}{s^2+2t^2}\Rightarrow\begin{cases}\lim\limits_{\mathbf{r}\to 0}g(\mathbf{r})=\begin{bmatrix}s=\sigma\\ t=0\end{bmatrix}=\lim\limits_{\sigma\to 0}\dfrac{0}{\sigma^2}=0\\ \lim\limits_{\mathbf{r}\to 0}g(\mathbf{r})=\begin{bmatrix}s=\sigma\\ t=\sigma\end{bmatrix}=\lim\limits_{\sigma\to 0}\dfrac{1}{3}=\dfrac{1}{3}\end{cases}\Rightarrow$$

$\Rightarrow$ gränsvärdet existerar inte.

b) $f(x,y)=\dfrac{xy^2-y^2}{x^2+2y-2x+1}=\dfrac{(x-1)y^2}{(x-1)^2+2y^2}$; $\mathbf{x}=(x,y)$; $\mathbf{a}=(1,0)$;

$\mathbf{z}=\mathbf{x}-\mathbf{a}=(x-1,y)=(\xi,\eta)\Rightarrow f(x,y)=f(1+\xi,\eta)=\dfrac{\xi\eta^2}{\xi^2+2\eta^2}$;

$|f(x,y)|=\left|\dfrac{\xi\eta^2}{\xi^2+2\eta^2}\right|=\dfrac{|\xi|\eta^2}{\xi^2+2\eta^2}\leqslant\dfrac{|\mathbf{z}|^3}{|\mathbf{z}|^2}=|\mathbf{z}|\xrightarrow[\mathbf{z}\to 0]{}0\Rightarrow$

$\lim\limits_{\mathbf{z}\to 0}\dfrac{\xi\eta^2}{\xi^2+2\eta^2}=0=\lim\limits_{\mathbf{x}\to\mathbf{a}}\dfrac{xy^2-y^2}{x^2+2y^2-2x+1}$.

Problem 1.23 (Sid. 2)

Lösning

$\mathbf{x}=(x,y,z)\Rightarrow|\mathbf{x}|^2=x^2+y^2+z^2\Rightarrow\begin{cases}|\mathbf{x}|^2\geqslant x^2\\|\mathbf{x}|^2\geqslant y^2\\|\mathbf{x}|^2\geqslant z^2\end{cases}\Leftrightarrow\begin{cases}|x|\leqslant|\mathbf{x}|\\|y|\leqslant|\mathbf{x}|\\|z|\leqslant|\mathbf{x}|\end{cases}$.

a) $|f(\mathbf{x})|=\left|\dfrac{xyz}{|\mathbf{x}|^2}\right|=\dfrac{|xyz|}{|\mathbf{x}|^2}=\dfrac{|x||y||z|}{|\mathbf{x}|^2}\leqslant\dfrac{|\mathbf{x}|^3}{|\mathbf{x}|^2}=|\mathbf{x}|\xrightarrow[\mathbf{x}\to 0]{}0$;

b) $|f(\mathbf{x})|=\left|\dfrac{3xz^2}{x^2+2y^2+3z^2}\right|=\dfrac{3|x|\cdot z^2}{x^2+2y^2+3z^2}\leqslant\dfrac{3|x|z^2}{x^2+y^2+z^2}\leqslant$

$\leqslant\dfrac{3|\mathbf{x}|\cdot|\mathbf{x}|^2}{|\mathbf{x}|^2}=3|\mathbf{x}|\xrightarrow[\mathbf{x}\to 0]{}0$.

c) Låt oss närma origo längs z-axeln:

$f(0,0,z)=-\dfrac{1}{z}\Rightarrow\left\{\begin{matrix}\lim\limits_{\mathbf{x}\to 0}f(\mathbf{x})=\lim\limits_{z\to 0^+}\left(-\frac{1}{z}\right)=-\infty\\\lim\limits_{\mathbf{x}\to 0}f(\mathbf{x})=\lim\limits_{z\to 0^-}\left(-\frac{1}{z}\right)=+\infty\end{matrix}\right\}\Rightarrow$

$\Rightarrow\lim\limits_{\mathbf{x}\to 0}f(\mathbf{x})$ existerar inte.

d) $|\sin xyz|\leqslant|xyz|=|x||y||z|\leqslant|\mathbf{x}|^3\Rightarrow\left|\dfrac{\sin(xyz)}{x^2+y^2+z^2}\right|=$

$=\dfrac{|\sin(xyz)|}{|\mathbf{x}|^2}\leqslant\dfrac{|\mathbf{x}|^3}{|\mathbf{x}|^2}=|\mathbf{x}|\xrightarrow[\mathbf{x}\to 0]{}0\Rightarrow\lim\limits_{\mathbf{x}\to 0}f(x,y,z)=$

$=\lim\limits_{\mathbf{x}\to 0}\dfrac{\ln(1+|\mathbf{x}|^2)}{|\mathbf{x}|^2}\cdot\lim\limits_{\mathbf{x}\to 0}\dfrac{1}{1+3\dfrac{\sin xyz}{|\mathbf{x}|^2}}=1\cdot 1=1$.

Problem 1.24 (Sid. 2)

Lösning

För alla $|t|$ gäller som bekant $|\sin t|\leqslant 1$ och för små $|t|$ gäller $|\sin t|\leqslant|t|$.

a) $\left|\dfrac{\sin(x^2y^2)}{2x^2+3y^2}\right|=\dfrac{|\sin(x^2y^2)|}{2x^2+3y^2}\leqslant\dfrac{1}{2x^2+3y^2}\leqslant\dfrac{1}{x^2+y^2}=\dfrac{1}{|\mathbf{x}|^2}\longrightarrow 0$,

när $|\mathbf{x}|\longrightarrow\infty$, dvs $\lim\limits_{|\mathbf{x}|\to\infty}\dfrac{\sin x^2y^2}{2x^2+3y^2}=0$.

b) $\mathbf{x}=(x,y)\Rightarrow\left|\dfrac{x}{|\mathbf{x}|^2}\right|=\dfrac{|x|}{|\mathbf{x}|^2}\leqslant\dfrac{|\mathbf{x}|}{|\mathbf{x}|^2}=\dfrac{1}{|\mathbf{x}|}\xrightarrow[|\mathbf{x}|\to\infty]{}0\Rightarrow|f(\mathbf{x})|=$

$=\left|\dfrac{|\mathbf{x}|^2}{x+|\mathbf{x}|^2}\right|=\dfrac{1}{|1+x/|\mathbf{x}|^2|}\xrightarrow[|\mathbf{x}|\to\infty]{}1\Leftrightarrow\lim\limits_{|\mathbf{x}|\to\infty}\dfrac{x^2+y^2}{x^2+x+y^2}=1$.

c) $|xye^{-x^2-y^2}|=|xy|e^{-x^2-y^2}=|x||y|e^{-(x^2+y^2)}\leqslant|\mathbf{x}|^2e^{-|\mathbf{x}|^2}$

$\xrightarrow[|\mathbf{x}|\to\infty]{}0\Leftrightarrow\lim\limits_{|\mathbf{x}|\to\infty}(xye^{-x^2-y^2})=0$.

Problem 1.25 (Sid. 2)

Lösning: Låt oss gå in mot origo längs en rät linje $y=kx$; $f(\mathbf{x})=f(x,kx)=\dfrac{k^4x^4}{k^4x^2+(k-x)^2} \Rightarrow$

$\lim\limits_{\mathbf{x}\to 0} f(\mathbf{x})=\lim\limits_{x\to 0} f(x,kx)=0.$

Låt oss nu nå origo längs parabeln $y=x^2$;

$\lim\limits_{\mathbf{x}\to 0} f(\mathbf{x})=\lim\limits_{x\to 0} f(x,x^2)=\lim\limits_{x\to 0}\dfrac{x^8}{x^8}=\lim\limits_{x\to 0} 1=1.$

Vi drar slutsatsen att $\lim\limits_{\mathbf{x}\to 0} f(\mathbf{x})$ inte existerar.

Problem 1.26 (Sid. 2)

Lösning

a) $\sqrt{1+x^2}-\sqrt{1-y^2}=\dfrac{(\sqrt{1+x^2}-\sqrt{1-y^2})(\sqrt{1+x^2}+\sqrt{1-y^2})}{\sqrt{1+x^2}+\sqrt{1+y^2}}=$

$=\dfrac{(\sqrt{1+x^2})^2-(\sqrt{1-y^2})^2}{\sqrt{1+x^2}+\sqrt{1+y^2}}=\dfrac{x^2+y^2}{\sqrt{1+x^2}+\sqrt{1+y^2}} \Rightarrow$

$\lim\limits_{\mathbf{x}\to 0}\dfrac{\sqrt{1+x^2}+\sqrt{1-y^2}}{x^2+y^2}=\lim\limits_{\mathbf{x}\to 0}\dfrac{1}{\sqrt{1+x^2}+\sqrt{1-y^2}}=\dfrac{1}{1+1}=\dfrac{1}{2}.$

b) $\dfrac{x^2+y^2}{|x|+|y|}\leq\dfrac{x^2+y^2+2|xy|}{|x|+|y|}=\dfrac{(|x|+|y|)^2}{|x|+|y|}=|x|+|y|\xrightarrow[\mathbf{x}\to 0]{} 0$

$\Rightarrow \lim\limits_{\mathbf{x}\to 0}\dfrac{x^2+y^2}{|x|+|y|}=0.$

c) Låt oss gå in mot origo längs linjen $y=x$:

$\lim\limits_{\mathbf{x}\to 0}\dfrac{xy^2}{x^2+y^4}=\lim\limits_{x\to 0}\dfrac{x^3}{x^2+x^4}=\lim\limits_{x\to 0}\dfrac{x}{1+x}=0.$ (*)

Låt oss nu se vad som händer om vi går in mot origo längs parabeln $x=y^2$:

$\lim\limits_{\mathbf{x}\to 0}\dfrac{xy^2}{x^2+y^4}=\lim\limits_{y\to 0}\dfrac{y^4}{y^4+y^4}=\lim\limits_{y\to 0}\dfrac{1}{2}=\dfrac{1}{2}\overset{(*)}{\neq} 0.$

Gränsvärdet existerar inte i detta fall.

Problem 1.27 (Sid. 2)

Lösning

En funktion f är kontinuerlig i en punkt $\mathbf{x}=\mathbf{a}$ om f är definierad i $\mathbf{x}=\mathbf{a}$ ($f(\mathbf{a})$ existerar, alt. $\mathbf{a}\in D_f$) och $\lim\limits_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})=f(\mathbf{a})$.

$f(x,y)$ är kontinuerlig för $(x,y)\neq(0,0)$;

$\lim\limits_{\mathbf{x}\to 0} f(\mathbf{x})=\lim\limits_{\mathbf{x}\to 0}|\mathbf{x}|^2=0\neq 1=f(\mathbf{0})$; dvs f är diskontinuerlig i origo; diskontinuiteten är dock hävbar; omdefinitionen $f(0,0)=0$ ger den kontinuerliga funktionen $g(\mathbf{x})=|\mathbf{x}|^2$.

Problem 1.28 (Sid. 3)

Lösning

a) $\lim\limits_{\mathbf{x}\to 0}\dfrac{\sin|\mathbf{x}|^2}{|\mathbf{x}|^2}=1$, har visats i Problem 1.20; $f(0)=1$ ger den kontinuerliga funktionen

$$g(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \mathbf{x} \neq \mathbf{0} \\ 1, & \mathbf{x} = \mathbf{0} \end{cases}$$

b) Låt oss sätta kursen in mot $(0,0)$ längs $y=kx$:

$\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x}) = \lim_{x\to 0} f(x,kx) = \lim_{x\to 0} \frac{k^2+4k+4}{3k^2+2k+1}$, existerar inte (det beror på k); f kan inte utvidgas till en kontinuerlig funktion.

c) $f(x,y) = \dfrac{6x^2+3y^2+x^2y^2}{2x^2+y^2} = 3 + \dfrac{x^2 \cdot y^2}{2x^2+y^2}$; $(*)$

$$0 \leq \frac{x^2y^2}{2x^2+y^2} \leq \frac{x^2y^2}{x^2+y^2} \leq \frac{|\mathbf{x}|^2 \cdot |\mathbf{x}|^2}{|\mathbf{x}|^2} = |\mathbf{x}|^2 \xrightarrow[\mathbf{x}\to\mathbf{0}]{} 0 \Rightarrow (*) \Rightarrow$$

$\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x}) = 3$. f:s kontinuerliga utvidgning är

$$g(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \mathbf{x} \neq \mathbf{0} \\ 3, & \mathbf{x} = \mathbf{0} \end{cases}$$

d) $|xe^{-1/|\mathbf{x}|}| = |x|e^{-1/|\mathbf{x}|} \leq |\mathbf{x}|e^{-1/|\mathbf{x}|} \xrightarrow[\mathbf{x}\to\mathbf{0}]{} 0$, ty

$t = |\mathbf{x}| = \sqrt{x^2+y^2} \Rightarrow \lim_{t\to 0^+} te^{-1/t} = \{u = \frac{1}{t}\} = \lim_{u\to\infty} \frac{1}{ue^u} = 0.$

$$g(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \mathbf{x} \neq \mathbf{0} \\ 0, & \mathbf{x} = \mathbf{0} \end{cases}$$

är en kontinuerlig funktion, en kontinuerlig utvidgning av $f(x,y) = xe^{-1/\sqrt{x^2+y^2}}$.

Problem 1.29 (Sid. 3)

Lösning

a) Låt oss gå in mot origo först längs x-axeln och sen längs y-axeln:

$$\left.\begin{array}{l} \lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x}) = \lim_{x\to 0} f(x,0,0) = \lim_{x\to 0} \frac{x^2}{2x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2} \\ \lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x}) = \lim_{y\to 0} f(0,y,0) = \lim_{y\to 0} \frac{y^3}{y^2} = \lim_{y\to 0} y = 0 \end{array}\right\} \Rightarrow$$

$\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x})$ existerar inte $\Rightarrow$ det går inte att utvidga f till en kontinuerlig funktion.

b) $f(\mathbf{x}) = \dfrac{xyz+yz}{x^2+y^2+z^2+2x+1} = \dfrac{(x+1)yz}{(x+1)^2+y^2+z^2}$;

$$\begin{cases} \mathbf{x} = (x,y,z) \\ \mathbf{a} = (-1,0,0) \end{cases} \Rightarrow \mathbf{u} = \mathbf{x} - \mathbf{a} = (x+1, y, z) \Rightarrow \begin{cases} u_1 = x+1 \\ u_2 = y \\ u_3 = z \end{cases} \Rightarrow$$

$g(\mathbf{u}) = f(\mathbf{a}+\mathbf{u}) = \dfrac{u_1u_2u_3}{u_1^2+u_2^2+u_3^2}$, studeras nära $\mathbf{u} = \mathbf{0}$;

$$\left|\frac{u_1u_2u_3}{u_1^2+u_2^2+u_3^2}\right| = \frac{|u_1||u_2||u_3|}{u_1^2+u_2^2+u_3^2} \leq \frac{|\mathbf{u}|^3}{|\mathbf{u}|^2} = |\mathbf{u}| \xrightarrow[\mathbf{u}\to\mathbf{0}]{} 0 \Rightarrow$$

$\lim_{\mathbf{u}\to\mathbf{0}} g(\mathbf{u}) = 0 \Rightarrow \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = 0$; med $f(\mathbf{a}) = 0$ fås en kontinuerlig $F(\mathbf{x}) = f(\mathbf{x})$, för $\mathbf{x} \neq \mathbf{a}$.

$$F(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \mathbf{x} \neq \mathbf{a} \\ 0, & \mathbf{x} = \mathbf{a} \end{cases}$$

2. Differentialkalkyl för reellvärda funktioner

Partiella derivator. Differentierbarhet. Differentialer.

Problem 2.1 (Sid. 3)

Lösning

$$\begin{cases} f'(x) = Df(x) = \dfrac{d}{dx}f(x) = d_x f(x). \\ f'_x(x,y) = D_x f(x,y) = \dfrac{\partial}{\partial x}f(x,y) = \partial_x f(x,y). \\ f'_y(x,y) = D_y f(x,y) = \dfrac{\partial}{\partial y}f(x,y) = \partial_y f(x,y). \end{cases}$$

a) $f(x,y) = x + x^3y + x^2y^3 + y^5, \quad D_f = \mathbb{R}^2.$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}f(x,y) = \frac{\partial}{\partial x}(x + x^3y + x^2y^3 + y^5) =$$
$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial x}(x^3y) + \frac{\partial}{\partial x}(x^2y^3) + \frac{\partial}{\partial x}(y^5) =$$
$$= \frac{d}{dx}(x) + \left(\frac{d}{dx}x^3\right)y + \left(\frac{d}{dx}x^2\right)y^3 + \frac{\partial}{\partial x}y^5 =$$
$$= 1 + (3x^2y + (2x)y^3 + 0 = \underline{1 + 3x^2y + 2xy^3}.$$

b) $$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}f(x,y) = \frac{\partial}{\partial y}(x + x^3y + x^2y^3 + y^5) =$$
$$= \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial y}(x^3y) + \frac{\partial}{\partial y}(x^2y^3) + \frac{\partial}{\partial y}(y^5) =$$
$$= \frac{\partial}{\partial y}(x) + x^3\frac{\partial}{\partial y}(y) + x^2\frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial y}(y^5) =$$
$$= 0 + x^3 \cdot 1 + x^2 \cdot (3y^2) + 5y^4 = \underline{x^3 + 3x^2y^2 + 5y^4}.$$

Så tänker man men så gör man:

$$f(x,y) = x + x^3y + x^2y^3 + y^5 \Rightarrow \begin{cases} \dfrac{\partial f}{\partial x} = 1 + 3x^2y + 2xy^3 \\ \dfrac{\partial f}{\partial y} = x^3 + 3x^2y^2 + 5y^4 \end{cases}$$

b) $f(x,y) = \ln(1 - x^2 - 2y^2), \quad x^2 + 2y^2 < 1.$

$$f(x,y) = \ln(1 - x^2 - 2y^2) \Rightarrow \begin{Bmatrix} f(x,y) = \ln u \\ u(x,y) = 1 - x^2 - 2y^2 \end{Bmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\ln u = \dfrac{1}{u}\dfrac{\partial u}{\partial x} = \dfrac{1}{1-x^2-2y^2}\cdot(-2x) = \dfrac{-2x}{1-x^2-2y^2}. \\ \dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\ln u = \dfrac{1}{u}\dfrac{\partial u}{\partial y} = \dfrac{1}{1-x^2-2y^2}\cdot(-4y) = \dfrac{-4y}{1-x^2-2y^2} \end{cases}$$

c) $f(x,y) = e^{-y^2}\arcsin 2y, \quad -\frac{1}{2} \le y \le \frac{1}{2}$ (Ober. av x).

$$\frac{\partial f}{\partial x} = 0;$$
$$\frac{\partial f}{\partial y} = \frac{d}{dy}e^{-y^2}\sin^{-1}2y = (-2ye^{-y^2})\sin^{-1}2y + e^{-y^2}\cdot\frac{2}{\sqrt{1-(2y)^2}} =$$
$$= -2ye^{-y^2}\arcsin 2y + 2\cdot\frac{e^{-y^2}}{\sqrt{1-4y^2}}.$$

d) $f(x,y) = \dfrac{x+y}{x-y}, \quad x \neq y$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x+y}{x-y}\right) = \frac{(x-y)\partial_x(x+y) - (x+y)\partial_x(x-y)}{(x-y)^2} =$$

$$= \frac{x-y-(x+y)}{(x-y)^2} = -\frac{2y}{(x-y)^2}.$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x+y}{x-y}\right) = \frac{(x-y)\partial_y(x+y)-(x+y)\partial_y(x-y)}{(x-y)^2} =$$

$$= \frac{x-y+(x+y)}{(x-y)^2} = \frac{2x}{(x-y)^2}.$$

Problem 2.2 (Sid. 3)

Lösning

a) $f(x,y,z) = \cos(xy-z^2)$, $D_f = \mathbb{R}^3$.

$$u = xy - z^2 \Rightarrow \begin{cases} \frac{\partial f}{\partial x} = -\sin u \frac{\partial u}{\partial x} = -y\cdot\sin(xy-z^2) \\ \frac{\partial f}{\partial y} = -\sin u \frac{\partial u}{\partial y} = -x\cdot\sin(xy-z^2). \\ \frac{\partial f}{\partial z} = -\sin u \frac{\partial u}{\partial z} = 2z\cdot\sin(xy-z^2) \end{cases}$$

b) $f(x,y,z) = \frac{1}{\sqrt{z}}\arctan\frac{y}{x}$.

(1) $u = \frac{y}{x} \Rightarrow \frac{\partial u}{\partial x}\left(\frac{y}{x}\right) = -\frac{y}{x^2} \wedge \frac{\partial u}{\partial y} = \frac{1}{x}$; $\frac{d}{dz}z^{-1/2} = -\frac{1}{2z\sqrt{z}}$.

(2) $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\frac{1}{\sqrt{z}}\arctan u = \frac{1}{\sqrt{z}}\frac{\partial}{\partial x}\arctan u =$

$$= \frac{1}{\sqrt{z}}\frac{1}{1+u^2}\frac{\partial u}{\partial x} = \frac{1}{\sqrt{z}}\cdot\frac{1}{1+(y/x)^2}\cdot\left(-\frac{y}{x^2}\right) = -\frac{1}{\sqrt{z}}\frac{y}{x^2+y^2}.$$

(3) $\frac{\partial f}{\partial y} = \frac{1}{\sqrt{z}}\frac{\partial}{\partial y}\arctan u = \frac{1}{\sqrt{z}}\frac{1}{1+u^2}\frac{\partial u}{\partial y} = \frac{1}{\sqrt{z}}\frac{x}{x^2+y^2}.$

(4) $\frac{\partial f}{\partial z} = \left(\frac{d}{dz}\frac{1}{\sqrt{z}}\right)\arctan\frac{y}{x} = -\frac{1}{2z\sqrt{z}}\arctan\frac{y}{x}.$

c) $f(x,y,z) = x^{y^z}$

$u = x^{y^z} \Leftrightarrow \ln u = \ln x^{y^z} = (\ln x)y^z = (\ln x)e^{z\ln y}.$

$\frac{\partial}{\partial x}\ln u = \left(\frac{d}{dx}\ln x\right)y^z \Leftrightarrow \frac{1}{u}\frac{\partial u}{\partial x} = \frac{1}{x}y^z \Leftrightarrow \frac{\partial u}{\partial x} = \frac{u}{x}y^z$

$\Leftrightarrow \frac{\partial f}{\partial x} = x^{(y^z)-1}\cdot y^z.$

$\frac{\partial}{\partial y}\ln u = (\ln x)\frac{\partial}{\partial y}y^z \Leftrightarrow \frac{1}{u}\frac{\partial u}{\partial y} = (\ln x)\cdot zy^{z-1} \Leftrightarrow$

$\Leftrightarrow \frac{\partial f}{\partial y} = u\cdot(\ln x)\cdot zy^{z-1} \Leftrightarrow \frac{\partial f}{\partial y} = x^{y^z}\cdot y^z\cdot z\cdot\ln x.$

$\frac{\partial}{\partial z}\ln u = (\ln x)\frac{\partial}{\partial z}e^{z\ln y} \Leftrightarrow \frac{1}{u}\frac{\partial u}{\partial z} = (\ln x)e^{z\ln y}\cdot\ln y$

$\Leftrightarrow \frac{\partial u}{\partial z} = u\cdot(\ln x)y^z\cdot\ln y \Leftrightarrow \frac{\partial f}{\partial z} = x^{y^z}\cdot y^z(\ln x)\ln y.$

Problem 2.3 (Sid. 3)

Lösning

a) $f(x,y) = x + x^3y + x^2y^3 + y^5$

$$\frac{\partial f}{\partial x} = 1 + 3x^2y + 2xy^3,\quad \frac{\partial f}{\partial y} = x^3 + 3x^2y^2 + 5y^4.$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(1+3x^2y+2xy^3) = 6xy + 2y^3.$$

$$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(1+3x^2y+2xy^3) = 3x^2 + 6xy^2.$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(x^3+3x^2y^2+5y^4) = 3x^2 + 6xy^2 = \frac{\partial^2 f}{\partial y\partial x}.$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(x^3+3x^2y^2+5y^4) = 6x^2y + 20y^3.$$

Problem 2.4 (Sid. 3)

Lösning

$$f(x,y)=\begin{cases}\dfrac{x^3+y^4}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

(1) $$\frac{\partial f}{\partial x}=\frac{(x^2+y^2)\partial_x(x^3+y^4)-(x^3+y^4)\partial_x(x^2+y^2)}{(x^2+y^2)^2}=$$
$$=\frac{(x^2+y^2)3x^2-(x^3+y^4)\cdot 2x}{(x^2+y^2)^2}=\frac{3x^4+3x^2y^2-2x^4-2xy^4}{(x^2+y^2)^2}=$$
$$=\frac{x^4+3x^2y^2-2xy^4}{(x^2+y^2)^2},\quad (x,y)\neq(0,0).$$

$$f'_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{h^3}{h^3}=\lim_{h\to 0}1=1.$$

(2) $$\frac{\partial f}{\partial y}=\frac{(x^2+y^2)\partial_y(x^3+y^4)-(x^3+y^4)\partial_y(x^2+y^2)}{(x^2+y^2)^2}=$$
$$=\frac{(x^2+y^2)\cdot 4y^3-(x^3+y^4)\cdot 2}{(x^2+y^2)^2}=\frac{4x^2y^3+4y^5-2x^3y-2y^5}{(x^2+y^2)^2}=$$
$$=\frac{4x^2y^3+2y^5-2x^3y}{(x^2+y^2)},\quad (x,y)\neq(0,0)$$

$$f'_y(0,0)=\lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k}=\lim_{k\to 0}\frac{k^4}{k^3}=\lim_{k\to 0}k=0.$$

Svar:
$$\begin{cases} f'_x(\mathbf{x})=\begin{cases}\dfrac{x^4+3x^2y^2-2xy^4}{(x^2+y^2)^2}, & \mathbf{x}\neq(0,0)\\ 1, & \mathbf{x}=(0,0)\end{cases}\\ f'_y(\mathbf{x})=\begin{cases}\dfrac{4x^2y^3+2y^5-2x^3y}{(x^2+y^2)^2}, & \mathbf{x}\neq(0,0)\\ 0, & \mathbf{x}=(0,0).\end{cases}\end{cases}$$

Problem 2.5 (Sid. 3)

Lösning

$$f(x,y)=\begin{cases}\dfrac{(x+y)^2}{x^2+y^2}, & x\neq 0\\ 1, & x=0\end{cases}$$

(1) $x\neq 0$: $$\frac{\partial f}{\partial x}=\frac{2(x+y)(x^2+y^2)-2x(x+y)^2}{(x^2+y^2)^2}=$$
$$=\frac{2(x^3+xy^2+yx^2+y^3)-2x(x^2+2xy+y^2)}{(x^2+y^2)^2}=$$
$$=\frac{\cancel{2x^3}+\cancel{2xy^2}+2x^2y+2y^3-\cancel{2x^3}-4x^2y-\cancel{2xy^2}}{(x^2+y^2)^2}$$
$$=\frac{2y^3-2x^2y}{(x^2+y^2)^2};$$

$$f'_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{1-1}{h}=0$$

$$f'_x(x,y)=\begin{cases}\dfrac{2y^3-2x^2y}{(x^2+y^2)^2}, & x\neq 0\\ 0, & x=0\end{cases}$$

(2) På samma sätt visas att
$$f'_y(x,y)=\begin{cases}\dfrac{2x^3-2y^2x}{(x^2+y^2)^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0).\end{cases}$$

(3) Låt oss gå in mot origo längs linjen $y=kx$:

$$\lim_{\mathbf{x}\to 0}f(\mathbf{x})=\lim_{x\to 0}f(x,kx)=\lim_{x\to 0}\frac{(1+k)^2}{1+k^2},$$ existerar inte,

dvs f är inte kontinuerlig i origo $(0,0)$.

Anm. f kontinuerlig i $x=a$ om $\lim_{x\to a} f(x)=f(a)$.

Problem 2.6 (Sid. 3)

Lösning

$$f(x,y)=\begin{cases} y^2\arctan\frac{x}{y}, & y\neq 0 \\ 0, & y=0 \end{cases}$$

a)
$$y\neq 0 \Rightarrow \begin{cases} \frac{\partial f}{\partial x}=y^2\frac{\partial}{\partial x}\arctan\frac{x}{y}=y^2\cdot\frac{1}{1+x^2/y^2}\cdot\frac{1}{y}=\frac{y^3}{x^2+y^2} \\ \frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(y^2\operatorname{atn}\frac{x}{y})=2y\cdot\operatorname{atn}\frac{x}{y}+y^2\cdot\frac{1}{1+x^2/y^2}\left(\frac{-x}{y^2}\right) \end{cases}$$
$$=2y\arctan\frac{x}{y}-\frac{xy^2}{x^2+y^2}.$$

$$f'_x(x,0)=\lim_{h\to 0}\frac{f(x+h,0)-f(x,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0;$$
$$f'_y(x,0)=\lim_{k\to 0}\frac{f(x,k)-f(0,0)}{k}=\lim_{k\to 0}k\cdot\operatorname{atn}\frac{x}{k}=0;$$

$$\left.\begin{array}{l} \lim_{y\to 0}f'_x(x,y)=\lim_{y\to 0}\frac{y^3}{x^2+y^2}=0=f'_x(x,0). \\ \lim_{y\to 0}f'_y(x,y)=\lim_{y\to 0}\left(2y\cdot\operatorname{atn}\frac{x}{y}-\frac{xy^2}{x^2+y^2}\right)=0=f'_y(x,0) \end{array}\right\}\Rightarrow$$

$\Rightarrow$ f är kontinuerligt deriverbar, dvs $f\in\mathcal{C}^1$.

b) $y\neq 0 \Rightarrow f''_{xy}(x)=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{3x^2y^2+y^4}{(x^2+y^2)^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=f''_{yx}(x)$;

$y=0$: $f''_{xy}(0,0)=\lim_{k\to 0}\frac{f'_x(0,k)-f'_x(0,0)}{k}=\lim_{k\to 0}\frac{k^3}{k^3}=1$;

$$f''_{yx}(0,0)=\lim_{h\to 0}\frac{f'_y(h,0)-f'_y(0,0)}{h}=0\neq f''_{xy}(0,0).$$

f är således inte en $\mathcal{C}^2$-funktion.

Anm. $\lim_{x\to 0}f''_{xy}(x)=\lim_{x\to 0}f''_{xy}(x,kx)=\frac{3k^2+k^4}{(1+k^2)^2}$.

Problem 2.7 (Sid. 3)

Lösning

a) $\frac{\partial z}{\partial x}=2x+y$ (1); $\frac{\partial f}{\partial z}=x+2y$ (2);

Kriteriet för existensen är $\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}$; detta är uppenbarligen uppfyllt.

$(1)\Leftrightarrow \frac{\partial z}{\partial x}=2x+y \Leftrightarrow z=x^2+xy+f(y) \Rightarrow \frac{\partial z}{\partial y}=x+f'(y)=$

$\overset{(2)}{=}x+2y \Leftrightarrow f'(y)=2y \Leftrightarrow f(y)=y^2+C$, C konstant;

Resultat: $z=x^2+xy+y^2+C$.

Anm. $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=(2x+y)dx+(x+2y)dy=$

$=2xdx+(ydx+xdy)+2ydy=dx^2+d(xy)+dy^2=$

$=d(x^2+xy+y^2)\Leftrightarrow z=x^2+xy+y^2+C$ (Se sid. 116)

b)
$$\left.\begin{cases} \frac{\partial z}{\partial x}=e^{xy} \Rightarrow \frac{\partial^2 z}{\partial y\partial x}=\frac{\partial}{\partial y}e^{xy}=xe^{xy} \\ \frac{\partial z}{\partial y}=e^{xy} \Rightarrow \frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}e^{xy}=ye^{xy} \end{cases}\right. \Rightarrow \text{kriteriet är}$$

inte uppfyllt; Exempel 26 konsulteras.

c) $\dfrac{\partial z}{\partial x} = ye^x$ (1); $\dfrac{\partial z}{\partial y} = e^x$ (2).

$\dfrac{\partial^2 z}{\partial x \partial y} = e^x = \dfrac{\partial^2 z}{\partial y \partial x} \Rightarrow$ lösning(ar) existerar säkert.

$\dfrac{\partial z}{\partial x} \overset{(1)}{=} ye^x \Rightarrow z = ye^x + f(y) \Rightarrow \dfrac{\partial z}{\partial y} = e^x + f'(y) \overset{(2)}{=} e^x \Leftrightarrow$

$f'(y) = 0 \Leftrightarrow f(y) = C$, C konstant.

Resultat: $z = ye^x + C$.

Anm. $dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy = ye^x dx + e^x dy = d(ye^x)$.

Problem 2.8 (Sid. 3)

Lösning

$$\begin{cases} \dfrac{\partial z}{\partial x} = ye^{x^2y^4} \Rightarrow \dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial}{\partial y}(ye^{x^2y^4}) = (1+4x^2y^4)e^{x^2y^4} \\ \dfrac{\partial z}{\partial y} = xe^{x^2y^4} \Rightarrow \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial}{\partial x}(xe^{x^2y^4}) = (1+2x^2y^4)e^{x^2y^4} \end{cases}$$

$\Rightarrow \dfrac{\partial^2 z}{\partial x \partial y} \neq \dfrac{\partial^2 z}{\partial y \partial x} \Rightarrow$ det finns ingen C^2-fkn z med förstaderivator som ovan. (Sats 9).

Problem 2.8 (Sid. 3)

Lösning

a) (1) $\dfrac{\partial u}{\partial x} = y + 3z - 3$; (2) $\dfrac{\partial u}{\partial y} = x + 2z - 2$; (3) $\dfrac{\partial u}{\partial z} = 2y + 3x - 1$.

Kriteriet för existensen blir i detta fall:

$\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$, $\dfrac{\partial^2 u}{\partial y \partial z} = \dfrac{\partial^2 u}{\partial z \partial y}$ och $\dfrac{\partial^2 u}{\partial z \partial x} = \dfrac{\partial^2 z}{\partial x \partial z}$.

Prövning visar att kriteriet är uppfyllt.

$\dfrac{\partial u}{\partial x} \overset{(1)}{=} y + 3z - 3 \Leftrightarrow u = xy + 3xz - 3x + f(y,z) \Rightarrow \dfrac{\partial u}{\partial y} =$
$= x + \dfrac{\partial f}{\partial y} \overset{(2)}{=} x + 2z - 2 \Leftrightarrow \dfrac{\partial f}{\partial y} = 2z - 2 \Leftrightarrow f(y,z) = 2yz -$
$- 2y + g(z) \Rightarrow u = xy + 3xz - 3x + 2zy - 2y + g(z) \Rightarrow \dfrac{\partial u}{\partial z} =$
$= 3x + 2y + g'(z) \overset{(3)}{=} 2y + 3x - 1 \Leftrightarrow g'(z) = -1 \Leftrightarrow g(z) = -z + C$

Resultat: $u = xy + 3xz + 2yz - 3x - 2y - z + C$.

b) (1) $\dfrac{\partial u}{\partial x} = 1 + y\sin xy$, (2) $\dfrac{\partial u}{\partial y} = e^z + x\sin xy$, (3) $\dfrac{\partial u}{\partial z} = (1+x+y)e^z$.

$\dfrac{\partial^2 u}{\partial x \partial z} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial z}\right) = e^z \neq 0 = \dfrac{\partial}{\partial z}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial^2 u}{\partial z \partial x} \Rightarrow$ kriteriet är inte uppfyllt; inga lösningar således.

c) (1) $\dfrac{\partial u}{\partial x} = z + xy^2$, (2) $\dfrac{\partial u}{\partial y} = x^2y$, (3) $\dfrac{\partial u}{\partial z} = yz$; (Test?)

$\dfrac{\partial u}{\partial x} \overset{(1)}{=} z + xy^2 \Leftrightarrow u = xz + \tfrac{1}{2}x^2y^2 + f(y,z) \Rightarrow \dfrac{\partial u}{\partial y} = x^2y +$
$+ \dfrac{\partial f}{\partial y} \overset{(2)}{=} x^2y \Leftrightarrow \dfrac{\partial f}{\partial y} = 0 \Leftrightarrow f(y,z) = g(z) \Rightarrow u = xz +$
$+ \tfrac{1}{2}x^2y^2 + g(z) \Rightarrow \dfrac{\partial u}{\partial z} = x + g'(z) \overset{(3)}{=} yz \Leftrightarrow g'(z) = yz - x$;
denna motsägelse beror på att systemet är inkonsistent.

Anm. $\dfrac{\partial^2 u}{\partial x \partial z} = 0 \neq 1 = \dfrac{\partial^2 u}{\partial z \partial x}$! Testa först.

Problem 2.10 (Sid. 3)

Lösning: $z=f(x,y)$

a) $\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}f(x,y)=0 \Leftrightarrow f(x,y)=\varphi(y),\ \varphi\in \mathcal{C}^1(\mathbb{R})$.

b) $\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}f(x,y)=0 \Leftrightarrow f(x,y)=\psi(x),\ \psi\in \mathcal{C}^1(\mathbb{R})$.

c) $\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=0 \Leftrightarrow \frac{\partial z}{\partial x}=u(y)=\frac{\partial}{\partial x}f(x,y) \Leftrightarrow$

$\Leftrightarrow f(x,y)=x\cdot u(y)+v(y)$.

d) $\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=0 \Leftrightarrow \frac{\partial z}{\partial y}=g(y)=\frac{\partial}{\partial y}f(x,y) \Leftrightarrow$

$\Leftrightarrow f(x,y)=G(y)+\mathcal{F}(x)$.

e) $\frac{\partial z}{\partial x}=z \Leftrightarrow \frac{\partial z}{\partial x}-z=0 \Leftrightarrow e^{-x}\frac{\partial z}{\partial x}-e^{-x}z=\frac{\partial}{\partial x}e^{-x}z=0$

$\Leftrightarrow e^{-x}\cdot z=\phi(y) \Leftrightarrow z=\phi(y)e^{x} \Leftrightarrow f(x,y)=\phi(y)e^{x}$.

f) $\frac{\partial z}{\partial x}=yz \Leftrightarrow \frac{\partial z}{\partial x}-yz=0 \Leftrightarrow e^{-xy}\frac{\partial z}{\partial x}-ye^{-xy}z=0 \Leftrightarrow$

$\frac{\partial}{\partial x}(e^{-xy}\cdot z)=0 \Leftrightarrow e^{-xy}z=\psi(y) \Leftrightarrow f(x,y)=\psi(y)e^{xy}$.

g) $\frac{\partial z}{\partial x}=xz \Leftrightarrow \frac{\partial z}{\partial x}-xz=0 \Leftrightarrow e^{-x^2/2}\frac{\partial z}{\partial x}-xe^{-x^2/2}z=0 \Leftrightarrow$

$\frac{\partial}{\partial x}(e^{-x^2/2}z)=0 \Leftrightarrow e^{-x^2/2}z=g(y) \Leftrightarrow f(x,y)=g(y)e^{x^2/2}$

h) $\frac{\partial^2 z}{\partial y^2}+e^{2x}z=0 \Leftrightarrow \left(\frac{\partial}{\partial y}+ie^{x}\right)\left(\frac{\partial}{\partial y}-ie^{x}\right)z=0 \quad (*)$

$u=\frac{\partial z}{\partial y}-ie^{x}z \overset{(*)}{\Rightarrow} \frac{\partial u}{\partial y}+ie^{x}u=0 \Leftrightarrow \frac{\partial}{\partial y}ue^{iye^{x}}=0 \Leftrightarrow$

$\Leftrightarrow ue^{iye^{x}}=\phi(x) \Leftrightarrow u=\frac{\partial z}{\partial y}-ie^{x}z=\phi(x)e^{-iye^{x}} \Leftrightarrow$

$\Leftrightarrow \frac{\partial}{\partial y}ze^{-iye^{x}}=\phi(x)e^{-i2ye^{x}} \Leftrightarrow ze^{-iye^{x}}=\phi(x)\frac{i}{2}e^{-x}e^{-i2ye^{x}}+$

$+\psi(x) \Leftrightarrow z=\frac{i}{2}\phi(x)e^{-x}e^{-iye^{x}}+\psi(x)e^{iye^{x}}=f(x,y)$.

Den reellvärda lösningen är

$$z=\mathcal{F}(x)\cos(ye^{x})+G(x)\sin(ye^{x}).$$

Problem 2.11 (Sid. 4)

Lösning

Tangentplanets ekvation i punkten $P:(a,b,c)$ är

$$z=c+f'_x(a,b)\cdot(x-a)+f'_y(a,b)(y-b).$$

För en funktionsyta är $c=f(a,b)$, så att

$$\pi:\ z=f(a,b)+f'_x(a,b)(x-a)+f'_y(a,b)(y-b).$$

a) $f(x,y)=x^3+xy^2 \Rightarrow \begin{cases}\frac{\partial f}{\partial x}=3x^2+y^2\\ \frac{\partial f}{\partial y}=2xy\end{cases} \Rightarrow \begin{cases}f'_x(1,2)=7\\ f'_y(1,2)=4\end{cases} \Rightarrow$

$\tau:\ z=5+7(x-1)+4(y-2) \Leftrightarrow \tau:\ 7x+4y-z=10.$

b) $f(x,y)=e^{2x}-1 \Rightarrow \begin{cases}\frac{\partial f}{\partial x}=2e^{2x}\\ \frac{\partial f}{\partial y}=0\end{cases} \Rightarrow \begin{cases}f'_x(0,2)=2\\ f'_y(0,2)=0\end{cases} \Rightarrow$

$\tau:\ z=0+2(x-0)+0\cdot(y-2) \Leftrightarrow \tau:\ 2x-z=0.$

c) $y=\arcsin(xz) \Leftrightarrow xz=\sin y \Leftrightarrow z=f(x,y)=\frac{\sin y}{x} \Rightarrow$

$$\Rightarrow \begin{cases} \frac{\partial f}{\partial x} = -\frac{\sin y}{x^2} \\ \frac{\partial f}{\partial y} = \frac{\cos y}{x} \end{cases} \Rightarrow \begin{cases} f'_x(1,\frac{\pi}{6}) = -\frac{1}{2} \\ f'_y(1,\frac{\pi}{6}) = \frac{\sqrt{3}}{2} \end{cases} \Rightarrow \tau:\ z = \frac{1}{2} - \frac{1}{2}(x-1) + \frac{\sqrt{3}}{2}\left(y - \frac{\pi}{6}\right)$$

$$\Leftrightarrow \underline{\tau:\ x - \sqrt{3}y + 2z = 2 - \sqrt{3}\pi/6}.$$

Problem 2.12 (Sid. 4)

Lösning

a) $f(x,y,z) = 2x - 3y + z \Rightarrow \begin{cases} \partial_x f = 2 \\ \partial_y f = -3 \\ \partial_z f = 1 \end{cases} \Rightarrow \underline{df = 2dx - 3dy + dz}.$

b) $\underline{f(x,y) = \sin(xy^2)}$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = dx\frac{\partial}{\partial x}\sin(xy^2) + dy\frac{\partial}{\partial y}\sin(xy^2) =$$

$$= dx\cos(xy^2)\frac{\partial}{\partial x}(xy^2) + dy\cos(xy^2)\frac{\partial}{\partial y}(xy^2) =$$

$$= dx\cdot\cos(xy^2)\cdot y^2 + dy\cdot\cos(xy^2)\cdot 2xy =$$

$$= \underline{\cos(xy^2)(y^2dx + 2xy\,dy)} = \cos(xy^2)\,d(xy^2).$$

c) $\underline{f(p,V,T) = PV/T}$

$$df = \frac{\partial f}{\partial p}dp + \frac{\partial f}{\partial V}dV + \frac{\partial f}{\partial T}dT = \underline{\frac{V}{T}dp + \frac{P}{T}dV - \frac{PV}{T^2}dT}.$$

Problem 2.13 (Sid. 4)

Lösning

$$P = \frac{U^2}{R} \Rightarrow dP = \frac{\partial P}{\partial U}dU + \frac{\partial P}{\partial R}dR = \frac{2U}{R}dU - \frac{U^2}{R^2}dR;$$

a) $\underline{U = 10,\ R = 2}$; $\underline{\Delta U = 0{,}3,\ \Delta R = 0{,}1}$

$$dP = 2\cdot\frac{10}{2}\cdot 0{,}3 - \frac{10^2}{2^2}\cdot 0{,}1 = 3 - 2{,}5 = 0{,}5 \text{ (watt)}.$$

b) $\underline{U = 10,\ R = 2}$; $\underline{\Delta U = 0{,}3,\ \Delta R = 0{,}2}$

$$dP = 2\cdot\frac{10}{2}\cdot 0{,}3 - \frac{10^2}{2^2}\cdot 0{,}2 = 3 - 5 = -2 \text{ (watt)}.$$

Svar: a) Den ökar med 0,5 W. b) Den minskar med 2 watt.

Problem 2.14 (Sid. 4)

Lösning

$\underline{R_1 = 200,\ \Delta R_1 = 0{,}5}$; $\underline{R_2 = 300,\ \Delta R_2 = 1}$.

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{R_1 + R_2}{R_1 R_2} \Leftrightarrow R = \frac{R_1 R_2}{R_1 + R_2} = \frac{200\cdot 300}{500} = 120;$$

$$dR = \frac{\partial R}{\partial R_1}dR_1 + \frac{\partial R}{\partial R_2}dR_2 = \frac{R_2^2 dR_1 + R_1^2 dR_2}{(R_1 + R_2)^2} = \dots = 0{,}34.$$

Resultat: $R = 120 \pm 0{,}34\ \Omega$.

Problem 2.15 (Sid. 4)

Lösning

$$V = \pi x^2 h \Rightarrow dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial h}dh = \pi(2xh\,dx + x^2 dh) \Rightarrow$$

$$\Rightarrow \frac{dV}{V} = \frac{\pi(2\pi x h\,dx + x^2 dh)}{\pi x^2 h} = 2\frac{dx}{x} + \frac{dh}{h} = 2\cdot 0{,}03 - 0{,}01 = 0{,}05.$$

Svar: Med ungefär 5%.

Problem 2.16 (Sid. 4)

Lösning

$f(x,y)=xy \Rightarrow \Delta f(1,2)=f(1+\Delta x, 2+\Delta y)-f(1,2)=$

$=(1+\Delta x)(2+\Delta y)-1\cdot 2 = 2+2\Delta x+\Delta y+\Delta x\cdot\Delta y-2=$

$=2\cdot\Delta x+1\cdot\Delta y+\Delta x\cdot\Delta y;$

$A_1=2$ och $A_2=1$ avläses direkt.

$|\Delta x\cdot\Delta y|=|\Delta x|\cdot|\Delta y|\leq|\Delta \mathbf{x}|\cdot|\Delta\mathbf{x}|=|\Delta\mathbf{x}|^2 \Rightarrow \frac{|\Delta x\cdot\Delta y|}{|\Delta\mathbf{x}|}\leq$

$\leq|\Delta\mathbf{x}|\xrightarrow[\Delta\mathbf{x}\to 0]{}0 \Rightarrow f$ differentierbar i $(1,2)$.

Problem 2.17 (Sid. 4)

Lösning

a) Låt oss utreda om f är differentierbar i origo.

f är deriverbar i $(0,0)$ och $A_1=1$, $A_2=0$ så att

$R(\Delta\mathbf{x})=f(\Delta\mathbf{x})-f(\mathbf{0})-1\cdot\Delta x-0\cdot\Delta y=f(\Delta\mathbf{x})-\Delta x=$

$=\frac{(\Delta x)^3+(\Delta y)^4}{(\Delta x)^2+(\Delta y)^2}-\Delta x=\frac{(\Delta x)^3+(\Delta y)^4-(\Delta x)^3-\Delta x\cdot(\Delta y)^2}{(\Delta x)^2+(\Delta y)^2}=$

$=\frac{(\Delta y)^4-\Delta x\cdot(\Delta y)^2}{(\Delta x)^2+(\Delta y)^2}.$

$|\rho(\Delta\mathbf{x})|=\frac{|R(\Delta\mathbf{x})|}{|\Delta\mathbf{x}|}=\frac{|\Delta x(\Delta y)^2-(\Delta y)^4|}{|\Delta\mathbf{x}|^3}=\begin{bmatrix}\Delta x=r\cos v\\ \Delta y=r\sin v\end{bmatrix}=$

$=|\cos v\sin^2 v-r\sin^4 v|\xrightarrow[r\to 0]{}|\cos v|\sin^2 v$ (beror av v)

$\Rightarrow f$ är inte differentierbar i origo.

b) $f'_x(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}h\cdot\sin\frac{1}{h^2}=0=A_1$, ty

$|h\cdot\sin\frac{1}{h^2}|=|h|\cdot|\sin\frac{1}{h^2}|\leq|h|\xrightarrow[h\to 0]{}0.$

Pss visas att $A_2=f'_y(0,0)=0$.

$R(\Delta\mathbf{x})=f(\Delta\mathbf{x})-f(0,0)-0\cdot\Delta x-0\cdot\Delta y=f(\Delta\mathbf{x})=|\Delta\mathbf{x}|^2\sin\frac{1}{|\Delta\mathbf{x}|^2}.$

$|\rho(\Delta\mathbf{x})|=\frac{|R(\Delta\mathbf{x})|}{|\Delta\mathbf{x}|^2}=|\Delta\mathbf{x}|\cdot|\sin\frac{1}{|\Delta\mathbf{x}|^2}|\leq|\Delta\mathbf{x}|\xrightarrow[\Delta\mathbf{x}\to 0]{}0 \Rightarrow$

$\Rightarrow \lim_{\Delta\mathbf{x}\to 0}\rho(\Delta\mathbf{x})=0 \Rightarrow$ <u>f differentierbar i origo.</u>

$\mathbf{x}\neq\mathbf{0} \Rightarrow \frac{\partial f}{\partial x}=2x\cdot\sin\frac{1}{x^2+y^2}+(x^2+y^2)\cos\frac{1}{x^2+y^2}\cdot\left(\frac{-2x}{(x^2+y^2)^2}\right)$

$=2x\left(\sin\frac{1}{|\mathbf{x}|^2}-\frac{1}{|\mathbf{x}|^2}\cos\frac{1}{|\mathbf{x}|^2}\right);$

(1) $|2x\cdot\sin(x^2+y^2)^{-1}|\leq 2|\mathbf{x}|\xrightarrow[\mathbf{x}\to\mathbf{0}]{}0;$

(2) $\lim_{\mathbf{x}\to\mathbf{0}}\left((x^2+y^2)\cos(x^2+y^2)^{-1}\left(\frac{-2x}{(x^2+y^2)^2}\right)\right)=\begin{bmatrix}x=r\cos v\\ y=r\sin v\end{bmatrix}=$

$=\lim_{r\to 0}2\cos v\cdot\frac{1}{r}\cos\frac{1}{r^2}$ existerar inte.

f är således <u>inte kontinuerligt deriverbar.</u>

Kedjeregeln.

Variabelbyten i partiella diff-ekvationer.

Problem 2.18 (Sid. 4)

vg vänd

Lösning

a) $z=\sin(x-y) \Rightarrow \left\{\begin{array}{l} \frac{\partial z}{\partial x}=\frac{\partial}{\partial x}\sin(x-y)=\cos(x-y) \\ \frac{\partial z}{\partial y}=\frac{\partial}{\partial y}\sin(x-y)=\cos(x-y)\cdot(-1) \end{array}\right\} \Rightarrow$

$\Rightarrow \frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=\cos(x-y)-\cos(x-y)=0.$

b) $z=1+(x-y)e^{-x}e^{y}=1+(x-y)e^{-(x-y)}=1+te^{-t}$, $\underline{t=x-y}$

$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=(\frac{\partial}{\partial x}+\frac{\partial}{\partial y})z=(\frac{\partial}{\partial x}+\frac{\partial}{\partial y})(1+te^{-t})=\frac{\partial}{\partial x}te^{-t}+$

$+\frac{\partial}{\partial y}te^{-t}=(\frac{d}{dt}te^{-t})\frac{\partial t}{\partial x}+(\frac{d}{dt}te^{-t})\frac{\partial t}{\partial y}=(\frac{d}{dt}te^{-t})(1-1)=0.$

b) $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=(\frac{\partial}{\partial x}+\frac{\partial}{\partial y})z=(\frac{\partial}{\partial x}+\frac{\partial}{\partial y})f(x-y)=\frac{\partial}{\partial x}f(x-y)+$

$+\frac{\partial}{\partial y}f(x-y)=f'(x-y)\frac{\partial}{\partial x}(x-y)+f'(x-y)\frac{\partial}{\partial y}(x-y)=$

$=f'(x-y)-f'(x-y)=0.$

Anm. I a) är $f(t)=\sin t$ och i b) är $f(t)=1+te^{-t}$.

Problem 2.19 (Sid. 4)

Lösning: $u=f(t)$, $t=x/y$.

$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y})u=(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y})f(t)=$

$=x\frac{\partial}{\partial x}f(t)+y\frac{\partial}{\partial y}f(t)=$

$=xf'(t)\frac{\partial t}{\partial x}+yf'(t)\frac{\partial t}{\partial y}=$

$=f'(t)(x\cdot\frac{1}{y}+y\cdot(-\frac{x}{y^2}))=f'(t)(\frac{x}{y}-\frac{x}{y})=0.$

$u=\frac{x^2-y^2}{xy}=\frac{x}{y}-(\frac{x}{y})^{-1}=f(\frac{x}{y})=f(t) \Rightarrow f(t)=t-\frac{1}{t}$. Ja!

Problem 2.20 (Sid. 4)

Lösning

a) $u=x+y \Rightarrow \frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=1$; $v=xy \Rightarrow \frac{\partial v}{\partial x}=y \wedge \frac{\partial v}{\partial y}=x.$

$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}+y\frac{\partial z}{\partial v}.$

$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}+x\frac{\partial z}{\partial v}.$

b) $u=x^2-y^2 \Rightarrow \frac{\partial u}{\partial x}=2x \wedge \frac{\partial u}{\partial y}=-2y$; $v=2xy \Rightarrow \left\{\begin{array}{l}\frac{\partial v}{\partial x}=2y \\ \frac{\partial v}{\partial y}=2x\end{array}\right.$;

$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=2x\frac{\partial z}{\partial u}+2y\frac{\partial z}{\partial v}$;

$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=-2y\frac{\partial z}{\partial u}+2x\frac{\partial z}{\partial v}$;

c) $u=2xy \Rightarrow \frac{\partial u}{\partial x}=2y \wedge \frac{\partial u}{\partial y}=2x$; $v=\frac{1}{y} \Rightarrow \frac{\partial v}{\partial x}=0 \wedge \frac{\partial v}{\partial y}=-\frac{1}{y^2}.$

$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=2y\frac{\partial z}{\partial u}$;

$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=2x\frac{\partial z}{\partial u}-\frac{1}{y^2}\frac{\partial z}{\partial v}.$

Problem 2.21 (Sid. 5)

Lösning: $u=x-y$, $v=x+y$

a) $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}\cdot 1+\frac{\partial z}{\partial v}\cdot 1+$

$+\frac{\partial z}{\partial u}(-1)+\frac{\partial z}{\partial v}\cdot 1=2\frac{\partial z}{\partial v}=0 \Leftrightarrow \frac{\partial z}{\partial v}=0 \Leftrightarrow z=f(u)=f(x-y).$

b) $z(0,y)=y-\cos y \Rightarrow f(-y)=y-\cos y \Leftrightarrow f(y)=-y-\cos y$

$\Rightarrow z=-(x-y)-\cos(x-y) \Leftrightarrow \underline{z=y-x-\cos(x-y)}.$

Problem 2.22 (Sid. 5)

Lösning

a) $\begin{cases} u=2x-3y \\ v=x \end{cases} \Leftrightarrow \begin{cases} x= \quad v \\ y=\dfrac{2v-u}{3} \end{cases} \Rightarrow \begin{cases} \dfrac{\partial u}{\partial x}=2 \\ \dfrac{\partial x}{\partial u}=0 \end{cases} \Rightarrow \dfrac{\partial u}{\partial x}\cdot\dfrac{\partial x}{\partial u} \overset{!}{=} 0.$

Det är uppenbarligen inte sant; $\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v$, dvs olika saker hålls konstanta.

b) $\underline{f(x,y)=g(u,v);\ u=2x-3y,\ v=x.}$

$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}f(x,y)=\frac{\partial}{\partial x}g(u,v)=\frac{\partial g}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial g}{\partial v}\frac{\partial v}{\partial x}=2\frac{\partial g}{\partial u}+\frac{\partial g}{\partial v}$

Distinktionen $f(x,y)=g(u,v)$ sker inte av en del författare. Se f.ö. Ex. 15 på sidan 70.

Problem 2.23 (Sid. 5)

Lösning

a) $\underline{z=g(t),\ t=3x-2y.}$

$2\frac{\partial z}{\partial x}+3\frac{\partial z}{\partial y}=(2\frac{\partial}{\partial x}+3\frac{\partial}{\partial y})z=(2\frac{\partial}{\partial x}+3\frac{\partial}{\partial y})g(t)=2\frac{\partial}{\partial x}g(t)+$
$+3\frac{\partial}{\partial y}g(t)=2g'(t)\frac{\partial t}{\partial x}+3g'(t)\frac{\partial t}{\partial y}=6g'(t)-6g'(t)=0.$

b) $\underline{z=f(u,v),\ u=2x+3y,\ v=3x-2y.}$

$2\frac{\partial z}{\partial x}+3\frac{\partial z}{\partial y}=(2\frac{\partial}{\partial x}+3\frac{\partial}{\partial y})z=(2\frac{\partial}{\partial x}+3\frac{\partial}{\partial y})f(u,v)=$
$=2\frac{\partial}{\partial x}f(u,v)+3\frac{\partial}{\partial y}f(u,v)=2(\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x})+$
$+3(\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y})=2(2\frac{\partial f}{\partial u}+3\frac{\partial f}{\partial v})+3(3\frac{\partial f}{\partial u}-2\frac{\partial f}{\partial v})=$
$=13\frac{\partial f}{\partial u}=0 \Leftrightarrow \frac{\partial f}{\partial u}=0 \Leftrightarrow f(u,v)=g(v) \Rightarrow z=g(3x-2y).$

c) Man sätter $a=2k$ och $b=3k$, helt enkelt.

Motsvarande koordinattransformation ges av

$u=ax+by,\ v=bx-ay\quad (k=1).$

Lösningen blir alltså $z=h(bx-ay)$.

d) $z=f(\frac{x}{a}+\frac{y}{b}+\frac{z}{c})=\varsigma$; $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=k$, k konstant.

Problem 2.24 (Sid. 5)

Lösning

$\underline{h(x,y,z)=f(u,v),\ u=x/y,\ v=y/z.}$

$x\frac{\partial h}{\partial x}+y\frac{\partial h}{\partial y}+z\frac{\partial h}{\partial z}=(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z})h(x,y,z)=$
$=(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z})f(u,v)=x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=$
$=x(\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x})+y(\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y})+z(\frac{\partial f}{\partial u}\frac{\partial u}{\partial z}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial z})=$
$=x(\frac{1}{y}\frac{\partial f}{\partial u})+y(-\frac{x}{y^2}\frac{\partial f}{\partial u}+\frac{1}{z}\frac{\partial f}{\partial v})+z(-\frac{y}{z^2}\frac{\partial f}{\partial v})=\frac{x}{y}\frac{\partial f}{\partial u}-\frac{x}{y}\frac{\partial f}{\partial u}+$
$+\frac{y}{z}\frac{\partial f}{\partial v}-\frac{y}{z}\frac{\partial f}{\partial v}=0.$

Problem 2.25 (Sid. 5)

Lösning

$T(\rho,t)=f(u),\ u=\rho^2/t.$

$$\frac{\partial T}{\partial t}=\frac{\partial}{\partial t}T(\rho,t)=\frac{\partial}{\partial t}f(u)=f'(u)\frac{\partial u}{\partial t}=-\frac{\rho^2}{t^2}f'(u);$$

$$\frac{\partial T}{\partial \rho}=\frac{\partial}{\partial \rho}T(\rho,t)=\frac{\partial}{\partial \rho}f(u)=f'(u)\frac{\partial u}{\partial \rho}=2\frac{\rho}{t}f'(u);$$

$$\frac{\partial^2 T}{\partial \rho^2}=\frac{\partial}{\partial \rho}\left(\frac{\partial T}{\partial \rho}\right)=\frac{\partial}{\partial \rho}2\frac{\rho}{t}f'(u)=2\left(\frac{1}{t}f'(u)+2\frac{\rho^2}{t^2}f''(u)\right);$$

$$\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial \rho^2}-\frac{1}{\rho}\frac{\partial T}{\partial \rho}\Rightarrow -\frac{\rho^2}{t^2}f'(u)=\cancel{\frac{2}{t}f'(u)}+4\frac{\rho^2}{t^2}f''(u)-\cancel{\frac{2}{t}f'(u)}$$

$$\Leftrightarrow 4f''(u)+f'(u)=0\Leftrightarrow f''(u)+\frac{1}{4}f'(u)=0\Leftrightarrow f(u)=$$

$$=A+Be^{-u/4}\Leftrightarrow T(\rho,t)=A+Be^{-\rho^2/4t}.$$

Problem 2.26 (Sid. 5)

Lösning: $z=f(u,v),\ u=xy^2,\ v=y.$

(1) $$2x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=\left(2x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)z=\left(2x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right)f(u,v)=$$

$$=2x\frac{\partial f}{\partial x}-y\frac{\partial f}{\partial y}=2x\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}\right)-y\left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}\right)=$$

$$=2x\left(\frac{\partial f}{\partial x}y^2+\frac{\partial f}{\partial v}\cdot 0\right)-y\left(\frac{\partial f}{\partial u}\cdot 2xy+\frac{\partial f}{\partial v}\cdot 1\right)=-y\frac{\partial f}{\partial v}=y+xy$$

$$\Leftrightarrow \frac{\partial f}{\partial v}=-1-x=-1-\frac{u}{v^2}\Leftrightarrow f(u,v)=-v+\frac{u}{v}+\phi(u)\quad\Leftrightarrow$$

$$\Leftrightarrow z=-y+xy+\phi(xy^2),\ \phi\in C^1.$$

(2) $$z(1,y)=e^{-y}\Rightarrow -y+y+\phi(y^2)=\phi(y^2)=e^{-y}\Leftrightarrow \phi(t)=e^{-\sqrt{t}}.$$

Svar: $z=xy-y+e^{-y\sqrt{x}},\ x,y>0.$

Problem 2.27 (Sid. 5)

Lösning

$f(x,y)=\tilde f(u,v),\ u=x/y,\ v=\phi(x,y)$

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)f(x,y)=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\tilde f(u,v)=$$

$$=x\frac{\partial \tilde f}{\partial x}+y\frac{\partial \tilde f}{\partial y}=x\left(\frac{\partial \tilde f}{\partial u}\frac{\partial u}{\partial x}-\frac{\partial \tilde f}{\partial v}\cdot\frac{\partial \phi}{\partial x}\right)+y\left(\frac{\partial \tilde f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial \tilde f}{\partial v}\frac{\partial \phi}{\partial y}\right)=$$

$$=x\left(\frac{\partial \tilde f}{\partial u}\frac{1}{y}\right)+x\frac{\partial \tilde f}{\partial v}\frac{\partial \phi}{\partial x}+y\left(-\frac{x}{y^2}\frac{\partial \tilde f}{\partial u}\right)+y\left(\frac{\partial \tilde f}{\partial v}\cdot\frac{\partial \phi}{\partial y}\right)=$$

$$=\left(x\frac{\partial \phi}{\partial x}+y\frac{\partial \phi}{\partial y}\right)\frac{\partial \tilde f}{\partial v}=-\tilde f\ ;\quad \phi(x,y)=v=\ln x \text{ duger}$$

$$\frac{\partial \tilde f}{\partial v}+\tilde f=0\Leftrightarrow e^v\frac{\partial \tilde f}{\partial v}+e^v\tilde f=\frac{\partial}{\partial v}(\tilde f e^v)=0\Leftrightarrow e^v\tilde f(u,v)=$$

$$=g(u)\Leftrightarrow \tilde f(u,v)=g(u)e^{-v}\Leftrightarrow f(x,y)=\frac{1}{x}g\left(\frac{x}{y}\right).$$

Problem 2.28 (Sid. 5)

Lösning

(1) $z=f(x,y),\ x=2t,\ y=t;$

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}\Rightarrow g'(t)=f'_x(2t,t)\cdot 2+f'_y(2t,t)\cdot 1$$

$$\Rightarrow g'(0)=2f'_x(0,0)+f'_y(0,0)\overset{!}{=}\alpha\ ;$$

(2) $z=f(x,y),\ x=t,\ y=-t;$

$$\frac{dz}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}\Rightarrow h'(t)=f'_x(t,-t)\cdot 1+f'_y(t,-t)\cdot(-1);$$

$$\Rightarrow h'(0)=f'_x(0,0)-f'_y(0,0)\stackrel{!}{=}b\,;$$

$$(3)\ \begin{cases}2f'_x(0,0)+f'_y(0,0)=a\\ f'_x(0,0)-f'_y(0,0)=b\end{cases}\Leftrightarrow\begin{cases}f'_x(0,0)=\dfrac{a+b}{3}\\ f'_y(0,0)=\dfrac{a-2b}{3}\end{cases}.$$

Problem 2.29 (Sid. 6)

Lösning

$z=f(x,y)=\tilde{f}(u,v)\,,\quad u=x^2-y,\ v=x+y^2.$

$$(1)\ \left\{\begin{aligned}\frac{\partial z}{\partial x}&=\frac{\partial}{\partial x}\tilde{f}(u,v)=\frac{\partial\tilde f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial\tilde f}{\partial v}\frac{\partial v}{\partial x}=2x\frac{\partial\tilde f}{\partial u}+\frac{\partial\tilde f}{\partial v}\\ \frac{\partial z}{\partial y}&=\frac{\partial}{\partial y}\tilde{f}(u,v)=\frac{\partial\tilde f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial\tilde f}{\partial v}\frac{\partial v}{\partial y}=-\frac{\partial\tilde f}{\partial u}+2y\frac{\partial\tilde f}{\partial v}\end{aligned}\right\}\Rightarrow VL=$$

$$=(1-2y)\frac{\partial z}{\partial x}+(1+2x)\frac{\partial z}{\partial y}=2x\frac{\partial\tilde f}{\partial u}+\frac{\partial\tilde f}{\partial v}-4xy\frac{\partial\tilde f}{\partial u}-2y\frac{\partial\tilde f}{\partial v}-\frac{\partial\tilde f}{\partial u}+$$

$$+4xy\frac{\partial\tilde f}{\partial v}-2x\frac{\partial\tilde f}{\partial u}=\frac{\partial\tilde f}{\partial u}-\frac{\partial\tilde f}{\partial v}=0=HL\Leftrightarrow\frac{\partial\tilde f}{\partial u}-\frac{\partial\tilde f}{\partial v}=0.$$

$$(2)\ \frac{\partial\tilde f}{\partial u}-\frac{\partial\tilde f}{\partial v}=0\Leftrightarrow\tilde f(u,v)=\phi(u+v)\Leftrightarrow f(x,y)=\phi(x^2+y^2+x-y).$$

Övning 2.30 (Sid. 6)

Lösning: $y(x)=\eta(u)$. (η utläses éta).

$$\frac{dy}{dx}=\frac{d}{dx}y(x)=\frac{d}{dx}\eta(u)=\frac{d\eta}{du}\frac{du}{dx}=\frac{1}{x}\frac{d\eta}{du};$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{1}{x}\frac{d\eta}{du}\right)=-\frac{1}{x^2}\frac{d\eta}{du}+\frac{1}{x}\frac{d}{dx}\left(\frac{d\eta}{du}\right)=$$

$$=-\frac{1}{x^2}\frac{d\eta}{du}+\frac{1}{x}\frac{d^2\eta}{du^2}\frac{du}{dx}=\frac{1}{x^2}\left(\frac{d^2\eta}{du^2}-\frac{d\eta}{du}\right);$$

$$x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}-3y=\frac{d^2\eta}{du^2}-\frac{d\eta}{du}+3\frac{d\eta}{du}-3\eta=5e^{2t}\Leftrightarrow$$

$$\Leftrightarrow\frac{d^2\eta}{du^2}+2\frac{d\eta}{du}-3\eta=5e^{2u}\quad\text{(konst. koefficienter).}$$

$$(1)\ \frac{d^2\eta}{du^2}+2\frac{d\eta}{du}-3\eta=0\Leftrightarrow r^2+2r-3=0\quad\text{(kar. ekv.)}$$

$$\Leftrightarrow(r-1)(r+3)=0\Leftrightarrow r=1\ \vee\ r=-3.$$

$\eta(u)=Ae^{u}+Be^{-3u}$ (homogenlösningen)

$$(2)\ \eta(u)=Ce^{2u}\Rightarrow\frac{d\eta}{du}=2\eta\Rightarrow\frac{d^2\eta}{du^2}=4\eta;$$

$$\frac{d^2\eta}{du^2}+2\frac{d\eta}{du}-3\eta=4\eta+4\eta-3\eta=5\eta=5e^{2u}\Leftrightarrow\eta=e^{2u}$$

är partikulärlösning.

$$(3)\ \eta=Ae^{u}+Be^{-3u}+e^{2u}\Leftrightarrow y=Ax+Bx^{-3}+x^2.$$

Övning 2.31 (Sid. 6)

Lösning

$$\frac{dz}{dx}=\frac{d}{dx}f(u,v)=\frac{\partial f}{\partial u}\frac{du}{\partial x}+\frac{\partial f}{\partial v}\frac{dv}{dx}=\frac{1}{x}\frac{\partial f}{\partial u}+2x\frac{\partial f}{\partial v}$$

$$\frac{d^2z}{dx^2}=\frac{d}{dx}\left(\frac{dz}{dx}\right)=\frac{d}{dx}\left(\frac{1}{x}\frac{\partial f}{\partial u}\right)+\frac{d}{dx}\left(2x\frac{\partial f}{\partial v}\right)=$$

$$=-\frac{1}{x^2}\frac{\partial f}{\partial u}+\frac{1}{x}\frac{d}{dx}\left(\frac{\partial f}{\partial u}\right)+2\frac{\partial f}{\partial v}+2x\frac{d}{dx}\left(\frac{\partial f}{\partial v}\right)=$$

$$=-\frac{1}{x^2}\frac{\partial f}{\partial u}-\frac{1}{x}\left(\frac{\partial^2 f}{\partial u^2}\frac{du}{dx}+\frac{\partial^2 f}{\partial u\partial v}\frac{dv}{dx}\right)+$$

$$+2\frac{\partial f}{\partial v}+2x\left(\frac{\partial^2 f}{\partial u\partial v}\frac{du}{dx}+\frac{\partial^2 f}{\partial v^2}\frac{dv}{dx}\right)=$$

$$=-\frac{1}{x^2}\frac{\partial f}{\partial u}-\frac{1}{x}\left(\frac{1}{x}\frac{\partial^2 f}{\partial u^2}+2x\frac{\partial^2 f}{\partial u\partial v}\right)+$$

$$+2\frac{\partial f}{\partial v}+2x\left(\frac{1}{x}\frac{\partial^2 f}{\partial u\partial v}+2x\frac{\partial^2 f}{\partial v^2}\right)=$$

$$=-\frac{1}{x^2}\frac{\partial f}{\partial u}+2\frac{\partial f}{\partial v}+\frac{1}{x^2}\frac{\partial^2 f}{\partial u^2}+4\frac{\partial^2 f}{\partial u\partial v}+4x^2\frac{\partial^2 f}{\partial v^2}.$$

Problem 2.32 (Sid. 6)

Lösning: $z = f(u,v)$, $u = 2x+y$, $v = x$.

a) $\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial x \partial y} + 4\dfrac{\partial^2 z}{\partial y^2} = \left(\dfrac{\partial^2}{\partial x^2} - 4\dfrac{\partial^2}{\partial x \partial y} + 4\dfrac{\partial^2}{\partial y^2}\right) z =$

$= \left(\dfrac{\partial}{\partial x} - 2\dfrac{\partial}{\partial y}\right)\left(\dfrac{\partial}{\partial x} - 2\dfrac{\partial}{\partial y}\right) z = \left(\dfrac{\partial}{\partial x} - 2\dfrac{\partial}{\partial y}\right)\left(\dfrac{\partial}{\partial x} - 2\dfrac{\partial}{\partial y}\right) f(u,v).$

$\dfrac{\partial z}{\partial x} - 2\dfrac{\partial z}{\partial y} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial x} - 2\left(\dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial y}\right) =$

$= 2\dfrac{\partial f}{\partial u} + \dfrac{\partial f}{\partial v} - 2\left(\dfrac{\partial f}{\partial u}\right) = \dfrac{\partial f}{\partial v}.$

$\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial x \partial y} + 4\dfrac{\partial^2 z}{\partial y^2} = \left(\dfrac{\partial}{\partial v}\right)^2 f(u,v) = \dfrac{\partial^2 f}{\partial v^2} = 6y = 6u - 12v.$

$\Leftrightarrow \dfrac{\partial}{\partial v}\left(\dfrac{\partial f}{\partial v}\right) = 6u - 12v \Leftrightarrow \dfrac{\partial f}{\partial v} = 6uv - 6v^2 + \phi(u) \Leftrightarrow$

$\Leftrightarrow f(u,v) = 3uv^2 - 2v^3 + \phi(u)v + \psi(u)$, $\phi, \psi \in C^1$;

$\Leftrightarrow \underline{z = 3(2x+y)x^2 - 2x^3 + x\phi(2x+y) + \psi(2x+y).}$

b) $z(0,y) = e^{-y^2} \Rightarrow \psi(y) = e^{-y^2} \Rightarrow \psi(2x+y) = e^{-(2x+y)^2}.$

$z = 6x^3 + 3x^2y - 2x^3 + x\phi(2x+y) + e^{-(2x+y)^2} =$

$= 4x^3 + 3x^2y + x\phi(2x+y) + e^{-(2x+y)^2}.$

c) $\dfrac{\partial z}{\partial x} = 12x^2 + 6xy + \phi(2x+y) + 2x\phi'(2x+y) - 4(2x+y)e^{-(2x+y)^2}$

$z'_x(0,y) = 0 \Rightarrow \phi(y) - 4ye^{-y^2} = 0 \Leftrightarrow \phi(y) = 4ye^{-y^2}$

$z = 4x^3 + 3x^2y + 4x(2x+y)e^{-(2x+y)^2} + e^{-(2x+y)^2}.$

Anm. $z(x,y) \neq z(u,v)$ i allmänhet; använd ζ i stället: $z(x,y) = \zeta(u,v)$ är ett bra val.

Problem 2.33 (Sid. 6)

Lösning

$f(x,y) = \tilde{f}(u,v)$, $u = x+y$, $v = xy$

$\dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\tilde{f}(u,v) = \dfrac{\partial \tilde{f}}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial \tilde{f}}{\partial v}\dfrac{\partial v}{\partial x} = \dfrac{\partial \tilde{f}}{\partial u} + y\dfrac{\partial \tilde{f}}{\partial v} \Rightarrow \dfrac{\partial^2 f}{\partial x \partial y} =$

$= \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial \tilde{f}}{\partial u} + y\dfrac{\partial \tilde{f}}{\partial v}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial \tilde{f}}{\partial u}\right) + \dfrac{\partial \tilde{f}}{\partial v} + y\dfrac{\partial}{\partial y}\left(\dfrac{\partial \tilde{f}}{\partial v}\right) =$

$= \dfrac{\partial^2 \tilde{f}}{\partial u^2}\dfrac{\partial u}{\partial y} + \dfrac{\partial^2 \tilde{f}}{\partial u \partial v}\dfrac{\partial v}{\partial y} + \dfrac{\partial \tilde{f}}{\partial v} + y\left(\dfrac{\partial^2 \tilde{f}}{\partial u \partial v}\dfrac{\partial u}{\partial y} + \dfrac{\partial^2 \tilde{f}}{\partial v^2}\dfrac{\partial v}{\partial y}\right) =$

$= \dfrac{\partial^2 \tilde{f}}{\partial u^2} + x\dfrac{\partial^2 \tilde{f}}{\partial u \partial v} + \dfrac{\partial \tilde{f}}{\partial v} + y\left(\dfrac{\partial^2 \tilde{f}}{\partial u \partial v} + x\dfrac{\partial^2 \tilde{f}}{\partial v^2}\right) =$

$= \dfrac{\partial^2 \tilde{f}}{\partial u^2} + (x+y)\dfrac{\partial^2 \tilde{f}}{\partial u \partial v} + xy\dfrac{\partial^2 \tilde{f}}{\partial v^2} + \dfrac{\partial \tilde{f}}{\partial v} = \underline{\dfrac{\partial^2 \tilde{f}}{\partial u^2} + u\dfrac{\partial^2 \tilde{f}}{\partial u \partial v} + v\dfrac{\partial^2 \tilde{f}}{\partial v^2} + \dfrac{\partial \tilde{f}}{\partial v}}.$

Problem 2.34 (Sid. 6)

Lösning

$\underline{z = f(u,v),\ u = 2xy,\ v = 1/y}$

(1) $\dfrac{\partial z}{\partial x} = \dfrac{\partial}{\partial x}f(u,v) = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial x} = 2y\dfrac{\partial f}{\partial u};$

(2) $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right) = 2y\dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial u}\right) = 4y^2\dfrac{\partial^2 f}{\partial u^2};$

(3) $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial}{\partial y}\left(2y\dfrac{\partial f}{\partial u}\right) = 2\dfrac{\partial f}{\partial u} + 2y\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial u}\right) =$

$= 2\dfrac{\partial f}{\partial u} + 2y\left(\dfrac{\partial^2 f}{\partial u^2}\dfrac{\partial u}{\partial y} + \dfrac{\partial^2 f}{\partial u \partial v}\dfrac{\partial v}{\partial y}\right) =$

$= 2\dfrac{\partial f}{\partial u} + 2y\left(2x\dfrac{\partial^2 f}{\partial u^2} - \dfrac{1}{y^2}\dfrac{\partial^2 f}{\partial u \partial v}\right) =$

$= 2\dfrac{\partial f}{\partial u} + 4xy\dfrac{\partial^2 f}{\partial u^2} - \dfrac{2}{y}\dfrac{\partial^2 f}{\partial u \partial v}$

(4) $x\frac{\partial^2 z}{\partial x^2}-y\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial z}{\partial x}=4xy^2\frac{\partial^2 f}{\partial u^2}-2y\frac{\partial f}{\partial u}-4xy^2\frac{\partial^2 f}{\partial u^2}+2\frac{\partial^2 f}{\partial u\partial v}+$
$+2y\frac{\partial f}{\partial u}=2\frac{\partial^2 f}{\partial u\partial v}.$

(5) $x\frac{\partial^2 z}{\partial x^2}-y\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial z}{\partial x}=x \Leftrightarrow 2\frac{\partial^2 f}{\partial u\partial v}=\frac{uv}{2}\Leftrightarrow\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial v}\right)=\frac{uv}{4}$

$\Leftrightarrow \frac{\partial f}{\partial v}=\frac{1}{8}u^2v+\phi(v)\Leftrightarrow f(u,v)=\frac{1}{16}u^2v^2+g(v)+h(u)$

$\Leftrightarrow \underline{z=\frac{1}{4}x^2+g\left(\frac{1}{y}\right)+h(2xy)}$

Problem 2.35 (Sid. 6)

Lösning

(1) $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial x\partial y}-6\frac{\partial^2 f}{\partial y^2}=\left(\frac{\partial}{\partial x}+3\frac{\partial}{\partial y}\right)\left(\frac{\partial f}{\partial x}-2\frac{\partial f}{\partial y}\right)=1$;

(2) $\begin{cases}u=x+ay\\ v=x+by\end{cases}\Rightarrow\left\{\begin{array}{l}\frac{\partial f}{\partial x}=\frac{\partial \tilde f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial \tilde f}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial \tilde f}{\partial u}+\frac{\partial \tilde f}{\partial v}\\ \frac{\partial f}{\partial y}=\frac{\partial \tilde f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial \tilde f}{\partial v}\frac{\partial v}{\partial y}=a\frac{\partial \tilde f}{\partial u}+b\frac{\partial \tilde f}{\partial v}\end{array}\right\}\Rightarrow$

$\Rightarrow VL=\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}+3a\frac{\partial}{\partial u}+3b\frac{\partial}{\partial v}\right)\left(\frac{\partial \tilde f}{\partial u}+\frac{\partial \tilde f}{\partial v}-2a\frac{\partial \tilde f}{\partial u}-2b\frac{\partial \tilde f}{\partial v}\right)=$

$=\left((1+3a)\frac{\partial}{\partial u}+(1+3b)\frac{\partial}{\partial v}\right)\left((1-2a)\frac{\partial \tilde f}{\partial u}+(1-2b)\frac{\partial \tilde f}{\partial v}\right)=1=HL$

(3) $\underline{a=\frac{1}{2}\wedge b=-\frac{1}{3}}.$

$\frac{5}{2}\frac{\partial}{\partial u}\left(\frac{5}{3}\frac{\partial \tilde f}{\partial v}\right)=1\Leftrightarrow\frac{\partial}{\partial u}\left(\frac{\partial \tilde f}{\partial v}\right)=\frac{6}{25}\Leftrightarrow\frac{\partial \tilde f}{\partial v}=\frac{6}{25}u+\phi(v)\Leftrightarrow$

$\Leftrightarrow \tilde f(u,v)=\frac{6}{25}uv+\Phi(v)+\Psi(u),\quad \Phi,\Psi\in C^2,$

$\Leftrightarrow \underline{f(x,y)=\frac{6}{25}\left(x+\frac{1}{2}y\right)\left(x-\frac{1}{3}y\right)+\Phi\left(x+\frac{1}{2}y\right)+\Psi\left(x-\frac{1}{3}y\right)}.$

Problem 2.36 (Sid. 6)

Lösning

$\underline{u(x)=f(\rho),\ \rho=\sqrt{x^2+y^2}}$

a) $\rho^2=x^2+y^2\Rightarrow\frac{\partial\rho^2}{\partial x}=\frac{\partial}{\partial x}(x^2+y^2)\Rightarrow 2\rho\frac{\partial\rho}{\partial x}=2x\Rightarrow\frac{\partial\rho}{\partial x}=\frac{x}{\rho}.$

På samma sätt fås $\frac{\partial\rho}{\partial y}=\frac{y}{\rho}.$

$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}f(\rho)=f'(\rho)\frac{\partial\rho}{\partial x}=f'(\rho)\frac{x}{\rho}=\frac{du}{d\rho}\cdot\frac{x}{\rho}=\frac{x}{\rho}\frac{du}{d\rho};$

$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}f(\rho)=f'(\rho)\frac{\partial\rho}{\partial y}=f'(\rho)\frac{y}{\rho}=\frac{du}{d\rho}\frac{y}{\rho}=\frac{y}{\rho}\frac{du}{d\rho};$

b) $\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{x}{\rho}f'(\rho)\right)=\frac{1}{\rho}f'(\rho)+x\left(\frac{-1}{\rho^2}\right)\frac{x}{\rho}f'(\rho)+$

$+\frac{x}{\rho}f''(\rho)\frac{x}{\rho}=\frac{1}{\rho}f'(\rho)-\frac{x^2}{\rho^3}f'(\rho)+\frac{x^2}{\rho^2}f''(\rho)=$

$=\frac{1}{\rho}\frac{du}{d\rho}-\frac{x^2}{\rho^3}\frac{du}{d\rho}+\frac{x^2}{\rho^2}\frac{d^2u}{d\rho^2}.$

$\frac{\partial^2 u}{\partial y^2}=\frac{1}{\rho}\frac{du}{d\rho}-\frac{x^2}{\rho^3}\frac{du}{d\rho}+\frac{x^2}{\rho^2}\frac{d^2u}{d\rho^2}$ fås på samma sätt.

c) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{2}{\rho}\frac{du}{d\rho}-\frac{x^2+y^2}{\rho^3}\frac{du}{d\rho}+\frac{x^2+y^2}{\rho^2}\frac{d^2u}{d\rho^2}=$

$=\frac{2}{\rho}\frac{du}{d\rho}-\frac{\rho^2}{\rho^3}\frac{du}{d\rho}+\frac{\rho^2}{\rho^2}\frac{d^2u}{d\rho^2}=$

$=\frac{2}{\rho}\frac{du}{d\rho}-\frac{1}{\rho}\frac{du}{d\rho}+\frac{d^2u}{d\rho^2}=$

$=\underline{\frac{1}{\rho}\frac{du}{d\rho}-\frac{d^2u}{d\rho^2}=0}.$

Problemet ovan uppvisar axialsymmetri.

Problem 2.37 (Sid. 7)

Lösning

$f(x,y)=g(u,v)$; $u=(\cos\varphi)x+(\sin\varphi)y$, $v=-(\sin\varphi)x+(\cos\varphi)y$

(1) $$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}g(u,v)=\frac{\partial g}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial g}{\partial v}\frac{\partial v}{\partial x}=\cos\varphi\frac{\partial g}{\partial u}-\sin\varphi\frac{\partial g}{\partial v}\Rightarrow$$

$$\Rightarrow\frac{\partial}{\partial x}=\cos\varphi\frac{\partial}{\partial u}-\sin\varphi\frac{\partial}{\partial v};$$

(2) $$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\left(\cos\varphi\frac{\partial}{\partial u}-\sin\varphi\frac{\partial}{\partial v}\right)\left(\cos\varphi\frac{\partial g}{\partial u}-\sin\varphi\frac{\partial g}{\partial v}\right)=$$

$$=\cos\varphi\frac{\partial}{\partial u}\left(\cos\varphi\frac{\partial g}{\partial u}-\sin\varphi\frac{\partial g}{\partial v}\right)-$$

$$-\sin\varphi\frac{\partial}{\partial v}\left(\cos\varphi\frac{\partial g}{\partial u}-\sin\varphi\frac{\partial g}{\partial v}\right)=$$

$$=\cos^2\varphi\frac{\partial^2 g}{\partial u^2}-\sin\varphi\cos\varphi\frac{\partial^2 g}{\partial u\partial v}-$$

$$-\sin\varphi\cos\varphi\frac{\partial^2 g}{\partial v\partial u}+\sin^2\varphi\frac{\partial^2 g}{\partial v^2}=$$

$$=\cos^2\varphi\frac{\partial^2 g}{\partial u^2}-2\sin\varphi\cos\varphi\frac{\partial^2 g}{\partial u\partial v}+\sin^2\varphi\frac{\partial^2 g}{\partial v^2}.$$

På samma sätt visas att

(3) $$\frac{\partial^2 f}{\partial y^2}=\sin^2\varphi\frac{\partial^2 g}{\partial u^2}+2\sin\varphi\cos\varphi\frac{\partial^2 g}{\partial u\partial v}+\cos^2\varphi\frac{\partial^2 g}{\partial v^2}.$$

(4) Ledvis addition av (2) och (3) ger

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=(\cos^2\varphi+\sin^2\varphi)\left(\frac{\partial^2 g}{\partial u^2}+\frac{\partial^2 g}{\partial v^2}\right)=\frac{\partial^2 g}{\partial u^2}+\frac{\partial^2 g}{\partial v^2}=0.$$

Problem 2.38 (Sid. 7)

Lösning: $x=\rho\cos\varphi$, $y=\rho\sin\varphi$.

a) $$\begin{cases} dx=\cos\varphi\,d\rho-\rho\sin\varphi\,d\varphi=\cos\varphi\,d\rho-\sin\varphi(\rho\,d\varphi)\\ dy=\sin\varphi\,d\rho+\rho\cos\varphi\,d\varphi=\sin\varphi\,d\rho+\cos\varphi(\rho\,d\varphi)\end{cases}\Rightarrow$$

$$\Leftrightarrow\begin{bmatrix}dx\\dy\end{bmatrix}=\begin{bmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{bmatrix}\begin{bmatrix}d\rho\\ \rho d\varphi\end{bmatrix}\Leftrightarrow\begin{bmatrix}d\rho\\ \rho d\varphi\end{bmatrix}=\begin{bmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{bmatrix}\begin{bmatrix}dx\\dy\end{bmatrix}$$

$$\Leftrightarrow\begin{cases}d\rho=\cos\varphi\,dx+\sin\varphi\,dy\\ \rho\,d\varphi=-\sin\varphi\,dx+\cos\varphi\,dy\end{cases}\Leftrightarrow\begin{cases}d\rho=\cos\varphi\,dx+\sin\varphi\,dy\\ d\varphi=-\frac{\sin\varphi}{\rho}dx+\frac{\cos\varphi}{\rho}dy\end{cases};$$

$$d\rho=\frac{\partial\rho}{\partial x}dx+\frac{\partial\rho}{\partial y}dy\Rightarrow\frac{\partial\rho}{\partial x}=\cos\varphi\text{ och }\frac{\partial\rho}{\partial y}=\sin\varphi.$$

$$d\varphi=\frac{\partial\varphi}{\partial x}dx+\frac{\partial\varphi}{\partial y}dy\Rightarrow\frac{\partial\varphi}{\partial x}=-\frac{\sin\varphi}{\rho}\text{ och }\frac{\partial\varphi}{\partial y}=\frac{\cos\varphi}{\rho}.$$

b) $u(x,y)=f(\rho,\varphi)$.

$$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}f(\rho,\varphi)=\frac{\partial f}{\partial\rho}\frac{\partial\rho}{\partial x}+\frac{\partial f}{\partial\varphi}\frac{\partial\varphi}{\partial x}=\cos\varphi\frac{\partial f}{\partial\rho}-\frac{\sin\varphi}{\rho}\frac{\partial f}{\partial\varphi}\Leftrightarrow$$

$$\Leftrightarrow\frac{\partial}{\partial x}=\cos\varphi\frac{\partial}{\partial\rho}-\frac{\sin\varphi}{\rho}\frac{\partial}{\partial\varphi};$$

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}f(\rho,\varphi)=\frac{\partial f}{\partial\rho}\frac{\partial\rho}{\partial y}+\frac{\partial f}{\partial\varphi}\frac{\partial\varphi}{\partial y}=\sin\varphi\frac{\partial f}{\partial\rho}+\frac{\cos\varphi}{\rho}\frac{\partial f}{\partial\varphi}\Leftrightarrow$$

$$\Leftrightarrow\frac{\partial}{\partial y}=\sin\varphi\frac{\partial}{\partial\rho}+\frac{\cos\varphi}{\rho}\frac{\partial}{\partial\varphi};$$

c) $$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\left(\cos\varphi\frac{\partial}{\partial\rho}-\frac{\sin\varphi}{\rho}\frac{\partial}{\partial\varphi}\right)\left(\cos\varphi\frac{\partial f}{\partial\rho}-\frac{\sin\varphi}{\rho}\frac{\partial f}{\partial\varphi}\right)=$$

$$=\cos\varphi\frac{\partial}{\partial\rho}\left(\cos\varphi\frac{\partial f}{\partial\rho}-\frac{\sin\varphi}{\rho}\frac{\partial f}{\partial\varphi}\right)-$$

$$-\frac{\sin\varphi}{\rho}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial f}{\partial\rho}-\frac{\sin\varphi}{\rho}\frac{\partial f}{\partial\varphi}\right)=$$

$$=\cos^2\varphi\frac{\partial^2 f}{\partial\rho^2}+\sin\varphi\cos\varphi\frac{1}{\rho^2}\frac{\partial f}{\partial\varphi}-\sin\varphi\cos\varphi\frac{1}{\rho}\frac{\partial^2 f}{\partial\rho\partial\varphi}+$$

$$+\sin^2\varphi\frac{1}{\rho^2}\frac{\partial f}{\partial\rho}-\sin\varphi\cos\varphi\frac{1}{\rho}\frac{\partial^2 f}{\partial\rho\partial\varphi}+\sin\varphi\cos\varphi\frac{1}{\rho^2}\frac{\partial f}{\partial\varphi}+$$
$$+\sin^2\varphi\frac{1}{\rho^2}\frac{\partial^2 f}{\partial\varphi^2}=\cos^2\varphi\frac{\partial^2 f}{\partial\rho^2}-\frac{\sin^2\varphi}{\rho}\frac{\partial f}{\partial\rho}-\frac{\sin^2\varphi}{\rho^2}\frac{\partial^2 f}{\partial\varphi^2}+$$
$$+\frac{\sin^2\varphi}{\rho^2}\frac{\partial f}{\partial\varphi}-\frac{\sin 2\varphi}{\rho}\frac{\partial^2 f}{\partial\rho\partial\varphi}.$$

$$\frac{\partial^2 u}{\partial y^2}=\sin^2\varphi\frac{\partial^2 f}{\partial\rho^2}+\frac{\cos^2\varphi}{\rho}\frac{\partial f}{\partial\varphi}+\frac{\cos^2\varphi}{\rho^2}\frac{\partial^2 f}{\partial\varphi^2}+\frac{\cos^2\varphi}{\rho^2}\frac{\partial f}{\partial\varphi}+$$
$$+\frac{\sin 2\varphi}{\rho}\frac{\partial^2 f}{\partial\rho\partial\varphi},$$ visas på samma sätt.

Ledvis addition ger

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=(\sin^2\varphi+\cos^2\varphi)\left(\frac{\partial^2 f}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial f}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 f}{\partial\varphi^2}\right)=$$
$$=\frac{\partial^2 f}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial f}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2 f}{\partial\varphi^2}=0.$$

Problem 2.39 (Sid. 7)

Lösning

$\underline{z=f(u,v),\ \ u=x,\ v=x/y}$

$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}f(u,v)=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial f}{\partial u}\cdot 0+\frac{-x}{y^2}\frac{\partial f}{\partial v}=-\frac{x}{y^2}\frac{\partial f}{\partial v};$$
$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(-\frac{x}{y^2}\frac{\partial f}{\partial v}\right)=\frac{2x}{y^3}\frac{\partial f}{\partial v}-\frac{x}{y^2}\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial v}\right)=$$
$$=\frac{2x}{y^3}\frac{\partial f}{\partial v}-\frac{x}{y^2}\cdot\left(-\frac{x}{y^2}\right)\frac{\partial^2 f}{\partial v^2}=\frac{2x}{y^3}\frac{\partial f}{\partial v}+\frac{x^2}{y^4}\frac{\partial^2 f}{\partial v^2};$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(-\frac{x}{y^2}\frac{\partial f}{\partial v}\right)=-\frac{1}{y^2}\frac{\partial f}{\partial v}-\frac{x}{y^2}\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial v}\right)=$$
$$=-\frac{1}{y^2}\frac{\partial f}{\partial v}-\frac{x}{y^2}\left(\frac{\partial^2 f}{\partial u\partial v}\frac{\partial u}{\partial x}+\frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial x}\right)=$$
$$=-\frac{1}{y^2}\frac{\partial f}{\partial v}-\frac{x}{y^2}\left(\frac{\partial^2 f}{\partial u\partial v}+\frac{1}{y}\frac{\partial^2 f}{\partial v^2}\right)=-\frac{1}{y^2}\frac{\partial f}{\partial v}-\frac{x}{y^2}\frac{\partial^2 f}{\partial u\partial v}-\frac{x}{y^3}\frac{\partial^2 f}{\partial v^2};$$

$$x\frac{\partial^2 z}{\partial x\partial y}+y\frac{\partial^2 z}{\partial y^2}=-\frac{x}{y^2}\frac{\partial f}{\partial v}-\frac{x^2}{y^2}\frac{\partial^2 f}{\partial u\partial v}-\frac{x^2}{y^3}\frac{\partial^2 f}{\partial v^2}+\frac{2x}{y^2}\frac{\partial f}{\partial v}+$$
$$+\frac{x^2}{y^3}\frac{\partial^2 f}{\partial v^2}=\frac{x}{y^2}\frac{\partial f}{\partial v}-\frac{x^2}{y^2}\frac{\partial^2 f}{\partial v\partial u}=0\Leftrightarrow\frac{\partial f}{\partial v}-x\frac{\partial^2 f}{\partial v\partial u}=0\Leftrightarrow$$
$$\Leftrightarrow\frac{\partial f}{\partial v}-u\frac{\partial^2 f}{\partial v\partial u}=0\Leftrightarrow\frac{\partial}{\partial v}\left(f-u\frac{\partial f}{\partial u}\right)=0\Leftrightarrow f-u\frac{\partial f}{\partial u}=$$
$$=g(u)\Leftrightarrow u\frac{\partial f}{\partial u}-f=-g(u)\Leftrightarrow\frac{1}{u}\frac{\partial f}{\partial u}-\frac{1}{u^2}f=-\frac{1}{u^2}g(u)\Leftrightarrow$$
$$\Leftrightarrow\frac{\partial}{\partial u}\left(\frac{1}{u}f\right)=-\frac{1}{u^2}g(u)\Leftrightarrow\frac{1}{u}f=\mathcal{F}(u)+G(v)\Leftrightarrow f(u,v)=$$
$$=u\mathcal{F}(u)+uG(v)\Leftrightarrow z=x\mathcal{F}(x)+xG\left(\frac{x}{y}\right)=\underline{H(x)+xG\left(\frac{x}{y}\right)}.$$

Problem 2.40 (Sid. 7)

Lösning

(1) $u=xy^2\Rightarrow du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=y^2dx+2xydy$

(2) $\left(\frac{\partial u}{\partial x}\right)_y=y^2$, ty $y=$ konstant $\Rightarrow dy=0$.

(3) $z=2x+y=$ konstant $\Rightarrow 2dx+dy=0\Leftrightarrow dy=-2dx\Rightarrow$

$\Rightarrow du=y^2dx+2xy\cdot(-2)dx=(y^2-4xy)dx\Rightarrow$

$\Rightarrow\left(\frac{\partial u}{\partial x}\right)_z=y^2-4xy.$

Problem 2.41 (Sid. 7)

Lösning: Uppgiften är rent fysikalisk.

Jag väljer koordinatsystemen (T,V) och (ξ,p), där $\xi=T$. Då är $\left(\frac{\partial E}{\partial T}\right)_V$ och $\left(\frac{\partial E}{\partial T}\right)_p$ samma sak som $\frac{\partial E}{\partial T}$ och $\frac{\partial E}{\partial \xi}$ (korta beteckningar) och enligt kedjeregeln är

$$\left(\frac{\partial E}{\partial T}\right)_p = \frac{\partial E}{\partial \xi} = \frac{\partial E}{\partial T}\frac{\partial T}{\partial \xi} + \frac{\partial E}{\partial V}\frac{\partial V}{\partial \xi} = \left(\frac{\partial E}{\partial T}\right)_p + \left(\frac{\partial E}{\partial V}\right)_T \cdot \left(\frac{\partial V}{\partial T}\right)_p.$$

Ovanstående resonemang kan vara rena rama grekiska på denna nivå... Tålamod!

Gradient. Riktningsderivata

Problem 2.42 (Sid. 7)

Lösning

a) $f(\mathbf{x}) = x+2y+3z \Rightarrow \operatorname{grad} f(\mathbf{x}) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (1,2,3).$

b) $f(\mathbf{x}) = xy^2e^{-xy}.$

$\frac{\partial f}{\partial x} = y^2e^{-xy} + xy^2\cdot(-y)e^{-xy} = y^2e^{-xy} - xy^3e^{-xy};$

$\frac{\partial f}{\partial y} = 2xye^{-xy} + xy^2\cdot(-x)e^{-xy} = 2xye^{-xy} - x^2y^2e^{-xy};$

$\operatorname{grad} f(\mathbf{x}) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (y^2e^{-xy} - xy^3e^{-xy}, 2xye^{-xy} - x^2y^2e^{-xy}).$

$= e^{-xy}(y^2-xy^3, 2xy-x^2y^2).$

Förenkla så långt som möjligt.

Problem 2.43 (Sid. 7)

Lösning

$C: x^3+xy+y^3=5$ är nivåkurvan till

$$f(x,y) = x^3+xy+y^3$$

genom punkten $P_0:(2,-1)$. En normalvektor till C genom P_0 är $\operatorname{grad} f(P_0) = (3x_0^2+y_0, x_0+3y_0^2) = (11,5)$, så en tangentvektor i samma plkt är $u=(5,-11)$. (bestäms med blotta ögat).

Normalens ekvation är $(5,-11)\cdot(x-2,y+1)=0$, dvs $5x-11y=21$. Tangentens ekvation blir $(11,5)\cdot(x-2,y+1)=0 \Leftrightarrow 11x+5y=17.$

Problem 2.44 (Sid. 7)

Lösning

$C: x^2+xy+y^2=1.$ $P_0:(\xi,\eta)\in C.$

Låt oss bestämma ekvationen för tangenten i P_0. En normalvektor i P_0 ges av $\operatorname{grad} f(P_0)$, där $f(x,y)=x^2+xy+y^2$; $C: f(P)=f(P_0)$ är nivåkurvan till f genom P_0.

$\operatorname{grad} f(\mathbf{x}) = (2x+y, x+2y) \Rightarrow \operatorname{grad} f(P_0) = (2\xi+\eta, \xi+2\eta)$.

Tangentens ekvation blir

$$(2\xi+\eta, \xi+2\eta)\cdot(x-\xi, y-\eta) = 0 \Leftrightarrow$$

$$\Leftrightarrow (2\xi+\eta)x + (\xi+2\eta)y = \xi(2\xi+\eta) + \eta(\xi+2\eta) = 2 \quad (1)$$

ty

$$\xi^2 + \xi\eta + \eta^2 = 1 \quad (2)$$

a) $P:(0,2)$ ligger på tangenten. Insättning av $x=0, y=2$ i ekvationen (1) $\Rightarrow \xi + 2\eta = 1$. Detta kombineras med (2) och vi får:

$$(1-2\eta)^2 + \eta(1-2\eta) + \eta^2 = 1 \Leftrightarrow 4\eta^2 - 4\eta + 1 + \eta - 2\eta^2 + \eta^2 = 1$$

$$\Leftrightarrow 3\eta^2 - 3\eta = 3\eta(\eta-1) = 0 \Leftrightarrow \eta = 0 \vee \eta = 1 \Leftrightarrow \xi = 1 \vee \xi = -1$$

$$\Rightarrow P_1:(1,0) \text{ och } P_2:(-1,1) \Rightarrow 2x+y=2 \text{ och } -x+y=2.$$

b) $P:(0,0)$ Insättning av $x=y=0$ i tangentens ekvation (1) $\Rightarrow 0=2$, dvs inga tangenter går genom origo.

c) $P:(-1,0)$ $-(2\xi+\eta) = 2 \Leftrightarrow \eta = -2-2\xi$ (sätts in i (2)).

$$\xi^2 + \xi(-2-2\xi) + 4(1+\xi)^2 = 1 \Leftrightarrow \xi^2 - 2\xi - 2\xi^2 + 4 + 8\xi + 4\xi^2 = 1$$

$$\Leftrightarrow 3\xi^2 + 6\xi + 3 = 0 \Leftrightarrow 3(\xi+1)^2 = 0 \Leftrightarrow \xi = -1 \Rightarrow \eta = 0 \Rightarrow$$

Genom $(-1,0)$ går endast en tangent: $2x+y=-2$.

Problem 2.45 (Sid. 7)

Lösning

$$(1)\ \begin{cases} x^2-y^2=3 \\ xy=2 \end{cases} \Leftrightarrow \left\{\begin{cases} x^2-y^2=3 \\ (x^2+y^2)^2 \equiv (x^2-y^2)^2 + 4x^2y^2 = 25 \\ xy=2 \end{cases}\right\} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} x^2-y^2=3 \\ x^2+y^2=5 \\ xy=2 \end{cases} \Leftrightarrow \begin{cases} x^2=4 \\ y^2=1 \\ xy=2 \end{cases} \Leftrightarrow \begin{cases} x=\pm 2 \\ y=\pm 1 \\ xy=2 \end{cases} \Leftrightarrow \begin{cases} x=2 \\ y=1 \end{cases} \vee \begin{cases} x=-2 \\ y=-1 \end{cases};$$

skärningspunkterna är $(2,1), (-2,-1)$.

(2) Vinkeln mellan tangenterna är lika med vinkeln mellan normalerna (gradienterna).

C_1: $x^2-y^2=3$ är en nivåkurva till funktionen

$$f(x,y) = x^2-y^2.$$

C_2: $xy=2$ är en nivåkurva till funktionen

$$g(x,y) = xy.$$

$\operatorname{grad} f(\mathbf{x}) = (2x, -2y) = 2(x,-y)$; $\operatorname{grad} g(\mathbf{x}) = (y,x)$

$\operatorname{grad} f(2,1)\cdot\operatorname{grad} g(2,1) = 2(2,-1)\cdot(1,2) = 2\cdot 0 = 0$

$\operatorname{grad} f(-2,-1)\cdot\operatorname{grad} g(-2,-1) = 0$ även i detta fall.

Resultat: Kurvorna (hyperblerna) skär varandra i $\pm(2,1)$ under rät vinkel.

Problem 2.46 (Sid. 8)

Lösning

(1). y-axeln har som bekant ekvationen $x=0$.

$$\begin{cases} x=0 \\ x^3+y^3+x-y=0 \end{cases} \Leftrightarrow \begin{cases} x=0 \\ y^3-y=0 \end{cases} \Leftrightarrow \begin{cases} x=0 \\ y=0,\pm 1 \end{cases} \Leftrightarrow \begin{cases} P_1:(0,0) \\ P_2:(0,1) \\ P_3:(0,-1) \end{cases}.$$

(2) $C: x^3+y^3+x-y=0$ är en nivåkurva till

$$f(x,y)=x^3+y^3+x-y.$$

Vinkeln bestäms med formeln $u_i \cdot e_y = u_i \cdot 1 \cdot \cos\theta$, där u_i är en tangentvektor till C i punkten P_i. En normal i samma punkt är som bekant $\text{grad} f(P_i)$, $i=1,2,3$. $\text{grad} f(\mathbf{x}) = (3x^2+1, 3y^2-1)$.

$\underline{P_1:(0,0)}$ $\quad \text{grad} f(P_1)=(1,-1) \Rightarrow u_1=(1,1). \Rightarrow$

$$\Rightarrow u_1 \cdot e_y = \sqrt{2}\cos\theta \Leftrightarrow \cos\theta = \frac{1}{\sqrt{2}} \Leftrightarrow \underline{\underline{\theta = \pi/4}}$$

$\underline{P_2:(0,1)}$ $\quad \text{grad} f(0,1)=(1,2) \Rightarrow u_2=(-2,1) \Rightarrow$

$$\Rightarrow u_2 \cdot e_y = \sqrt{5}\cos\theta \Leftrightarrow \cos\theta = \frac{1}{\sqrt{5}} \Rightarrow \theta = \cos^{-1}\frac{1}{\sqrt{5}}$$

$\underline{P_3:(0,-1)}$ $\quad \text{grad} f(P_3)=(1,2) \Rightarrow u_3=(-2,1)$

Detta fall är detsamma som det förra; $\theta = \arccos(\frac{1}{\sqrt{5}})$ således.

Problem 2.47 (Sid. 8)

Lösning

$\underline{C: xy^2=2}$ är en nivåkurva till funktionen

$$f(x,y)=xy^2, \quad x>0.$$

Normalen i $P_0:(\xi,\eta)\in C$ är parallell med $\underline{\text{gradientvektorn}}$ där: $\text{grad} f(P_0)=(\eta^2, 2\xi\eta)$.

Normalen genom P_0 har riktningskoefficienten

$$k=\frac{2\xi\eta}{\eta^2}=\frac{2\xi}{\eta} \overset{!}{=} \frac{\eta}{\xi} \Leftrightarrow \left(\frac{\eta}{\xi}\right)^2=2 \Leftrightarrow \eta=\pm\sqrt{2}\xi \; ; \; \xi>0.$$

Insättning av $\eta^2=2\xi^2$ i $\xi\eta^2=2$ ger $\xi^3=1 \Rightarrow \xi=1$

Svar: I punkterna $(1,\sqrt{2})$ och $(1,-\sqrt{2})$.

Anm. I $\overset{!}{\Leftrightarrow}$ underförstås följande: En linje genom origo och $P_0:(\xi,\eta)$ har riktningskoefficienten $k=\frac{\eta}{\xi}$.

Problem 2.48 (Sid. 8)

Lösning

a) $\underline{S: x^2+2y^2+3z^2=6, \quad P_0:(1,1,1).}$

S är nivåytan till $f(\mathbf{x})=x^2+2y^2+3z^2$ genom P_0.

En normalvektor för tangeringsplanet i pkn P_0 är som bekant gradienten där; $\operatorname{grad} f(P_0)$;

$\operatorname{grad} f(\mathbf{x}) = (2x, 4y, 6z) \Rightarrow \operatorname{grad} f(P_0) = 2(1,2,3) = 2\mathbf{v}$.

Tangentplanets ekvation blir alltså

$$\mathbf{v}\cdot\overrightarrow{P_0P} = 0 \Leftrightarrow \mathbf{v}\cdot\overrightarrow{OP} = \mathbf{v}\cdot\overrightarrow{OP_0} \Leftrightarrow x+2y+3z=6.$$

<u>Anm.</u> Planets ekvation härleds i algebran.

b) Ekvationen för tangentplanet till en funktionsyta $z=f(x,y)$ i punkten $P_0:(a,b)$ ges av

$$\underline{T:\ z = f(P_0) + f'_x(P_0)(x-a) + f'_y(P_0)(y-b)}.$$

$f(x,y) = x^2y \Rightarrow \frac{\partial f}{\partial x} = 2xy \wedge \frac{\partial f}{\partial y} = x^2 \Rightarrow f'_x(P_0) = -4 \wedge$
$f_y(P_0) = 4 \Rightarrow T: z = 4 - 4(x+2) + 4(y-1) = -4x+4y-8$

Dess normalform är

$$\underline{4x-4y+z+8=0}$$

<u>Svar</u>: a) $x+2y+3z=6$; b) $4x-4y+z+8=0$.

<u>Anm.</u> Endast vid funktionsytor $z=f(x,y)$ gäller för tangeringspunkten $P:(a,b,c)$ att $c=f(a,b)$; yta är något mer generaliserat.

Problem 2.49 (Sid. 8)

<u>Lösning</u>

$$\underline{S: x^2+2y^2+3z^2+2xy+2yz=1;\ \pi: x-y+2z=0.}$$

Tangeringspunkten kallas $P_0:(\lambda,\mu,\nu)$.

S är en nivayta till funktionen

$$f(\mathbf{x}) = x^2+2y^2+3z^2+2xy+2yz.$$

En normalvektor till π är $\mathbf{n} = (1,-1,2)$, vilket ger sambandet $\operatorname{grad} f(P_0) \parallel \mathbf{n}$.

$\operatorname{grad} f(\mathbf{x}) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2x+2y,\ 2x+4y+2z,\ 2y+6z)$.

$\operatorname{grad} f(P_0) = 2(\lambda+\mu,\ \lambda+2\mu+\nu,\ \mu+3\nu) = 2k(1,-1,2)$

$$\Leftrightarrow \begin{cases} \lambda+\mu = k & -1 \\ \lambda+2\mu+\nu = -k \\ \mu+3\nu = 2k \end{cases} \Leftrightarrow \begin{cases} \lambda+\mu = k \\ \mu+\nu = -2k & -1 \\ \mu+3\nu = 2k \end{cases} \Leftrightarrow \begin{cases} \lambda+\mu = k \\ \mu+\nu = -2k \\ 2\nu = 4k \end{cases}$$

$$\Leftrightarrow \begin{cases} \lambda+\mu = k \\ \mu+\nu = -2k \\ \nu = 2k \end{cases} \Leftrightarrow \begin{cases} \lambda = 5k \\ \mu = -4k \\ \nu = 2k \end{cases} \Rightarrow f(\lambda,\mu,\nu) = 25k^2+32k^2+$$

$$+12k^2-40k^2-16k^2 = 13k^2 = 1 \Leftrightarrow k = \pm\frac{1}{\sqrt{3}} \Rightarrow \begin{cases} P_1: \left(\frac{5}{\sqrt{13}}, \frac{-4}{\sqrt{13}}, \frac{2}{\sqrt{13}}\right) \\ P_2: \left(\frac{-5}{\sqrt{13}}, \frac{4}{\sqrt{13}}, \frac{-2}{\sqrt{13}}\right) \end{cases}$$

Problem 2.50 (Sid. 8)

<u>Lösning</u>

$$\underline{S: x^2+2y^2+3z^2=6,\ P_1:(6,0,0),\ P_2:(0,3,0).}$$

Kalla tangeringspunkten $P_0 : (\lambda, \mu, \nu)$.

Jag behöver grad $f(P_0)$, normalen i P_0, där

$$f(x,y,z) = x^2+2y^2+3z^2.$$

$\text{grad} f(x) = (2x, 4y, 6z) \Rightarrow \text{grad} f(P_0) = 2(\lambda, 2\mu, 3\nu)$.

Planets ekvation är

$(\lambda, 2\mu, 3\nu)\cdot(x-\lambda, y-\mu, z-\nu) = 0 \Leftrightarrow \lambda x + 2\mu y + 3\nu z =$

$= \lambda^2+2\mu^2+3\nu^2 = 6 \quad \Leftrightarrow \quad \underline{\pi : \lambda x + 2\mu y + 3\nu z = 6.}$

$$\left.\begin{array}{l} P_1 \in \pi \Rightarrow 6\lambda = 6 \Leftrightarrow \underline{\lambda = 1} \\ P_2 \in \pi \Rightarrow 6\mu = 6 \Leftrightarrow \underline{\mu = 1} \end{array}\right\} \Rightarrow f(\lambda,\mu,\nu) = f(1,1,\nu) =$$

$= 3+3\nu^2 = 6 \Leftrightarrow 3\nu^2 = 3 \Leftrightarrow \nu^2 = 1 \Leftrightarrow \underline{\nu = \pm 1}$.

Svar: $x+2y+3z=6$ och $x+2y-3z=6$.

Problem 2.51 (Sid. 8)

Lösning

$\underline{S: x^2+3y^2+4z^2 = C;\ P_1 : (0,1,2),\ P_2 : (1,3,0),\ P_3 : (5,-1,1).}$

(1) Jag bestämmer planets ekvation gm P_1, P_2, P_3.

$$\pi : Ax+By+Cz = D \Rightarrow \begin{cases} P_1 \in \pi \Rightarrow B+2C = D \\ P_2 \in \pi \Rightarrow A+3B = D \\ P_3 \in \pi \Rightarrow 5A-B+C = D \end{cases} \Leftrightarrow \begin{cases} A = 2D/11 \\ B = 3D/11 \\ C = 4D/11 \end{cases}$$

$\Rightarrow \pi : 2x+3y+4z = 11. \qquad (n = (2,3,4)$ normal$)$.

(2) S är en nivåyta till funktionen

$$f(x,y,z) = x^2+3y^2+4z^2.$$

Om $P_0 : (\lambda, \mu, \nu)$ är tangeringspunkten, så är $\text{grad} f(P_0) = (2\lambda, 6\mu, 8\nu) = k\cdot(2,3,4)$, $k \neq 0$,

$\Leftrightarrow k = \lambda = 2\mu = 2\nu \Leftrightarrow P_0 : (k, \frac{k}{2}, \frac{k}{2})$;

$P_0 \in \pi \Rightarrow 2k + \frac{3}{2}k + 2k = \frac{11}{2}k = 11 \Leftrightarrow k = 2 \Rightarrow P_0 : (2,1,1)$.

Svar: $C = f(2,1,1) = 1$.

Problem 2.52 (Sid. 8)

Lösning

Låt oss bestämma eventuella skärningspunkter för olika värden på konstanten C.

$$\begin{cases} 2x^2+y^2 = C \\ x^2-2y^2 = 1 \end{cases} \Leftrightarrow \begin{cases} 5x^2 = 1-2C \geq 0 \\ 5y^2 = C-2 \geq 0 \end{cases} \Leftrightarrow \begin{cases} 2C \leq 1 \\ C \geq 2 \end{cases} \Leftrightarrow \begin{cases} C \leq \frac{1}{2} \\ C \geq 2 \end{cases}$$

Sådana $C > 0$ existerar inte så kurvorna saknar gemensamma punkter, de skär inte varandra helt enkelt.

Problem 2.53 (Sid. 8)

Lösning: a) $\nabla f(x,y) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ är en tvådimensionell gradient, en normalvektor till en nivåkurva $f(x,y)=C$, C konstant.

b) $\nabla F(x,y,z)=(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},-1)$ är en tredimensionell gradient, en normalvektor till funktionsytan $z=f(x,y)$; ∇F är nedåtriktad.

Anm. ∇f är ∇F:s projektion i xy-planet.

Problem 2.54 (Sid. 8)

Lösning

$$f(x,y)=\ln(x^2+2y^2),\ P_0:(2,1)$$

a) $\text{grad} f(x)=(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y})=(\frac{2x}{x^2+2y^2},\frac{4y}{x^2+2y^2})$; $\hat{v}=(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$;

$f'_v(P_0)=\text{grad} f(P_0)\cdot\hat{v}=(\frac{4}{6},\frac{4}{6})\cdot(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})=2\cdot\frac{2}{3}\cdot\frac{1}{\sqrt{2}}=\frac{2\sqrt{2}}{3}$.

b) $v=(1,2)\Rightarrow |v|=\sqrt{5}\Rightarrow \hat{v}=\frac{v}{|v|}=(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}})$.

$f'_v(P_0)=(\frac{4}{6},\frac{4}{6})\cdot(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}})=\frac{2}{3}\cdot\frac{3}{\sqrt{5}}=\frac{2}{\sqrt{5}}=\frac{2\sqrt{5}}{5}$.

Anm. Riktningsderivatan skrivs ofta $\frac{\partial f}{\partial v}$.

Problem 2.55 (Sid. 8)

Lösning: $f(x,y,z)=xy^2z^3$, $P_0:(3,2,1)$.

Tillväxthastighet fås mha riktningsderivatan. Det gäller att bestämma $f'_v(P_0)$ i riktningen $v=\overrightarrow{P_0O}=-\overrightarrow{OP_0}=(-3,-2,-1)$.

$\text{grad} f(x)=(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z})=(y^2z^3,2xyz^3,3xy^2z^2)$;

$\text{grad} f(P_0)=(4,12,36)=4(1,3,9)$;

$v=(-3,-2,-1)\Rightarrow \hat{v}=\frac{(-3,-2,-1)}{\sqrt{3^2+2^2+1^2}}=-\frac{1}{\sqrt{14}}(3,2,1)$;

$f'_v(P_0)=\text{grad} f(P_0)\cdot\hat{v}=4\cdot(1,3,9)\cdot\frac{-1}{\sqrt{14}}(3,2,1)=-\frac{72}{\sqrt{14}}$.

Resultat: Den avtar med hastigheten $\frac{72}{\sqrt{14}}$.

Problem 2.56 (Sid. 8)

Lösning

Med y-axeln pekande mot norr och x-axeln pekande mot öster blir temperaturgradienten $\nabla T=(2,-3)$.

a) Rakt åt vänster pekar vektorn $\hat{v}=(-1,0)$ s.a.

$\frac{\partial T}{\partial v}=\nabla T\cdot\hat{v}=(2,-3)\cdot(-1,0)=-2°C/km$.

b) Sydost bestäms av riktningen $\hat{v}=(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$

s.a. $\frac{\partial T}{\partial v}=\nabla T\cdot\hat{v}=(2,-3)\cdot(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})=\frac{5}{\sqrt{2}}\approx 3{,}54°C/km$.

c) Nordost bestäms av riktningen $\hat{v}=(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$ s.a.

$$\frac{\partial T}{\partial v} = \nabla T\cdot\hat{v} = (2,-3)\cdot\left(\frac{1}{\sqrt2},\frac{1}{\sqrt2}\right) = -\frac{1}{\sqrt2} \approx -0{,}7\,^\circ\mathrm{C/km}.$$

Svar: a) Faller med 2°C/km; b) Stiger med 3,52°C/km; c) Faller med 0,7°C/km.

Problem 2.57 (Sid. 8)

Lösning: $T(x,y) = 3\arctan(x^2+y) - 10 - \frac{6}{1+x^2+y^2}$.

Gradienten i en punkt pekar i den riktning i vilken T växer snabbast. Vi behöver bestämma tillväxthastigheten i riktningen $-\mathrm{grad}\,T(1,-2)$, ty avkylning eftersträvas.

$$\nabla T(x,y) = \left(\frac{\partial T}{\partial x},\frac{\partial T}{\partial y}\right) = \left(\frac{6x}{1+(x^2+y)^2} + \frac{12x}{(1+x^2+y^2)^2},\ \frac{3}{1+(x^2+y)^2} + \frac{12y}{(1+|\mathbf{x}|^2)^2}\right)$$

$$\nabla T(1,-2) = \left(3+\frac13,\ \frac32-\frac23\right) = \left(\frac{10}{3},\ \frac56\right) = \frac56\cdot(4,1);$$

Riktningen ifråga bestäms av $\hat{v} = \frac{1}{\sqrt{17}}(-4,-1)$.

$$\frac{\partial T}{\partial v} = \nabla T(1,-2)\cdot\hat{v} = -|\nabla T(1,-2)| = \frac56\sqrt{17} \approx 3{,}44\,^\circ\mathrm{C/km}$$

Motsvarande "tidshastighet" är $3\sqrt{17}/20$ °C/min.

Anm: $1\,^\circ\mathrm{C/km} = 1\,^\circ\mathrm{C}/10^3\,\mathrm{m} = (10^{-3})\,^\circ\mathrm{C/m}$.

$3\,\mathrm{m/s} = 3\,\mathrm{m}/\frac{1}{60}\,\mathrm{min} = 180\,\mathrm{m/min}$;

$\frac56\sqrt{17}\,^\circ\mathrm{C/km}\cdot 180\cdot 10^{-3}\,\mathrm{km/min} = \left(\frac{3}{20}\sqrt{17}\right)^\circ\mathrm{C/min}$.

Övning 2.58 (Sid. 8)

Lösning: $f(x,y) = x+2y-(x-1)^3$; $P:(1,-1)$.

a) $\mathrm{grad}\,f(\mathbf{x}) = \left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right) = (1-3(x-1)^2,\ 2) \Rightarrow \mathrm{grad}\,f(P) = (1,2)$

$\Rightarrow f'_v(P) = \mathrm{grad}(P)\cdot\hat{v} = a+2b$

$f'_v(P) > 0 \Leftrightarrow a+2b>0$; $f'_v(P)<0 \Leftrightarrow a+2b<0$;

$f'_v(P) = 0 \Leftrightarrow a = -2b \neq 0$.

b) I riktningarna $\hat{v} = (a,b)$ som gör $f'_v(P) > 0$ växer f initialt, dvs för $a+2b>0$.

Problem 2.59 (Sid. 8)

Lösning

Det gäller att finna det största värdet av gradientens belopp.

$$f(\mathbf{x}) = \frac{4}{1+x^2+y^2} \Rightarrow \mathrm{grad}\,f(\mathbf{x}) = \left(-\frac{8x}{(1+|\mathbf{x}|^2)^2},\ -\frac{8y}{(1+|\mathbf{x}|^2)^2}\right) \Rightarrow$$

$$\Rightarrow \phi(\mathbf{x}) = |\mathrm{grad}\,f(\mathbf{x})| = \frac{8|\mathbf{x}|}{(1+|\mathbf{x}|^2)^2} = g(|\mathbf{x}|) = g(r),\ r = |\mathbf{x}|.$$

$$g(r) = \frac{8r}{(1+r^2)^2} \Rightarrow g'(r) = 8\,\frac{1-3r^2}{(1+r^2)^3} = 0 \Leftrightarrow r^2 = x^2+y^2 = \frac13.$$

Anm. Kullen är rotationssymmetrisk, så den är brantast på höjden $4/\frac43 = 3$.

Problem 2.60 (Sid. 8)

Lösning

S: $z=4-x^2-2y^2$ är nivåytan till $f:\mathbb{R}\to\mathbb{R}$,

$$f(x,y,z)=x^2+2y^2+z=f(1,1,1)$$

genom punkten $P_0:(1,1,1)$.

Antag att vägens projektion på xy-planet ges av $y=f(x)$. En tangentvektor till $y=f(x)$ är $v=(1,f'(x))$. Då är gradientens projektion på xy-planet parallell med v. Alltså är

$$\operatorname{grad} f(x)|_{xy}=k\cdot(1,f'(x)) \Rightarrow (2x,4y)=k(1,f'(x)) \Leftrightarrow$$

$$\Leftrightarrow \frac{f'(x)}{4y}=\frac{1}{2x} \Leftrightarrow \frac{f'(x)}{f(x)}=\frac{2}{x} \Rightarrow \ln f(x)=\ln Cx^2 \Leftrightarrow$$

$$\Leftrightarrow f(x)=Cx^2.$$

Kurvan $y=f(x)$ ska gå genom P_0:s projektion i xy-planet; $f(1)=1 \Rightarrow C=1 \Rightarrow \underline{f(x)=x^2}$

Anm. Konsultera lösningen till 2.53.

Lokala undersökningar

Problem 2.61 (Sid. 8)

Lösning

a) $f(x,y)=(x^2+y^2-1)e^y=$

$$=(x^2+y^2-1)(1+y+\tfrac{1}{2}y^2+O(r^3))=$$

$$=x^2+y^2-1-y-\tfrac{1}{2}y^2+O(r^3)=$$

$$=\underline{-1-y+x^2+\tfrac{1}{2}y^2+O(r^3)},\quad r=\sqrt{x^2+y^2}.$$

b) $f(x,y)=\sin(x+y)\cdot\ln(1+2x+y)-xy=$

$$=(x+y+O(r^3))(2x+y-\tfrac{1}{2}(2x+y)^2+O(r^3))-xy=$$

$$=(x+y)(2x+y)-xy+O(r^3)=$$

$$=2x^2+3xy+y^2-xy+O(r^3)=$$

$$=\underline{2x^2+2xy+y^2+O(r^3)},\quad r=\sqrt{x^2+y^2}.$$

c) $f(x,y,z)=2\sqrt{1+x^2+y}-\cos(x-z)-y=$

$$=2(1+\tfrac{1}{2}(x^2+y)-\tfrac{3}{8}(x^2+y)^2)-(1-\tfrac{1}{2}(x-z)^2)-y+O(r^3)$$

$$=2+x^2+y-\tfrac{3}{4}(x^2+y)^2-1+\tfrac{1}{2}(x-z)^2-y+O(r^3)=$$

$$=2+x^2+y-\tfrac{3}{4}y^2-1+\tfrac{1}{2}x^2+\tfrac{1}{2}z^2-xz-y+O(r^3)=$$

$$=\underline{1+\tfrac{3}{2}x^2-\tfrac{3}{4}y^2+\tfrac{1}{2}z^2-xz+O(r^3)},\quad r=\sqrt{x^2+y^2+z^2}$$

Problem 2.62 (Sid. 9)

Lösning

$$z=f(x,y)=\ln(2x^2+xy+y^2)$$

$$\begin{cases} r=x-2 \\ s=y+1 \end{cases} \Leftrightarrow \begin{cases} x=2+r \\ y=-1+s \end{cases} \Rightarrow g(r,s)=f(2+r,-1+s)=$$

$$=\ln(2(r+2)^2+(r+2)(s-1)+(s-1)^2)=$$

$$=\ln(2r^2+8r+8+rs-r+2s-2+s^2-2s+1)=$$

$$=\ln(7+7r+2r^2+rs+s^2)=$$

$$=\ln 7(1+\tfrac{1}{7}(7r+2r^2+rs+s^2))=$$

$$=\ln 7+\ln(1+\tfrac{1}{7}(7r+2r^2+rs+s^2))=$$

$$=\ln 7+\tfrac{1}{7}(7r+2r^2+rs+s^2)-\tfrac{1}{98}\cdot(7r)^2+O(\rho^3)=$$

$$=\ln 7+r-\tfrac{3}{14}r^2+\tfrac{1}{7}rs+\tfrac{1}{7}s^2+O(\rho^3)=$$

$$=\ln 7+(x-2)-\tfrac{3}{14}(x-2)^2+\tfrac{1}{7}(x-2)(y+1)+\tfrac{1}{7}(y+1)^2+O(\rho^3),$$

där $\rho=\sqrt{(x-2)^2+(y+1)^2}$.

Problem 2.63 (Sid. 9)

Lösning

$$f(x,y)=(y-x^2)(y-3x^2)$$

$$x=ky \Rightarrow g(y)=f(ky,y)=(y-k^2y^2)(y-3k^2y^2)=$$

$$=y(1-k^2y)\cdot y(1-3k^2y)=y^2(1+O(|x|))\approx y^2$$

För små $|x|$ och $|y|$ är $f(x,y)\geq 0$, dvs f har ett lokalt minimum i origo längs varje linje genom origo.

b) I området $\Omega_1=\{(x,y):\ x^2<y<3x^2\}$ är $f(x,y)<0$ och i området $\Omega_2=\{(x,y):\ y>3x^2\}$ är $f(x,y)>0$. f antar både positiva och negativa värden i varje öppen omgivning omfattande origo, så origo är ingen extrempunkt i lokal mening.

Problem 2.64 (Sid. 9)

Lösning

a) För alla (x,y) gäller att $|x|+y^2\geq 0$, så att

$$f(x,y)-f(0,0)=-(|x|+y^2)\leq 0 \Leftrightarrow f(x,y)\leq f(0,0) \Rightarrow$$

$\Rightarrow$ origo är en lokal maximipunkt.

b) $f(x,y)-f(0,0)=|x|-1-\cos y=|x|-1-(1-\frac{y^2}{2})+O(r^4)=$

$=|x|+\frac{1}{2}y^2+O(r^4)\geq 0$, för små $|x|=r$, dvs

$f(x,y)\geq f(0,0) \Rightarrow$ origo är en lokal min/pkt.

c) $f(x,y)-f(0,0)=|x|+\cos y-1=g(x,y)$; $g(1,0)\cdot g(0,1)<0$

$\Rightarrow$ (0,0) ingen lokal extrempunkt.

d) $f(x,y,z)-f(0,0,0)=x^2-yz=g(x,y,z) \Rightarrow g(1,0,0)>0$

och $g(0,1,1)<0$; (0,0,0) ingen lokal extrem-punkt.

e) $f(x,y,z)-f(0,0,0)=\cos(xyz)-1\leq 0 \Rightarrow f(x,y,z)\leq f(0,0,0)$;

(0,0,0) ger lokalt maximum.

f) $f(x,y,z)-f(0,0,0)=(1+x^2)e^{-(y^2+z^2)}=g(x,y,z) \Rightarrow$

$\Rightarrow g(0,1,1)\cdot g(1,0,0)=(e^{-2}-1)\cdot(1)<0 \Rightarrow$ (0,0,0) ger inget lokalt extremvärde.

g) $f(x,y)-f(0,0)=(x+y)^2-xy^3=g(x,y)$; $g(\frac{1}{2},-\frac{1}{2})<0$

och $g(1,0)>0$; (0,0,0) är ingen extrempunkt i detta fall heller.

h) $f(x,y)-f(0,0)=|x|^2\sin(|x|^{-2})=g(x,y)$; $g(1,0)>0$

men $g(0,\frac{1}{2})<0$ så (0,0) är ingen (lokal) extrempunkt.

i) $f(x,y)-f(0,0)=(x-y)^2+O(r^4)\geq 0 \Rightarrow f(x,y)\geq f(0,0)$

$\Rightarrow$ (0,0) lokal minimipunkt.

Problem 2.65 (Sid. 9)

Lösning

a) $\left.\begin{array}{l} Q(h,k)=h^2-hk+k^2=(h-\frac{1}{2}k)^2+\frac{3}{4}k^2\geq 0 \\ Q(h,k)=0 \Leftrightarrow h-\frac{1}{2}k=0 \wedge k=0 \Leftrightarrow h=k=0 \end{array}\right\} \Rightarrow$

$\Rightarrow Q$ positivt definit.

b) $Q(h,k)=h^2+hk-k^2=[h\ k]\begin{bmatrix}1 & 1/2\\ 1/2 & -2\end{bmatrix}\begin{bmatrix}h\\ k\end{bmatrix} \Rightarrow A=A^t=$

$=\begin{bmatrix}1 & 1/2\\ 1/2 & -2\end{bmatrix} \Rightarrow \chi_A(\lambda)=\begin{vmatrix}1-\lambda & 1/2\\ 1/2 & -2-\lambda\end{vmatrix}=(\lambda-1)(\lambda+2)-\frac{1}{4}=$

$=\lambda^2+\lambda-\frac{9}{4}=(\lambda+\frac{1}{2})^2-\frac{5}{4}=(\lambda+\frac{1+\sqrt{5}}{2})(\lambda+\frac{1-\sqrt{5}}{2})$;

$\chi_A(\lambda)=0 \Leftrightarrow \lambda_1=-\frac{1+\sqrt{5}}{2}$ och $\lambda_2=-\frac{1-\sqrt{5}}{2}$; $\lambda_1\cdot\lambda_2<0$.

Q är indefinit.

c) $Q(h,k)=hk \Rightarrow Q(1,1)\cdot Q(1,-1)<0 \Rightarrow Q$ indefinit.

d) $(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc$ (Analys A).

$a=l \Rightarrow (l+b+c)^2=l^2+b^2+c^2+2lb+2lc+2bc$;

$b=2k \Rightarrow (l+2k+c)^2=l^2+4k^2+c^2+4kl+2lc+4kc$;

$c=-h \Rightarrow (l+2k-h)^2=l^2+4k^2+h^2+4kl-2hl-4hk$;

$Q(h,k,l)=(h-2k-l)^2-h^2-2hk=(h-2k-l)^2-(h-k)^2+k^2$.

$h-2k-l=h-k=k=0 \Leftrightarrow h=k=l=0 \Rightarrow Q$ indefinit.

e) $Q(h,k,l) = 4hk + 4kl - 2h^2 - 3k^2 - 4l^2 =$

$$= [h\ k\ l]\begin{bmatrix} -2 & 2 & 0 \\ 2 & -3 & 2 \\ 0 & 2 & -4 \end{bmatrix}\begin{bmatrix} h \\ k \\ l \end{bmatrix} \Rightarrow A = \begin{bmatrix} -2 & 2 & 0 \\ 2 & -3 & 2 \\ 0 & 2 & -4 \end{bmatrix} = A^t$$

$$\Rightarrow \chi_A(\lambda) = \begin{vmatrix} -2-\lambda & 2 & 0 \\ 2 & -3-\lambda & 2 \\ 0 & 2 & -4-\lambda \end{vmatrix} = -(\lambda+2)(\lambda+3)(\lambda+4) + 4(\lambda+2) +$$

$$+4(\lambda+4) = -(\lambda+2)(\lambda+3)(\lambda+4) + 8(\lambda+3) =$$

$$= -(\lambda+3)((\lambda+2)(\lambda+4) - 8) =$$

$$= -(\lambda+3)(\lambda^2+6\lambda) = -\lambda(\lambda+3)(\lambda+6);$$

$\chi_A(\lambda) = 0 \Leftrightarrow \lambda = 0 \vee \lambda = -3 \vee \lambda = -6$, så Q är en <u>negativ semidefinit form.</u>

f) $Q(h,k,l) = h^2 + 2hk + 2k^2 = (h+k)^2 + k^2 \geq 0$;

$Q(0,0,1) = 0$ så Q är <u>positivt semidefinit.</u>

g) $Q(h,k,l) \geq 0$; $Q(1,1,1) = 0$; <u>positivt semidefinit.</u>

Problem 2.66 (Sid. 9)

Lösning

$$Q(h,k) = [h\ k]\begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}\begin{bmatrix} h \\ k \end{bmatrix} \Rightarrow A = \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix} = A^t \Rightarrow$$

$$\Rightarrow \chi_A(\lambda) = \det(A - \lambda E) = (\lambda-1)^2 - a^2 = \lambda^2 - 2\lambda + 1 - a^2;$$

sambandet mellan nollställen och koefficienter ger $\underline{\lambda_1 \cdot \lambda_2 = 1 - a^2}$.

(1) $\lambda_1 \cdot \lambda_2 > 0 \Leftrightarrow 1 - a^2 > 0 \Leftrightarrow a^2 < 1 \Leftrightarrow -1 < a < 1$.

(2) $\lambda_1 \cdot \lambda_2 = 0 \Leftrightarrow a = \pm 1$.

(3) $\lambda_1 \cdot \lambda_2 < 0 \Leftrightarrow |a| > 1 \Leftrightarrow a < -1 \vee a > 1$.

<u>Svar</u>: $|a| < 1 \Rightarrow Q$ positivt definit.

$|a| = 1 \Rightarrow Q$ positivt semidefinit.

$|a| > 1 \Rightarrow Q$ indefinit.

Problem 2.67 (Sid. 9)

Lösning

a) $f(x,y) - f(0,0) = -y + x^2 + \frac{1}{2}y^2 + O(r^3)$; $f'_y(0,0) = -1 \neq 0$;

Ingetdera. (För lokalt maximum/minimum krävs det att $f'_x = f'_y = 0$).

b) För små $|x|, |y|$ är $f(x,y) \approx (x+y)^2 \geq 0$, så origo är en lokal min/pkt.

c) $f(x,y,z) - f(0,0,0) = \frac{1}{4}(6x^2 - 3y^2 + 2z^2 - 4xz) + O(r^3)$;

$$Q = x^t\begin{bmatrix} 6 & 0 & -2 \\ 0 & -3 & 0 \\ -2 & 0 & 2 \end{bmatrix}x \Rightarrow A = \begin{bmatrix} 6 & 0 & -2 \\ 0 & -3 & 0 \\ -2 & 0 & 2 \end{bmatrix} \Rightarrow \chi_A(\lambda) = |A - \lambda E| =$$

$= -(\lambda-2)(\lambda+3)(\lambda-6)+4(\lambda+3) = -(\lambda+3)((\lambda-2)(\lambda-6)-4) =$

$= -(\lambda+3)(\lambda^2-8\lambda+8)$; $\chi_A(\lambda)=0 \Rightarrow \lambda_1=-3$ och $\lambda_{2,3}=4\pm\sqrt{8}$; formen är indefinit så origo är ingen extrempunkt.

Problem 2.68 (Sid. 9)

Lösning

a) $\frac{\partial f}{\partial x} = 4x^3-3x^2y+2x-2y$, $\frac{\partial f}{\partial y} = -x^3-2x+4y$;

$\frac{\partial^2 f}{\partial x^2} = 12x^2-6xy+2$, $\frac{\partial^2 f}{\partial x\partial y} = -3x^2-2$, $\frac{\partial^2 f}{\partial y^2} = 4$;

$Q(h,k) = f''_{xx}(0,0)h^2+2f''_{xy}(0,0)hk+f''_{yy}(0,0)k^2 =$

$= 2h^2-4hk+4k^2 = 2(h^2-2hk+k^2)+2k^2 =$

$= 2(h-k)^2+2k^2$, pos. definit; min/pkt.

b) $f(x,y)-f(0,0) = (x-y)^2+x^3 = g(x,y) \Rightarrow g(1,1)\cdot g(-1,-1)<0$

$\Rightarrow$ origo ger inget av intresse.

c) $f(x,y,z) = 2\cos(x+y+z)+e^{xy}+e^{yz}+e^{xz}$;

$f(x,y,z)-f(0,0,0) = 2\cos(x+y+z)+e^{xy}+e^{yz}+e^{xz}-5 =$

$= 2(1-\frac{1}{2}(x+y+z)^2)+1+xy+1+yz+1+yz-5+O(r^4) =$

$= -(x^2+y^2+z^2+2xy+2yz+2xz)+xy+yz+xz+O(r^4) =$

$= -(x^2+y^2+z^2+xy+xz+yz)+O(r^4) = -\frac{1}{2}((x+y+z)^2+$

$+x^2+y^2+z^2)+O(r^4) \Rightarrow$ origo lokal min/pkt.

d) $f(x,y,z)-f(0,0,0) = \cos(x+y+z)+\cos x-2 =$

$= 1-\frac{1}{2}(x+y+z)^2+1-\frac{1}{2}x^2-2+O(r^4) =$

$= -\frac{1}{2}((x+y+z)^2+x^2)+O(r^4)$.

Origo är således lokal maximipunkt

Problem 2.69 (Sid. 9)

Lösning

$$f(x,y) = 4x^2e^y-2x^4-e^{4y}$$

a) $\frac{\partial f}{\partial x} = 8xe^y-8x^3$, $\frac{\partial f}{\partial y} = 4x^2e^y-4e^{4y}$.

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Rightarrow \begin{cases} 8x(e^y-x^2)=0 \\ 4e^y(x^2-e^{3y})=0 \end{cases} \Leftrightarrow \begin{cases} e^y = x^2 \\ e^{3y}=x^2 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} e^y = x^2 \\ e^{3y}=e^y \end{cases} \Leftrightarrow \begin{cases} x^2=e^y \\ y=0 \end{cases} \Leftrightarrow \begin{cases} x^2=1 \\ y=0 \end{cases} \Leftrightarrow \begin{cases} x=1 \\ y=0 \end{cases} \vee \begin{cases} x=-1 \\ y=0 \end{cases}$$

$\frac{\partial^2 f}{\partial x^2} = 8e^y-24x^2$, $\frac{\partial^2 f}{\partial y^2} = 4x^2e^y-16e^{4y}$, $\frac{\partial^2 f}{\partial x\partial y} = 8xe^y$

(1) $f''_{xx}(1,0)=-16$, $f''_{yy}(1,0)=-12$, $f''_{xy}(1,0)=8$.

$Q(h,k) = -16h^2+16hk-12k^2 = -4(4h^2-4hk+3k^2) =$

$= -4((2h-k)^2+2k^2)$, negativt definit, så

(1,0) är en lokal max/pkt.

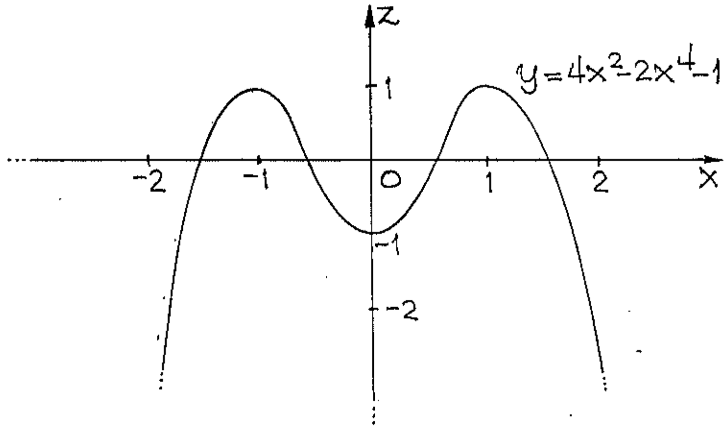
(2) $f''_{xx}(-1,0)=-16,\ f''_{yy}(-1,0)=-12,\ f''_{xy}(-1,0)=-8$;

$Q(h,k)=-16h^2-16hk-12k^2=-(4h-k)^2-11k^2$, negativt definit, dvs $(-1,0)$ lokal max/pkt.

c) $g(x)=f(x,0)=4x^2-2x^4-1$ (enkelspårig)

$g'(x)=8x-8x^3=8x(1-x^2)=8x(1+x)(1-x)$;

x	$-\infty$		-1		0		1		∞
$\operatorname{sgn} g'(x)$		$+$	0	$-$	0	$+$	0	$-$	
$g(x)$	$-\infty$	$\nearrow$	1	$\searrow$	-1	$\nearrow$	1	$\searrow$	$-\infty$



d) $h(y)=f(\pm1,y)=4e^y-2-e^{4y}$ (enkelspårig)

$h'(y)=4e^y-4e^{4y}=4e^y(1-e^{3y})=0 \Leftrightarrow y=0$

$$\begin{cases} y<0 \Rightarrow h'(y)>0 \Rightarrow h \text{ växande} \\ y>0 \Rightarrow h'(y)<0 \Rightarrow h \text{ avtagande} \end{cases} \Rightarrow \underline{h(y)\le h(0)=1.}$$

$\lim_{y\to-\infty} h(y)=-2$; $\lim_{y\to\infty} h(y)=-\infty \Rightarrow h(0)=1$ globalt.

Problem 2.70 (Sid. 9)

Lösning

Stationära punkter är derivatsystemets nollställen:

$$\frac{\partial T}{\partial x}=\frac{\partial T}{\partial y}=0 \Rightarrow \begin{cases} \dfrac{6x}{1+(x^2+y)^2}=-\dfrac{12x}{(1+x^2+y^2)^2} \\ \dfrac{3}{1+(x^2+y)^2}=-\dfrac{12y}{(1+x^2+y^2)^2} \end{cases} \Rightarrow \frac{6x}{3}=\frac{12x}{12y}=\frac{x}{y}$$

$\Leftrightarrow 2x=x/y \Leftrightarrow \underline{x=0} \vee \underline{y=\frac{1}{2}}.$

$x=0 \Rightarrow \left(\frac{\partial T}{\partial y}=0\right) \Rightarrow \frac{-3}{1+y^2}=\frac{12y}{(1+y^2)^2} \Leftrightarrow 1+y^2=-4y \Leftrightarrow$

$\Leftrightarrow y^2+4y=-1 \Leftrightarrow y=-2\pm\sqrt{3}.$

Resultat: Stationära är punkterna $(0,-2-\sqrt{3})$ och $(0,-2+\sqrt{3})$.

Problem 2.71 (Sid. 10)

Lösning

a) $\underline{f(x,y)=3+4x-4y-x^2-2y^2}$

(1) Stationära punkter

$\frac{\partial f}{\partial x}=4-2x,\ \frac{\partial f}{\partial y}=-4-4y;$

$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0 \Leftrightarrow 4-2x=0 \wedge -4-4y=0 \Leftrightarrow \underline{(x,y)=(2,-1).}$

(2) Lokala extrempunkter

$\frac{\partial^2 f}{\partial x^2}=-2,\ \frac{\partial^2 f}{\partial y^2}=-4,\ \frac{\partial^2 f}{\partial x\partial y}=0;$

$Q(h,k)=-2h^2-4k^2$, negativ definit; $(2,1)$ lokal maximipunkt.

b) $$\underline{f(x,y,z)=x^2+y^2+z^2-xy+2z+x}$$

(1) Stationära punkter

$\frac{\partial f}{\partial x}=2x-y+1,\ \frac{\partial f}{\partial y}=2y-x,\ \frac{\partial f}{\partial z}=2z+2;$

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=\frac{\partial f}{\partial z}=0 \Leftrightarrow \begin{cases}2x-y=-1\\ x-2y=0\\ z+1=0\end{cases} \Leftrightarrow \begin{cases}x=-1/3\\ y=-1/3\\ z=-1\end{cases};\ P_0:(-\tfrac{1}{3},-\tfrac{1}{3},-1)$$

(2) Lokala extrempunkter

$\frac{\partial^2 f}{\partial x^2}=2,\ \frac{\partial^2 f}{\partial y^2}=2,\ \frac{\partial^2 f}{\partial x\partial y}=-1,\ \frac{\partial^2 f}{\partial z^2}=2;\ \frac{\partial^2 f}{\partial x\partial z}=\frac{\partial^2 f}{\partial y\partial z}=0.$

$$Q(h,k,l)=2h^2+2k^2+2l^2-2hk=$$
$$=2(h^2+k^2+l^2-hk)=$$
$$=2\left((h-\tfrac{1}{2}k)^2+\tfrac{3}{4}k^2+l^2\right)\ \text{pos. definit.}$$

Punkten $P_0:(-\frac{1}{3},-\frac{1}{3},-1)$ är lokal min/pkt.

c) $$\underline{f(x,y)=xe^{-2x^2-y^2}}$$

(1) Stationära punkter

$\frac{\partial f}{\partial x}=e^{-2x^2-y^2}-4x^2e^{-2x^2-y^2}=(1-4x^2)e^{-2x^2-y^2};$

$\frac{\partial f}{\partial y}=-2xye^{-2x^2-y^2};$

forts

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0 \Rightarrow \begin{cases}1-4x^2=0\\ 2xy=0\end{cases} \Leftrightarrow \begin{cases}x=\frac{1}{2}\\ y=0\end{cases} \vee \begin{cases}x=-\frac{1}{2}\\ y=0\end{cases};$$

(2) Lokala extrempunkter

$\frac{\partial^2 f}{\partial x^2}=-8xe^{-2x^2-y^2}+4x(4x^2-1)e^{-2x^2-y^2}=4x(4x^2-3)e^{-2x^2-y^2};$

$\frac{\partial^2 f}{\partial y^2}=-2xe^{-2x^2-y^2}+4xy^2e^{-2x^2-y^2}=2x(2y^2-1)e^{-2x^2-y^2};$

$\frac{\partial^2 f}{\partial x\partial y}=-2ye^{-2x^2-y^2}+8x^2ye^{-2x^2-y^2}=2y(4x^2-1)e^{-2x^2-y^2};$

$\underline{(x,y)=(\frac{1}{2},0)}$: $f''_{xx}=-4e^{-1/2},\ f''_{yy}=-e^{1/2},\ f''_{xy}=0;$

$Q(h,k)=-e^{-1/2}(4h^2+k^2)$, negativt definit; $(\frac{1}{2},0)$ är en lokal max/pkt.

$\underline{(x,y)=(-\frac{1}{2},0)}$: $f''_{xx}=4e^{-1/2},\ f''_{yy}=1\cdot e^{-1/2},\ f''_{xy}=0.$

$Q(h,k)=e^{-1/2}(4h^2+k^2)$, positivt definit; $(-\frac{1}{2},0)$ är en lokal min/pkt.

d) $$\underline{f(x,y)=x+y-3\ln(2+xy),\ x,y>0.}$$

(1) Stationära punkter

$\frac{\partial f}{\partial x}=1-\frac{3y}{2+xy},\ \frac{\partial f}{\partial y}=1-\frac{3x}{2+xy};$

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0 \Rightarrow \begin{cases}\frac{3y}{2+xy}=1\\ \frac{3x}{2+xy}=1\end{cases} \Rightarrow \underline{(y=x)} \Rightarrow 3x=2+x^2 \Leftrightarrow$$

$$\Leftrightarrow x^2-3x+2=0 \Leftrightarrow x=1 \vee x=2 \Rightarrow (x,y)=(1,1),(2,2).$$