Kurskod: TAMS65 Provkod: TEN2

Solutions to the exam on 08 June 2020, 08:00-12:00

1 (3 points)

Solution. (1.1) Rose: $X \sim Bin(2000, p)$, $x = 125$ and Jack: $Y \sim Bin(3000, p)$, $y = 175$.
The likelihood function is

$$L(p) = L_1(p) \cdot L_2(p) = \binom{2000}{x} p^x (1-p)^{2000-x} \cdot \binom{3000}{y} p^y (1-p)^{3000-y}$$

Maximizing $L(p)$ is equivalent to maximize $\ln L(p)$ where

$$\ln L(p) = \ln\left(\binom{2000}{x}\binom{3000}{y}\right) + (x+y)\ln p + (2000-x)\ln(1-p) + (3000-y)\ln(1-p)$$

By taking $\frac{d \ln L(p)}{dp} = 0$, we get $\frac{x+y}{p} - \frac{5000-(x+y)}{1-p} = 0$.
Therefore $\hat{p}_{ML} = \frac{x+y}{5000} = 0.06$, since $\frac{d^2 \ln L(p)}{dp^2} < 0$.
(1.2) $\hat{p}_{ML} = \frac{x+y}{5000}$, so $\hat{P}_{ML} = \frac{X+Y}{5000}$.

$$E(\hat{P}_{ML}) = E(\frac{X+Y}{5000}) = p, \quad \text{it is unbiased.}$$

□

2 (2 points)

Solution. (2.1) $H_0 : \mu_1 = \mu_2$ $H_1 : \mu_1 \neq \mu_2$
(2.2)

$$\begin{aligned}
\beta(\mu_1 - \mu_2 = 0.5) &= P(H_0 \text{ is false, don't reject } H_0, \text{ if } \mu_1 - \mu_2 = 0.5) \\
&= P(TS \notin C, \text{ if } \mu_1 - \mu_2 = 0.5)) \\
&= P(-\lambda_{0.005} < \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{3^2}{100} + \frac{4^2}{100}}} < \lambda_{0.005}, \text{ if } \mu_1 - \mu_2 = 0.5) \\
&= P(\frac{-\lambda_{0.005}(0.5) - 0.5}{0.5} < \frac{\bar{X} - \bar{Y} - 0.5}{0.5} < \frac{\lambda_{0.005}(0.5) - 0.5}{0.5}) \\
&\approx \Phi(1.58) - \Phi(-3.58) \approx 0.9427.
\end{aligned}$$

□

3 (3 points)

Solution. (3.1) The sampling distribution is

$$\frac{\bar{X} - \mu}{\mu/\sqrt{n}} \approx N(0,1)$$

95% confidence interval for μ is

$$I_\mu = \left(\frac{\bar{x}}{1 + \frac{\lambda_{0.025}}{\sqrt{n}}}, \quad \frac{\bar{x}}{1 - \frac{\lambda_{0.025}}{\sqrt{n}}}\right) \approx (29.75, 52.57).$$

(3.2) $p = P(X > 80) = \int_{80}^{\infty} \frac{1}{\mu} e^{-\frac{x}{\mu}} dx = e^{-\frac{80}{\mu}}$.
We can get 95% one-sided lower bound confidence interval for μ is

$$I_\mu = \left(\frac{\bar{x}}{1 + \frac{\lambda_{0.05}}{\sqrt{n}}}, \quad \infty\right) = (30.83, \infty).$$

Therefore, 95% one-sided lower bound confidence interval for p is $I_p = (a, \infty) = (e^{-\frac{80}{30.83}}, 1) \approx (0.07, 1)$.

□

4 (4 points)

Solution. (4.1) Since population standard deviation σ is related to three samples, so

$$\hat{\sigma}^2 = s^2 = \frac{(n_1-1)s_1^2 + (n_2-1)s_2^2 + (n_3-1)s_3^2}{n_1+n_2+n_3-3} = 32.06.$$

Thus the sampling distribution for μ_1 is $\frac{\bar{X}_1-\mu_1}{S/\sqrt{n_1}} \sim t(n_1+n_2+n_3-3)$.
So 95% confidence interval for μ_1 is

$$I_{\mu_1} = \bar{x}_1 \mp t_{0.025}(15)\cdot\frac{s}{\sqrt{n_1}} = 249.2 \mp (2.13)\cdot\sqrt{\frac{32.06}{5}} \approx (243.81, 254.59).$$

(4.2) The sampling distribution for $\mu_1 - 1.5\mu_2$ is

$$\frac{(\bar{X}_1 - 1.5\bar{X}_2) - (\mu_1 - 1.5\mu_2)}{S\cdot\sqrt{\frac{1}{n_1}+\frac{1.5^2}{n_2}}} \sim t(n_1+n_2+n_3-3)$$

Method I: Use 99% one-sided lower bound confidence interval for $\mu_1 - 1.5\mu_2$

$$I_{\mu_1-1.5\mu_2} = (\bar{x}_1 - 1.5\bar{x}_2) - t_{0.01}(15)\cdot s\cdot\sqrt{\frac{1}{n_1}+\frac{1.5^2}{n_2}}, \infty) \approx (40.12, \infty).$$

We can see that $\mu_1 - 1.5\mu_2 \geq 40.12 > 0$, so it is possible that $\mu_1 > 1.5\mu_2$ with 99% confidence.
Method II: Make a test. □

5 (3 points)

Solution. Note: $n = 100$

X	0	1	2	3	4	5	6
Groups	1	2	3	4	5	6	7
N_i	10	13	25	25	14	9	4
p_i	0.072	0.189	0.249	0.218	0.144	0.076	0.033
np_i	7.2	18.9	24.9	21.8	14.4	7.6	3.3

Note that, $\hat{\mu} = \bar{x} = \frac{0\cdot10+1\cdot13+\ldots+6\cdot4}{100} \approx 2.63$.

$$p_1 = P(X=0) = \frac{\mu^0}{0!}e^{-\mu} \approx e^{-2.63} \approx 0.072$$
$$p_2 = P(X=1) = \frac{\mu^1}{1!}e^{-\mu} \approx 2.63\cdot e^{-2.63} \approx 0.189$$
$$\ldots$$
$$p_7 = P(X=6) = \frac{\mu^6}{6!}e^{-\mu} \approx \frac{2.63^6}{6!}\cdot e^{-2.63} \approx 0.033$$

Because $\sum_{i=1}^{7} p_i = 0.981 < 1$, so we add group 8,

$$\text{Group 8} \quad X \geq 7, \quad N_8 = 0, \quad p_8 = 0.019, \quad np_8 = 1.9.$$

Because $np_7 < 5$ and $np_8 < 5$, so we combine these groups. Thus we get new group 7 which is

$$\text{Group 7} \quad X \geq 6, \quad N_7 = 4, \quad p_7 = 0.052, \quad np_7 = 5.2,$$

which gives $k = 7$. Then

$$TS = \sum_{i=1}^{7} \frac{(N_i - np_i)^2}{np_i} \approx 3.93 \text{ and } C = (\chi^2_{0.05}(7-1-1), \infty) = (11.07, \infty)$$

$TS \notin C$, don't reject H_0, i.e. it is possible to assume $X \sim Po(\mu)$. □