Chalmers University of Technology
Department of Signals and Systems
Automatic Control Group

Exam questions for Linear Control System Design, SSY285

April 2^{nd}, 2016

Cover page

1. Timeframe: 4 hours.
2. Examiner: Balazs Kulcsar, ☎1785, `kulcsar@chalmers.se`
3. Necessary condition to obtain the exam grade is to have the course's mandatory project (all assignments and the lab) approved. Without approved mandatory project, the archived exam results are invalid.
4. 20 points can be reached in total (with 0.5 point resolution), Lic/PhD students have to reach at least 12 points to pass. For Msc students Table 1 shows the grading system.

Table 1: Grading for Msc students

Points	Grade
10 . . . 11.5	3
12 . . . 15.5	4
16 . . . 20	5

5. During this written exam, it is *optionally* permitted to use printed materials such as:
   - *Either* of the course textbooks: (hardcopy or plain printed version, without notes inside!)
     Feedback systems, an introduction for scientists and engineers by K. J. Åström and R. M. Murray, ISBN-13: 978-0-691-13576-2
     OR
     Reglerteknikens grunder by Bengt Lennartson, ISBN: 91-44-02416-9.
   - 1 piece of A4 paper, with *hand written* notes on both sides. Copied sheet can not be used.
   - Pocket calculator (non-programmable, cleared memory, without graphical plotting function).
   - Mathematical handbook Beta (without notes inside!).
6. Note that phones, tablets, computers, any other communication devices are not allowed to use during the exam session. In scheduled exam session for the course at Chalmers, teacher(s) will show up in person in the first and last 60 min.
7. Examination results will be advertised not later than on April 11th 2016 (pingpong.chalmers.se). Inspection of results in person, April 11th, 10-11 am, E-building floor 5, room 5315.

Good luck!

Questions

1. Given the following transfer function matrix by,

$$G(s) = \begin{bmatrix} \frac{2}{(2s^2+3s+1)} & \frac{2s-3}{(2s+1)(s+1)} \\ \frac{4s+5}{(2s+1)(s+1)} & \frac{-2s-7}{2s^2+3s+1} \end{bmatrix}$$

a) Is the system IO stable and of minimum phase one? (**1 point**)

$$det(G(s)) = \frac{2}{(2s+1)(s+1)} \frac{-2s-7}{(2s+1)(s+1)} - \frac{2s-3}{(2s+1)(s+1)} \frac{4s+5}{(2s+1)(s+1)} = -\frac{8s^2+2s+1}{(2s+1)(s+1)}$$

Poles are at $-0.5, -1$, so the transfer function related system representation is IO stable and has also 2 zeros at $-\frac{1}{2}$ and $\frac{1}{4}$. The latter makes the system of non-minimum phase.

b) Is the following state space realization equivalent to the transfer function (only check)? (**1 point**)

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The quickest way the prove the mismatch (i.e. they can not be described by the same transfer fucntion) is to check the eigenvalues of A, $eig(A) = (-1, \frac{1}{2})$. Otherwise

$$\tilde{G}(s) = C(sI - A)^{-1}B = \begin{bmatrix} \frac{2s^2+3s}{(2s^2+s-1)} & \frac{2s-5}{(2s^2+s-1)} \\ \frac{4s+3}{(2s^2+s-1)} & \frac{-2s-9}{(2s^2+s+1)} \end{bmatrix}$$

c) Find and draw a transfer function equivalent state-space representation's block diagram including integrators, gains, signal streams! (**2 point**)

$$y_1(s) = \left(\frac{4}{2s+1} - \frac{2}{s+1}\right) u_1(s) + \left(\frac{5}{s+1} - \frac{8}{s+1}\right) u_2(s) = \frac{4}{2s+1}(u_1 - 2u_2) + \frac{5}{s+1}(-2u_1 + 5u_2)$$

$$y_2(s) = \frac{6}{2s+1}(u_1 - 2u_2) + \frac{1}{s+1}(-u_1 - 5u_2) = 6x_1(s) + x_2(s)$$

$$y_1 = 4x_1(s) + x_2(s) - \frac{u_1}{s+1}$$

$$y_2 = x_1 + x_2$$

$$2\dot{x}_1 = -x_1 + u_1 - 2u_2, \quad \dot{x}_2 = -u_1 - 5u_2, \quad \dot{x}_3 = -x_3 + u_1, \quad y_1 = 4x_1 + x_2 - x_3, \quad y_2 = x_1 + x_2$$

2. Given the following state-space representation by,

$$x(k+1) = \begin{bmatrix} 1 & \alpha + \beta \\ 0.5 & \alpha \cdot \beta \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k)$$

with $|\alpha| < \infty$, $|\beta| < \infty$.

a) Is the state-space representation reachable for all finite values of α, β? (**1 point**)

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 1 + 0.5(\alpha + \beta) \\ 0.5 & 0.5(1 + \alpha\beta) \end{bmatrix}$$

$$det(\mathcal{C}) = 0.5(\alpha + \beta) + \alpha\beta = 0$$

If α, β satisfies the above equation, the system gets unreachable.

b) Is the state-space representation observable for all finite values of α, β? (**1 point**)

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Observable system ($det(\mathcal{O} = 1)$), independent of any values for α, β.

c) With $\alpha = 0.5$, $\beta = 1$, find $y(2)$ if $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $u(0) = u(1) = 1$, i.e. step response value at sample 2. (**1 point**)

$$x_1 = \begin{bmatrix} 1 & 1.5 \\ 0.5 & 0.5 \end{bmatrix} x(0) + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u(0)$$
$$x_2 = \begin{bmatrix} 1 & 1.5 \\ 0.5 & 0.5 \end{bmatrix} x(2) + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u(1)$$
$$y(2) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(2) = 6.75$$

d) With $\alpha = 0.5$, $\beta = 1$, find the reachable (similarity) state transformation matrix T. With the help of T transform the system into a reachable canonical representation, $(\tilde{A}, \tilde{B}, \tilde{C})$! (**2 point**) Find the pole polynomial first ($det(sI - A) = s^2 - 1.5s - 0.25$),

$$\tilde{\mathcal{R}} = \begin{bmatrix} 1 & 1.5 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{R} = \begin{bmatrix} 1 & 1.75 \\ 0.5 & 0.75 \end{bmatrix}, \quad \mathcal{R}^{-1} = \begin{bmatrix} -6 & 14 \\ 4 & -8 \end{bmatrix}, \quad T = \tilde{\mathcal{R}}\mathcal{R}^{-1} = \begin{bmatrix} 0 & 2 \\ 4 & -8 \end{bmatrix}$$

3. Consider the following time discrete state observer problem as

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} v(k),$$
$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) + w(k),$$

where the uncorrelated scalar noise intensities are $r_v = 1$ and $r_w = 0$ standing for the normally distributed random signals v and w, respectively.

a) Find the optimal steady-state Kalman filter gain $\bar{L}$ to minimize the "delayed" state error covariance $\bar{P} = E\{\tilde{x}(k|k-1)\tilde{x}(k|k-1)^T\} \geq 0$ (**2 points**).

$$\bar{P} - A\bar{P}A^T - NR_vN^T + A\bar{P}C^T(R_w + C\bar{C}^T)^{-1}CPA^T$$
$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, R_v = 1, R_w = 0, N = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \bar{P} = \begin{bmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{bmatrix} \geq 0$$
$$p_1 = p_2 = p_{12} = 1 \Rightarrow \bar{L} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$A - \bar{L}C = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

b) Find the observer's poles with the previously obtained gain $\bar{L}$ (**1 point**).
They are both at zero.

4. Consider the following time discrete LQR design problem

$$x(k+1) = x(k) + u_1(k) + u_2(k)$$
$$J = \frac{1}{2}\sum_{j=0}^{\infty}\left(x^2(j) + u_1^2(j) + u_2^2(j)\right)$$

a) Find the optimal steady-state LQR gain $\bar{K}$ to minimize the cost function J (**2 points**).

$$A = 1,\;\; B = \begin{bmatrix} 1 & 1 \end{bmatrix},\;\; Q_x = 1,\;\; Q_u = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\;\; \bar{K} = ?$$

$$A^T \bar{A} - \bar{P} - A^T \bar{P} B (Q_u + B^T \bar{P} B)^{-1} B^T \bar{P} A + Q_x = 0 \Rightarrow 2p^2 - 2p - 1 \Rightarrow \bar{P} = p = 1,36$$

$$\bar{K} = (Q_u + B^T \bar{P} B)^{-1} B^T \bar{P} A = \begin{bmatrix} 0,36 \\ 0,36 \end{bmatrix}$$

b) Find the closed-loop pole with the previously obtained gain $\bar{K}$ and draw the block diagram for the closed-loop system (**1 point**).

$$A - B\bar{K} = 1 - 20,36 = 0,267$$

5. In Figure 1 is given a closed-loop block-diagram.

(a) Find the complementary sensitivity function of the closed-loop system. (**1 point**)

$$T(s) = \frac{Y(s)}{R(s)} = \frac{\frac{1}{2(s+1)}}{1 + \frac{3}{2(s+1)}} = \frac{1}{2s+5}$$

(b) Compute the value for F such that $r_\infty = y_\infty$. (**1 point**)

$$R(s) = \frac{1}{s},\;\; r_\infty = 1 = y_\infty = \lim_{s \mapsto 0} sY(s) = \lim_{s \mapsto 0} s \frac{1}{s} T(s) = \frac{F}{5} = 1 \Rightarrow F = 5$$

(c) Find the sensitivity function and compute the Bode sensitivity integral criteria for the above system. (**1 point**)

$$1 - T(s) = S(s) = \frac{Y(s)}{D(s)} = \frac{1}{1 + \frac{3}{2(s+1)}} = \frac{2s+2}{2s+5}$$

$$\int_{-\infty}^{\infty} = \log |S(i\omega)| d\omega = 0$$

since, $S(s)$ has stable poles.

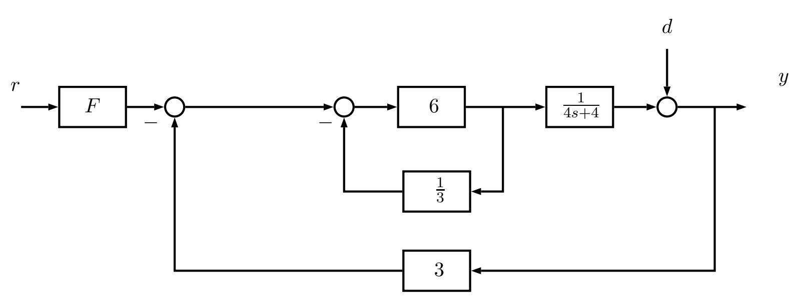

Figure 1: Blockdiagram

6. Given the following state-space representation and cost functional by,

$$\dot{x}(t) = 0.5x(t) + u(t)$$
$$J(u) = \frac{1}{2} \int_0^{\infty} \left(x^T(\tau)x(\tau) + u^T(\tau) Q_u u(\tau) \right) d\tau$$

The closed-loop system's pole is located at -1. Find Q_u and $\bar{P}$ (**2 point**).

$$A = 0,5, \quad B = -1, \quad Q_x = 1, \quad \Rightarrow Q_u = ?, \bar{P} = ?$$
$$A - B\bar{K} = -1 = 0,5 - 1Q_u^{-1}\bar{P}, \quad \bar{P} = 1,5Q_u$$
$$\bar{P}A + A^T\bar{P} - \bar{P}BQ_u^{-1}B\bar{P} + Q_x$$
$$\bar{P} + 1 - \bar{P}\frac{1,5}{\bar{P}}\bar{P} = 0, \quad \bar{P} = 2, \quad Q_u = \frac{4}{3}$$