Exam in Statistical Methods, 2017-11-27

Time allowed: kl: 8-12

Allowed aids: Calculator. One handwritten A4 paper (both sides) with the students own notes.

Assisting teacher: Hector Rodriguez-Deniz

Grades: A=19-20 points, B=17-18p, C=14-16p, D=12-13p, E=10-11p, F=0-9

Provide a detailed report that shows motivation of the results.

1

Let $f(y) = ce^{-2y},\ 0 \le y \le \infty$ be the density function of the random variable Y.

a) Find c so that $f(y)$ is a density function . 1p

b) Find the cumulative distribution function $F(y)$, and show that $\lim_{y\to 0} F(y) = 0$ and $\lim_{y\to\infty} F(y) = 1$. 1p

c) Calculate the probability $P(1 \le Y \le 2)$. 1p

2

Let the bivariate random variable (Y_1, Y_2) have joint density function:

$$f(y_1, y_2) = \begin{cases} 2, & 0 \le y_1 \le y_2 \le 1 \\ 0, & elsewhere \end{cases}$$

a) Find the marginal distributions $f(y_1)$ and $f(y_2)$. Check if y_1, y_2 are independent, by using the definition of independent random variables. 2p

b) Find $f(y_1|y_2)$. Can you identify the result as a known distribution?. 1p

c) Find $P(Y_2 \ge 0.8)$. 1p

3

Let $Y_1, Y_2, \dots, Y_n$ denote n independent and identically distributed random variables from the following distribution with parameter α,

$$f(y|\alpha) = \begin{cases} \alpha y^{\alpha-1}/3^{\alpha}, & 0 \le y \le 3 \\ 0, & elsewhere \end{cases}$$

a) Show that $E[Y] = 3\alpha/(\alpha+1)$. 2p

b) Calculate the method-of-moments estimator for α 1p

4

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a Bernoulli distribution. The probability function is given by $p(y|\theta) = \theta^y(1-\theta)^{1-y}$, where $y = \{0,1\}$ and $0 < \theta < 1$ is the success probability.

a) Estimate θ using Maximum Likelihood method. 3p

5

We are interested in study the relationship between body fat and several possible predictor variables. The sample consists of 20 females with ages between 25-30 years, including the following variables:

Y = Amount of body fat (%)
X1 = Triceps skinfold thickness (mm)
X2 = Thigh circumference (cm)
X3 = Midarm circumference (cm)

We set up the following multiple linear regression model, $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon$, and estimated it using ordinary least-squares (OLS), obtaining the following estimates and results:

$$\hat{\boldsymbol{\beta}}' = [117.085\,, 4.334\,, -2.857\,, -2.186]$$

$$SSE = 98.405$$

$$(X'X)^{-1} = \begin{pmatrix} 1618.867 & 48.810 & -41.849 & -25.799 \\ 48.810 & 1.479 & -1.265 & -0.779 \\ -41.849 & -1.265 & 1.084 & 0.666 \\ -25.799 & -0.779 & 0.666 & 0.414 \end{pmatrix}$$

a) Test $H_0: \beta_2 = 0$ at 5% significance level. Interpret your result. 3p

6

We want to test whether a 5 kronor coin is fair by tossing it n times. If we assume independent tosses, we can model the experiment as a single Binomial sample with the following likelihood

$$p(y|\theta) = \binom{n}{y}\theta^y(1-\theta)^{n-y}$$

Where n is the number of tosses, y is the number of heads (or successes) and the probability of heads is $\theta = p$. Now we place a Beta prior on θ, i.e. $\theta \sim Beta(\alpha, \beta)$.

a) Find the posterior distribution of θ, and an expression for the estimator $\hat{\theta}_{Bayes}$ 2p

b) In order to collect some data, we toss the coin $n = 10$ times, and we obtain $y = 5$ number of heads. Let's now define two different prior distributions for θ,

$p1(\theta) \sim Beta(\alpha_1 = 2, \beta_1 = 2)$
$p2(\theta) \sim Beta(\alpha_2 = 2, \beta_2 = 20)$

See Figure 1 for a graphical representation of their density. Your task is to calculate two Bayes estimates $\hat{\theta}_{Bayes\_1}, \hat{\theta}_{Bayes\_2}$ by using each prior. Do you think our prior beliefs have influenced the posterior estimates of θ here?

1p

c) Now we collect much more data: we toss the coin $n = 500$ times and obtain $y = 240$ heads. Recalculate the Bayes estimates $\hat{\theta}_{Bayes\_1}, \hat{\theta}_{Bayes\_2}$ using the same priors as in (b). What happened now with the posterior estimates after collecting so many data?

1p

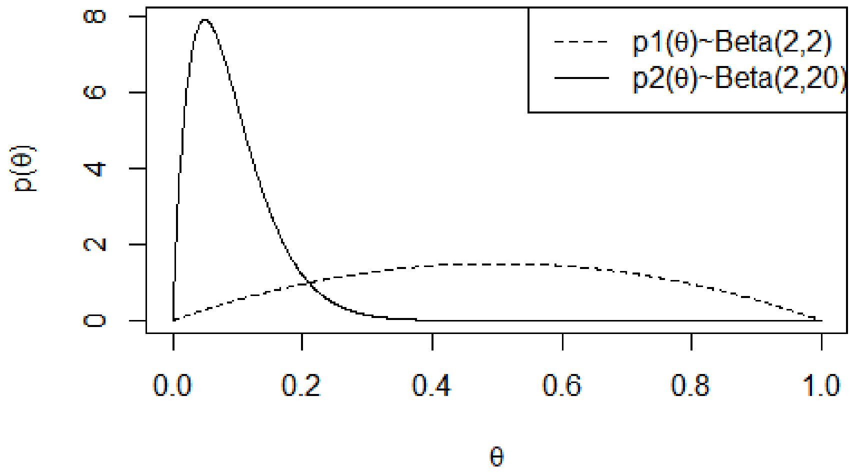


Figure 1 – Prior distributions for Binomial parameter $\theta = p$

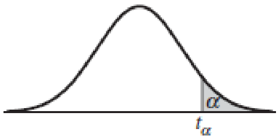
Discrete Distributions

Distribution	Probability Function	Mean	Variance
Binomial	$p(y) = \binom{n}{y} p^y (1-p)^{n-y};$ $y = 0, 1, \ldots, n$	np	$np(1-p)$
Geometric	$p(y) = p(1-p)^{y-1};$ $y = 1, 2, \ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Hypergeometric	$p(y) = \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}};$ $y = 0, 1, \ldots, n$ if $n \le r$, $y = 0, 1, \ldots, r$ if $n > r$	$\frac{nr}{N}$	$n\left(\frac{r}{N}\right)\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right)$
Poisson	$p(y) = \frac{\lambda^y e^{-\lambda}}{y!};$ $y = 0, 1, 2, \ldots$	λ	λ
Negative binomial	$p(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r};$ $y = r, r+1, \ldots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$

Continuous Distributions

Distribution	Probability Function	Mean	Variance
Uniform	$f(y) = \frac{1}{\theta_2 - \theta_1};\ \theta_1 \le y \le \theta_2$	$\frac{\theta_1 + \theta_2}{2}$	$\frac{(\theta_2 - \theta_1)^2}{12}$
Normal	$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\left(\frac{1}{2\sigma^2}\right)(y-\mu)^2\right]$ $-\infty < y < +\infty$	μ	σ^2
Exponential	$f(y) = \frac{1}{\beta} e^{-y/\beta};\quad \beta > 0$ $0 < y < \infty$	β	β^2
Gamma	$f(y) = \left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\right] y^{\alpha-1} e^{-y/\beta};$ $0 < y < \infty$	$\alpha\beta$	$\alpha\beta^2$
Chi-square	$f(y) = \frac{(y)^{(v/2)-1} e^{-y/2}}{2^{v/2}\Gamma(v/2)};$ $y^2 > 0$	v	$2v$
Beta	$f(y) = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] y^{\alpha-1}(1-y)^{\beta-1};$ $0 < y < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Table 5 Percentage Points of the t Distributions



$t_{.100}$	$t_{.050}$	$t_{.025}$	$t_{.010}$	$t_{.005}$	df
3.078	6.314	12.706	31.821	63.657	1
1.886	2.920	4.303	6.965	9.925	2
1.638	2.353	3.182	4.541	5.841	3
1.533	2.132	2.776	3.747	4.604	4
1.476	2.015	2.571	3.365	4.032	5
1.440	1.943	2.447	3.143	3.707	6
1.415	1.895	2.365	2.998	3.499	7
1.397	1.860	2.306	2.896	3.355	8
1.383	1.833	2.262	2.821	3.250	9
1.372	1.812	2.228	2.764	3.169	10
1.363	1.796	2.201	2.718	3.106	11
1.356	1.782	2.179	2.681	3.055	12
1.350	1.771	2.160	2.650	3.012	13
1.345	1.761	2.145	2.624	2.977	14
1.341	1.753	2.131	2.602	2.947	15
1.337	1.746	2.120	2.583	2.921	16
1.333	1.740	2.110	2.567	2.898	17
1.330	1.734	2.101	2.552	2.878	18
1.328	1.729	2.093	2.539	2.861	19
1.325	1.725	2.086	2.528	2.845	20
1.323	1.721	2.080	2.518	2.831	21
1.321	1.717	2.074	2.508	2.819	22
1.319	1.714	2.069	2.500	2.807	23
1.318	1.711	2.064	2.492	2.797	24
1.316	1.708	2.060	2.485	2.787	25
1.315	1.706	2.056	2.479	2.779	26
1.314	1.703	2.052	2.473	2.771	27
1.313	1.701	2.048	2.467	2.763	28
1.311	1.699	2.045	2.462	2.756	29
1.282	1.645	1.960	2.326	2.576	inf.