Examination Probability Theory

Linköpings Universitet, IDA, Statistik

Course code and name:	732A63 Probability Theory
Date:	2018/10/23, 8–12
Examinator:	Krzysztof Bartoszek phone 013–281 885
Allowed aids:	Pocket calculator
	Table with common formulae and moment generating functions (distributed with the exam)
	Table of integrals (distributed with the exam)
	Table with distributions from Appendix B in the course book (distributed with the exam)
Grades:	A= $[19-20]$ points
	B= $[17-19)$ points
	C= $[12-17)$ points
	D= $[10-12)$ points
	E= $[8-10)$ points
	F= $[0-8)$ points
Instructions:	Write clear and concise answers to the questions.
	Make sure to specify the support region for all density functions.

Problem 1 (5p)

Let X and Y be jointly distributed random variables such that

$$Y|X=x \ \sim \text{Binomial}(n,x) \ \text{ with } X \sim \text{Uniform}[0,1].$$

Compute EY, $\text{Var}\left[Y\right]$ and $\text{Cov}\left[X,Y\right]$ (without using the formula for the probability distribution nor the cumulative distribution function of Y).

Problem 2 (5p)

Let $X_1, X_2, X_3, \ldots, X_n$ be independent, continuous random variables with common cumulative distribution function $F(x)$, and consider the order statistic $(X_{(1)}, X_{(2)}, \ldots, X_{(n)})$. Compute $\mathbb{E}\left[F(X_{(n)}) - F(X_{(1)})\right]$

TIP: Recall that the joint density of $X_{(1)}$ and $X_{(n)}$ is given by

$$f_{X_{(1)},X_{(n)}}(x,y) = \begin{cases} n(n-1)(F(y)-F(x))^{n-2}f(y)f(x) & \text{for } x<y, \\ 0, & \text{otherwise.} \end{cases}$$

Problem 3 (5p)

Let Z be a random variable with finite first two moments. Denote by $\mathbb{M}[X]$ a median of X, i.e. a value such that $P(X \geq \mathbb{M}[X]) \geq 0.5$ and $P(X \leq \mathbb{M}[X]) \geq 0.5$. It is known that $\mathbb{M}[X]$ is the minimizer of $\mathbb{E}[|X - c|]$. Show that

$$|\,\mathbb{M}[X] - \mathbb{E}[X]\,| \leq \sqrt{\mathrm{Var}[X]}.$$

TIP: Jensen's inequality: for a convex function $\phi : \mathbb{R} \to \mathbb{R}$ it holds that

$$\phi(\mathbb{E}[X]) \leq \mathbb{E}[\phi(X)].$$

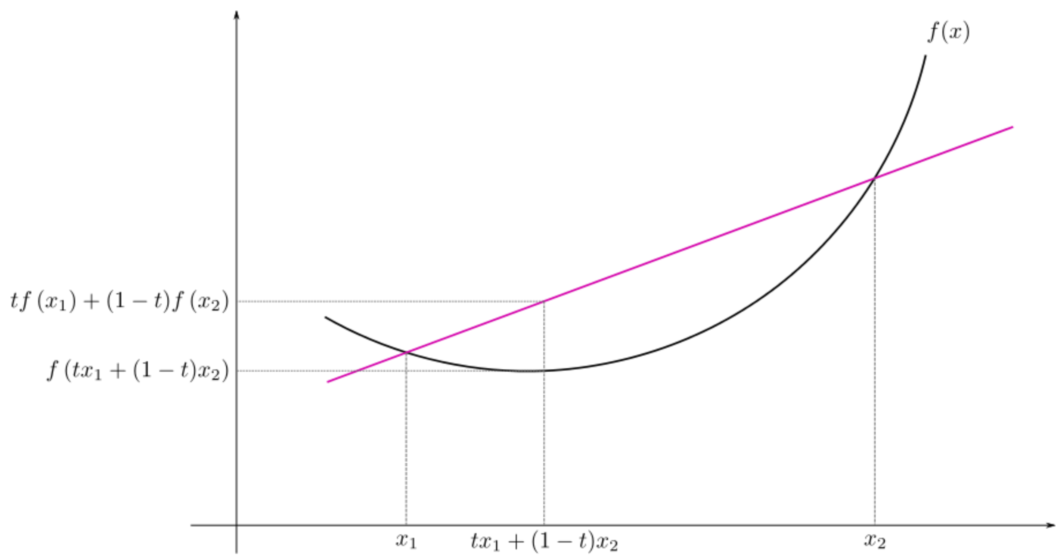

Figure 1: A function $f : \mathbb{R} \to \mathbb{R}$ is convex if "if the line segment between any two points on the graph of the function lies above or on the graph" (`https://en.wikipedia.org/wiki/Convex_function`). Graphic by "Eli Osherovich - Own work, CC BY-SA 3.0, `https://commons.wikimedia.org/w/index.php?curid=10764763`".

Problem 4 (5p)

a) (4p) Show that the average,

$$Z = \frac{1}{n} \sum_{i=1}^{n} X_i$$

of n independent Cauchy random variables has the Cauchy distribution too.

b) (1p) Why does the above result not violate the law of large numbers?