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Automatic Control Group

Solution guide to exam questions for Linear Control System Design, SSY285

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Questions

1. Given the following transfer function matrix by,

$$G(s) = \begin{bmatrix} \frac{s}{s+1} & \frac{2(s+2)}{s+1} \\ \frac{1}{s+2} & \frac{2}{s+2} \end{bmatrix}$$

   a) Is this an input-output stable and minimum phase transfer function matrix? Motivate your answer! (**1 point**)

$$det(G(s)) = \frac{s}{s+1}\frac{2}{s+2} - \frac{2(s+2)}{s+1}\frac{1}{s+2} = \frac{2s-2(s+2)}{(s+1)(s+2)} = \frac{-4}{(s+1)(s+2)}$$

   Stable poles at $-1, -2$, so that is an IO stable system. No zeros, the system is of minimum phase.

   b) Find a state-space representation (hint: use only 2 states)! (**1 point**)

$$y_1(s) = \frac{s+1-1}{s+1}u_1(s) + \frac{2(s+1)+2}{s+1}u_2(s) = \frac{1}{s+1}(-u_1(s) + 2u_2(s)) + 2u_2(s) + u_1(s) \Rightarrow \ \mathcal{L}^{-1}\{\}$$
$$\dot{x}_1(t) = -x_1(t) - u_1(t) + 2u_2(t), \ \ y_1(t) = x_1(t) + 2u_2(t) + u_1(t)$$
$$y_2(s) = \frac{1}{s+2}u_1(s) + \frac{2}{s+2}u_2(s) = \frac{1}{s+2}(u_1(s) + 2u_2(s)) \Rightarrow \ \mathcal{L}^{-1}\{\}$$
$$\dot{x}_2(t) = -2x_2(t) + u_2(t) + 2u_2(t), \ \ y_2(t) = x_2(t)$$
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$
$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

   c) Is the derived state-space representation internally stable in Lyapunov sense? Draw its block diagram including integrators, gains, signal streams! (**1 point**)

   $eig(A) = -1, -2$ internally Lyapunov and asymptotically stable continuous time state space representation + Block diagram

2. Given the following state-space representation by,

$$x(k+1) = \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ \beta \end{bmatrix} u(k)$$
$$y(k) = \begin{bmatrix} 0 & \beta \end{bmatrix} x(k)$$

   with $|\alpha| < \infty$, $|\beta| < \infty$.

a) Is the state-space representation reachable for all finite values of α, β? (**1 point**)

$$\mathcal{R} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & \alpha \\ \beta & \beta \end{bmatrix} \Rightarrow \; det(\mathcal{R}) = \beta(1-\alpha)$$

For $\beta = 0$ and/or $\alpha = 1$ the system becomes unreachable otherwise not.

b) Is the state-space representation observable for all finite values of α, β? (**1 point**)

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & \beta \\ 0 & \beta \end{bmatrix} \Rightarrow \; det(\mathcal{R}) = 0$$

Regardless of the value for β and/or α the system is unobservable.

c) With $\alpha = 1$, $\beta = 0$ is the representation controllable? (**1 point**)

$$rank(\begin{bmatrix} B & AB \end{bmatrix}) = rank(\begin{bmatrix} A^2 & B & AB \end{bmatrix})$$

$$\mathcal{R} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow rank(\mathcal{R}) = 1 \text{ Only one independent row/column}$$

$$rank(\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}) = 2 \text{ Two independent rows}$$

$$1 = rank(\mathcal{R}) \neq rank(\begin{bmatrix} A^2 & B & AB \end{bmatrix} = 2$$

The system is not controllable neither.

d) With $\alpha = 1$, $\beta = 0$, find $y(2)$ if $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $u(0) = u(1) = 1$, i.e. step response value at sample 2. (**1 point**) Since $\beta = 0$ and there is no direct feedthrough term, any value of the state/input will be anihilated, so $y(2) = 0$.

e) With $\alpha = 1$, $\beta = 0$, find the reachable (similarity) state transformation matrix T. With the help of T transform the system into a reachable canonical representation, $(\tilde{A}, \tilde{B}, \tilde{C})$! (**1 point**)
Since the system is not reachable, the reachability similarity transformation matrix is not "existing" (its inverse is singular due to lack of reachability). The state space representation does not have reachable form.

3. Consider the following time discrete LQR design problem

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix}$$

$$J = \frac{1}{2} \sum_{j=0}^{\infty} \left(x_1^2(j) + x_2^2(j) + u_1^2(j) + u_2^2(j) \right)$$

a) Find the optimal steady-state LQR gain $\bar{K}$ to minimize the cost function J provided that the steady-state Ricatti solution has the structure $\bar{P} = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix} > 0$ (**2 points**).
The solution by means of the discrete time CARE boils down to 2 independent scalar valued second order polynomial.

$$A = B = Q_u = Q_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \bar{P} = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix} > 0$$

$$P - A^T P - Q_x + A^T PB(B^T PB + Q_u)^{-1} B^T PA = 0$$

$$p_1^2 - p_1 - 1 = 0$$

$$p_2^2 - p_2 - 1 = 0$$

$$p_1 = p_2 = 1.618$$

$$\bar{P} = \begin{bmatrix} 1.618 & 0 \\ 0 & 1.618 \end{bmatrix} > 0$$

Note, with cost weight $Q_x = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$

$$
\begin{aligned}
p_1^2 - p_1 &= 0 \\
p_2^2 - 2p_2 - 2 &= 0 \\
\bar{P} &= \begin{bmatrix} 0 & 0 \\ 0 & 2.7 \end{bmatrix} > 0
\end{aligned}
$$

b) Find the closed-loop poles with the previously obtained gain $\bar{K}$ (**1 point**).

$$
\bar{K} = (B^T P B + Q_u)^{-1} B^T P A = \begin{bmatrix} \frac{p_1}{p_1+1} & 0 \\ 0 & \frac{p_1}{p_1+1} \end{bmatrix}
$$

$$
A - B\bar{K} = \begin{bmatrix} 1 - \frac{p_1}{p_1+1} & 0 \\ 0 & 1 - \frac{p_1}{p_1+1} \end{bmatrix} \Rightarrow \quad p_1 = p_2 = 0.382
$$

Note, with cost weight $Q_x = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$

$$
\begin{aligned}
\bar{P} &= \begin{bmatrix} 0 & 0 \\ 0 & 2.7 \end{bmatrix} > 0 \\
\bar{K} &= \begin{bmatrix} 0 & 0 \\ 0 & 0.74 \end{bmatrix} \\
A - B\bar{K} &= \begin{bmatrix} 1 & 0 \\ 0 & 0.267 \end{bmatrix}
\end{aligned}
$$

4. Given the following system and quadratic cost functional to minimize

$$
\begin{aligned}
&\dot{x}(t) = \frac{1}{2}x(t) + u(t) + w(t) \\
&y(t) = x(t) + v(t) \\
&J = \lim_{T \mapsto \infty} E\left\{ \frac{1}{2} \int_0^T \left(u^2(\tau) + 2x^2(\tau) d\tau \right) \right\}
\end{aligned}
$$

where v and w are normally distributed, zero mean and independent random processes with covariance intensity matrices $R_v = 30$, $R_w = 1$. Design a continuous time LQG controller.

a) Find the optimal control input gain $\bar{K}$ by LQR(**1 point**).

$$
\begin{aligned}
&A = 0.5, B = C = Q_u = R_w = N = 1, R_v = 30, Q_x = 2 \\
&\bar{P}_K A + A^T \bar{P}_K - \bar{P}_K B Q_u^{-1} B^T \bar{P}_K + Q_x = 0 \\
&\bar{P}_K - \bar{P}_K^2 + 2 = 0 \Rightarrow \bar{P}_K^2 - \bar{P}_K - 2 = (\bar{P}_K + 1)(\bar{P}_K - 2) = 0, \Rightarrow \bar{P}_K = 2 > 0 \\
&\bar{K} = Q_u^{-1} B^T \bar{P}_K = 2
\end{aligned}
$$

b) Find the optimal steady-state Kalman gain $\bar{L}$ (**1 point**).

$$
\begin{aligned}
&A\bar{P}_L + \bar{P}_L A^T - \bar{P}_L C^T R_v^{-1} C \bar{P}_L + N R_w N^T = 0, \\
&\bar{P}_L - \bar{P}_L^2/30 + 1 = 0, \Rightarrow P_L = 30.97 > 0 \\
&\bar{L} = \bar{P}_L C^T R_v^{-1} = 1.03
\end{aligned}
$$

c) By using $\tilde{x}(t) = x(t) - \hat{x}(t)$ with numerical values found for $\bar{K}$ and $\bar{L}$ previously, find the closed-loop dynamics, $\begin{bmatrix} \dot{x}(t) \\ \dot{\tilde{x}}(t) \end{bmatrix}$=? (**1 point**).

With $r = 0$

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\tilde{x}}(t) \end{bmatrix} = \begin{bmatrix} A - B\bar{K} & B\bar{K} \\ 0 & A - \bar{L}C \end{bmatrix} \begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix} = \begin{bmatrix} -1.5 & 2 \\ 0 & -0.53 \end{bmatrix} \begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix}$$

d) Draw the LQG controlled system's block diagram, including integrators, gains and signal steams (**1 point**).

Block diagram

e) Suppose $R_w \mapsto \infty$. Explain in a few sentences what are the consequences of the increase in the process covariance intensity (no numerical calculations needed) (**1 point**).

$R_w \mapsto \infty$, LQG/LTR the LQR loop transfer behaviour is reconstructed.

5. Given the closed-loop system as shown in Figure **??**

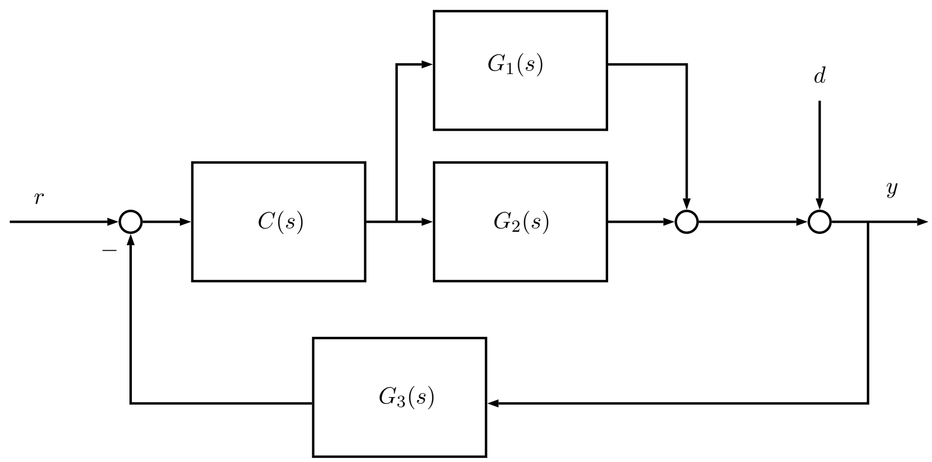

Figure 1: Closed-loop block diagram

$$G_1(s) = \frac{s(s+3)}{(s+2)(s+1)}, \quad C(s) = s + 3 + \frac{2}{s}, \quad G_3(s) = \frac{1}{5}, \quad G_2(s) = \frac{s}{s+3}$$

a) Find the sensitivity transfer function from about the input signal d onto y with $r = 0$ (**1 point**).

$$S(s) = \frac{1}{1 + C(s)(G_1(s) + G_2(s))G_3(s)}$$

$$C(s)(G_1(s) + G_2(s))G_3(s) = \frac{1}{s}(s^2 + 3s + 2) \cdot \left(\frac{s(s+3)}{(s+2)(s+1)} + \frac{s}{s+3} \right) \cdot \frac{1}{5} =$$

$$\frac{1}{s}((s+2)(s+1)) \cdot \left(s\frac{(s+3)^2 + (s+2)(s+1)}{(s+2)(s+1)(s+3)} \right) \frac{1}{5} = \frac{1}{5} \left(\frac{(s+3)^2 + (s+2)(s+1)}{(s+3)} \right)$$

$$S(s) = \frac{1}{1 + C(s)(G_1(s) + G_2(s))G_3(s)} = \frac{5(s+3)}{5(s+3) + (s+3)^2 + (s+2)(s+1)}$$

$$S(s) = \frac{5(s+3)}{5s + 15 + s^2 + 9 + 6s + s^2 + 3s + 2} = \frac{5s + 15}{2s^2 + 14s + 26} = \frac{1}{2} \frac{5s + 15}{(s + 3.5 - 0.866i)(s + 3.5 + 0.866i)}$$

Stable poles for $S(s)$.

b) What is the value of $\int_0^\infty \log |S(i\omega)| d\omega$ (**1 point**). Due to the fact $S(s)$ has stable poles, the Bode integral gets 0.

6. Given the following state-space representation and cost functional by,

$$\dot{x}(t) = 0.5x(t) + u(t)$$
$$J(u) = \frac{1}{2}\int_0^\infty \left(x^T(\tau)x(\tau) + u^T(\tau)Q_u u(\tau)\right) d\tau$$

a) The closed-loop system's pole is located at -1. Find Q_u and $\bar{P}$ (**2 point**).

$$A = 0.5, B = 1, Q_x = 1$$
$$Q_u = ?,\ \bar{P} = ?$$
$$A - B\bar{K} = -1 = 0.5 - Q_u^{-1}\bar{P} \Rightarrow \bar{P} = 1.5Q_u$$
$$\bar{P}A + A^T\bar{P} - \bar{P}BQ_u^{-1}B^T\bar{P} + Q_x = 0$$
$$\bar{P} + 1 - \bar{P}\frac{1.5}{\bar{P}}\bar{P} = 0 \Rightarrow \bar{P} = 2,\ \Rightarrow Q_u = \frac{4}{3}$$