Exam in Statistical Methods, 2017-10-25

Time allowed: kl: 8-12

Allowed aids: Calculator. One handwritten A4 paper (both sides) with the students own notes.

Assisting teacher: Hector Rodriguez-Deniz

Grades: A=19-20 points, B=17-18p, C=14-16p, D=12-13p, E=10-11p

Provide a detailed report that shows motivation of the results.

1

Let a uniformly distributed random variable Y with parameters θ_1, θ_2 such that

$$f(y) = \begin{cases} \frac{1}{\theta_2 - \theta_1}, & \theta_1 \le y \le \theta_2 \\ 0 & , elsewhere \end{cases}$$ be the density function of Y.

a) Show that $E[Y] = \frac{\theta_1 + \theta_2}{2}$. 1p

b) Find the cumulative distribution function $F(y)$, and show that $F(y) = y$ when $Y \sim Uniform(0,1)$. 1p

c) Calculate the probability $P(5 \le Y \le 10)$ when $Y \sim Uniform(0,20)$. 1p

2

Let the bivariate random variable (Y_1, Y_2) have joint density function:

$$f(y_1, y_2) = \begin{cases} c y_1 y_2, & 0 \le y_2 \le y_1 \le 1 \\ 0, & elsewhere \end{cases}$$

a) Find c so that $f(y_1, y_2)$ is a density function. 2p

b) Find the marginal distribution $f(y_1)$ 1p

c) Find $P(Y_1 > 0.5, Y_2 > 0.5)$. 2p

3

Let $Y_1, Y_2, \dots, Y_n$ be a random sample from a normal distribution with mean μ and variance 1. The density is given by $f(y|\mu) = \frac{1}{\sqrt{2\pi}} e^{-0.5(y-\mu)^2}$

a) Estimate μ using Maximum Likelihood. 3p

b) Show that $\hat{\mu}_{MLE}$ is an unbiased and consistent estimator of μ. 2p

4

We collect the following random sample (y_i,x_i) of size n = 7 where Y is the dependent (output) variable and X the independent (input) variable.

Y	X
5	-2
4	-1
4	0
3	1
2	2
1	3
1	4

a) Set up the simple linear regression model and estimate β_0 and β_1. 2p

b) Test if the slope is zero. You may use without showing any calculations that SSE= 0,571. Use 5% significance level. Interpret your result. 1p

c) Find a 90% confidence interval for $E[Y] = \beta_0 + \beta_1 x^*$ at $x^* = 5$. 1p

5

We want to estimate the number of patients arriving in an emergency room at Linköping's Hospital during 22:00-23:00 on Fridays. We collect a sample of size n, and we use a Poisson distribution for modeling the data, i.e. $Y \sim Poisson(\lambda)$. Now we place a Gamma prior on λ, i.e. $\lambda \sim Gamma(\alpha, \beta)$.

a) Find the posterior distribution of λ, including the proportionality constant. 2p

b) If we collect a sample such that $n = 10, \sum y = 71$, what would be your Bayes estimate for the number of patients after using a prior $Gamma(4,2)$?. 1p

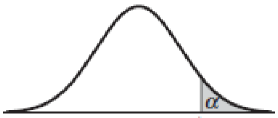
Discrete Distributions

Distribution	Probability Function	Mean	Variance
Binomial	$p(y) = \binom{n}{y} p^y (1-p)^{n-y};$ $y = 0, 1, \ldots, n$	np	$np(1-p)$
Geometric	$p(y) = p(1-p)^{y-1};$ $y = 1, 2, \ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Hypergeometric	$p(y) = \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}};$ $y = 0, 1, \ldots, n$ if $n \le r$, $y = 0, 1, \ldots, r$ if $n > r$	$\frac{nr}{N}$	$n\left(\frac{r}{N}\right)\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right)$
Poisson	$p(y) = \frac{\lambda^y e^{-\lambda}}{y!};$ $y = 0, 1, 2, \ldots$	λ	λ
Negative binomial	$p(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r};$ $y = r, r+1, \ldots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$

Continuous Distributions

Distribution	Probability Function	Mean	Variance
Uniform	$f(y) = \frac{1}{\theta_2 - \theta_1};\ \theta_1 \le y \le \theta_2$	$\frac{\theta_1 + \theta_2}{2}$	$\frac{(\theta_2 - \theta_1)^2}{12}$
Normal	$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\left(\frac{1}{2\sigma^2}\right)(y-\mu)^2\right]$ $-\infty < y < +\infty$	μ	σ^2
Exponential	$f(y) = \frac{1}{\beta} e^{-y/\beta};\quad \beta > 0$ $0 < y < \infty$	β	β^2
Gamma	$f(y) = \left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\right] y^{\alpha-1} e^{-y/\beta};$ $0 < y < \infty$	$\alpha\beta$	$\alpha\beta^2$
Chi-square	$f(y) = \frac{(y)^{(v/2)-1} e^{-y/2}}{2^{v/2}\Gamma(v/2)};$ $y^2 > 0$	v	$2v$
Beta	$f(y) = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] y^{\alpha-1}(1-y)^{\beta-1};$ $0 < y < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Table 5 Percentage Points of the *t* Distributions



$t_{.100}$	$t_{.050}$	$t_{.025}$	$t_{.010}$	$t_{.005}$	df
3.078	6.314	12.706	31.821	63.657	1
1.886	2.920	4.303	6.965	9.925	2
1.638	2.353	3.182	4.541	5.841	3
1.533	2.132	2.776	3.747	4.604	4
1.476	2.015	2.571	3.365	4.032	5
1.440	1.943	2.447	3.143	3.707	6
1.415	1.895	2.365	2.998	3.499	7
1.397	1.860	2.306	2.896	3.355	8
1.383	1.833	2.262	2.821	3.250	9
1.372	1.812	2.228	2.764	3.169	10
1.363	1.796	2.201	2.718	3.106	11
1.356	1.782	2.179	2.681	3.055	12
1.350	1.771	2.160	2.650	3.012	13
1.345	1.761	2.145	2.624	2.977	14
1.341	1.753	2.131	2.602	2.947	15
1.337	1.746	2.120	2.583	2.921	16
1.333	1.740	2.110	2.567	2.898	17
1.330	1.734	2.101	2.552	2.878	18
1.328	1.729	2.093	2.539	2.861	19
1.325	1.725	2.086	2.528	2.845	20
1.323	1.721	2.080	2.518	2.831	21
1.321	1.717	2.074	2.508	2.819	22
1.319	1.714	2.069	2.500	2.807	23
1.318	1.711	2.064	2.492	2.797	24
1.316	1.708	2.060	2.485	2.787	25
1.315	1.706	2.056	2.479	2.779	26
1.314	1.703	2.052	2.473	2.771	27
1.313	1.701	2.048	2.467	2.763	28
1.311	1.699	2.045	2.462	2.756	29
1.282	1.645	1.960	2.326	2.576	inf.