CHALMERS UNIVERSITY OF TECHNOLOGY
Department of Signals and Systems
Control, Automation and Mechatronics

SSY280 Model Predictive Control
Exam 2015-03-19

14:00 – 18:00

Teacher: Bo Egardt, tel 3721.

The following items are allowed to bring to the exam:

- Chalmers approved calculator.
- One A4 sheet (front+back page) with your own notes.
- Mathematics Handbook (Beta).

Note: Solutions should be given in **English**! They may be short, but should always be **clear, readable and well motivated**!

Grading: The exam consists of 5 problems of in total 30 points. The nominal grading is 12 (3), 18 (4) and 24 (5).
Review of the grading is offered on April 2 at 12.00 – 13.00. If you cannot attend at this occasion, any objections concerning the grading must be filed in written form not later than two weeks after the regular review occasion.

GOOD LUCK!

Problem 1.

a. The constrained LQ type MPC studied during the course gives a nonlinear control law with a special structure. Describe this particular structure! (2 p)

b. A classmate of yours shows you a piece of code to be used for MPC of the constrained LQ type. The code is claimed to solve the constrained optimization problem

$$\begin{aligned}
\text{minimize} \quad & f(z) = \frac{1}{2} z^T Q z + p^T z + \varepsilon \|\Delta h\|^2, \quad Q > 0 \\
\text{subject to} \quad & Gz \leq h + \Delta h \\
& Az = b \\
& \Delta h \geq 0
\end{aligned}$$

What is this type of constrained optimization problem called? Which particular feature does the code provide? Explain your answer well! (2 p)

c. You are planning to apply an MPC to a system, which has 2 inputs and 2 measured/controlled outputs. The following statements also hold:

1. The plant model is a linear state space model of order 3
2. ~~All three states are measurable~~

However, you have realised that the two outputs are affected by unknown, slowly-varying load disturbances, which requires a modification of the plant model. Modify the two statements above so that they apply to the revised model. Explain! (2 p)

d. One attractive feature of MPC is the ability to impose constraints on critical states or outputs. If you want to use this feature, you should be aware of a possible disadvantage that requires some consideration. Please explain! (2 p)

e. The terminal cost plays an important role when proving stability of a receding horizon control strategy, and this can be made very explicit in the *unconstrained* LQ case. Thus, show by a brief calculation how an infinite-horizon LQ objective can be transformed into an equivalent finite-horizon objective. (2 p)

Solution:

a. *The control law is a piecewise affine (or piecewise linear) feedback in the state x.*

b. *The code is useful, because it solves a standard quadratic programming (QP) problem, defined by a quadratic objective and affine equality and inequality constraints. This is what is needed to implement an MPC of the constrained LQ type. The algorithm cited contains a specific feature, namely that the inequality constraints are soft: by the use of the slack variable Δh, the right hand side of the inequality can be increased. This can help avoiding unfeasibility at the cost of an added cost to the objective function.*

c. *This problem was modified due to an ambiguous formulation. After augmenting the plant model with two constant load disturbances, the model order is 5.*

d. *Constraints on states or outputs have the implication that the finite-time optimal control problem, which is solved as part of the RHC, may turn out to be infeasible for some initial states. In such cases you need to devise some work-around like softening the constraints or switch to a fall-back controller.*

e. *The optimal infinite-horizon objective can be re-written as*

$$V_\infty^0 = \min_{u(0:\infty)} \sum_{i=0}^{\infty} (x^T(i)Qx(i) + u^T(i)Ru(i))$$

$$= \min_{u(0:N-1)} \big(\sum_{i=0}^{N-1} (x^T(i)Qx(i)+u^T(i)Ru(i)) + \min_{u(N:\infty)} \sum_{i=N}^{\infty} (x^T(i)Qx(i)+u^T(i)Ru(i)) \big)$$

$$= \min_{u(0:N-1)} \big(\sum_{i=0}^{N-1} (x^T(i)Qx(i) + u^T(i)Ru(i)) + x^T(N)Px(N) \big)$$

where the last step follows from the fact that the optimal cost-to-go is quadratic in the initial state with P being the solution of the algebraic Riccati equation.

Problem 2.
Consider an MPC application with one control input u with lower and upper limits

$$-2 \leq u \leq 1 \tag{1}$$

and rate limits on the control moves according to

$$-0.5 \leq \Delta u \leq 0.5 \tag{2}$$

We want to apply an MPC based on control moves Δu and a control horizon $M = 2$. The control constraints (1) and (2) can be expressed as $F\Delta\mathcal{U} \leq e$, where $\Delta\mathcal{U}$ is the incremental control vector over the control horizon. Determine the matrix F and the vector e. (4 p)

Solution: *With k as the current sampling instant, the constraints on u imply constraints on $\mathcal{U} = [u(k)\ u(k+1)]^T$ as*

$$-2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leq \mathcal{U} \leq 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

which gives constraints on $\Delta\mathcal{U} = [u(k) - u(k-1);\ u(k+1) - u(k)]^T$ as

$$-2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k-1) + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \Delta\mathcal{U} \leq 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Compiling all these inequalities with the inequalities for the control rates gives the following result:

$$F\Delta\mathcal{U} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 0 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \Delta\mathcal{U} \leq \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(k-1) = e$$

Problem 3.
Consider the following constrained quadratic program:

$$\begin{aligned} \text{minimize} \quad & f(x) = \frac{1}{2}\|x\|^2 \\ \text{subject to} \quad & a^T x = b \end{aligned}$$

where a is a column vector and b is a scalar.

a. Compute the optimal solution and give a geometrical interpretation of the result when x is a 2-dimensional vector. (4 p)

b. Formulate and solve the dual problem and conclude that strong duality holds. (2 p)

Solution:

a. *The KKT conditions for the problem become*

$$\begin{bmatrix} I & a \\ a^T & 0 \end{bmatrix} \begin{bmatrix} x^0 \\ \nu^0 \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix}$$

By multiplying the first block row by a^T and subtracting the result from the last row, the following is obtained:

$$\begin{bmatrix} I & a \\ 0 & -a^T a \end{bmatrix} \begin{bmatrix} x^0 \\ \nu^0 \end{bmatrix} = \begin{bmatrix} 0 \\ b \end{bmatrix}$$

From this follows immediately that

$$x^0 = -\nu^0 a = \frac{b}{a^T a} \cdot a$$

When $n = 2$, x^0 is the point on the straight line $a^T x = b$, for which the corresponding vector from the origin is parallel to a. This vector is thus orthogonal to the line, i.e. x^0 is the point on the line that is closest to the origin.

b. *The Lagrangian becomes*

$$L = \frac{1}{2} x^T x + \nu(a^T x - b),$$

which is minimized by differentiating and setting the derivative equal to 0, giving $x = -\nu a$. Inserting this into the Lagrangian gives the dual function

$$q = -\frac{1}{2} a^T a \nu^2 - b\nu,$$

which can be seen to be concave. The dual problem is solved by again differentiating with respect to ν, giving $\nu^0 = -\frac{b}{a^T a}$ (which coincides with what was obtained in (a)) and the dual optimal value

$$d^0 = -\frac{1}{2}\frac{b^2}{a^T a} + \frac{b^2}{a^T a} = \frac{1}{2}\frac{b^2}{a^T a},$$

which is identical to the primal optimal value $p^0 = f(x^0)$. Hence, strong duality holds.

Problem 4.
You want to apply an MPC to a system described by a linear state space model (A, B, C) with 3 inputs and 4 outputs. Outputs number 1 and 2 are important controlled outputs with specified setpoints that you would like to attain in steady state. In addition, you would like to keep both control input number 1 and output number 3 small in steady state. Suggest a formulation of how to compute steady state targets, disregarding any constraints (in practice your solution would serve as a starting point for some trial-and-error tuning). (4 p)

Solution: *Define outputs 1 and 2 as controlled outputs, i.e.* $z = C_z x$ *with* $C_z = HC$*, where* $H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. *You need two inputs to attain the corresponding two setpoints* z_{sp}*, so one input remains "free". The desired values (in steady state) of* u_1 *and* y_3 *are both 0, so that by minimizing a weighted combination of* $u_{s,1}^2$ *and* $y_{s,3}^2$ *(*u_s *and* y_s *are steady state values), a compromise of the two remaining objectives can be achieved. The steady state target problem is thus as follows:*

$$\min_{x_s, u_s} \left(u_s^T R_s u_s + x_s^T Q_s x_s \right)$$

subject to

$$\begin{bmatrix} I - A & -B \\ HC & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ z_{sp} \end{bmatrix}$$

where

$$R_s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad Q_s = C^T \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} C$$

Problem 5.
A sampled data model for a DC motor is given by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) = \begin{bmatrix} 1 & 0.14 \\ 0 & 0.86 \end{bmatrix} x(k) + \begin{bmatrix} 0.21 \\ 2.79 \end{bmatrix} u(k) \\ y(k) &= Cx(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \end{aligned}$$

We will investigate an MPC for this system, based on the minimisation of the quadratic criterion

$$V(x, u(0), u(1)) = \sum_{i=1}^{4} \hat{y}^2(i)$$

with respect to the control signals $u(0)$ and $u(1)$, where we have (as usual) placed the time origin at the current sampling time. It can be seen that the prediction horizon is 4 and the control horizon is 2.

a. The predicted outputs over the prediction horizon can be computed from the equation

$$\begin{bmatrix} \hat{y}(1) \\ \hat{y}(2) \\ \hat{y}(3) \\ \hat{y}(4) \end{bmatrix} = \Psi x + \Phi \mathcal{U}$$

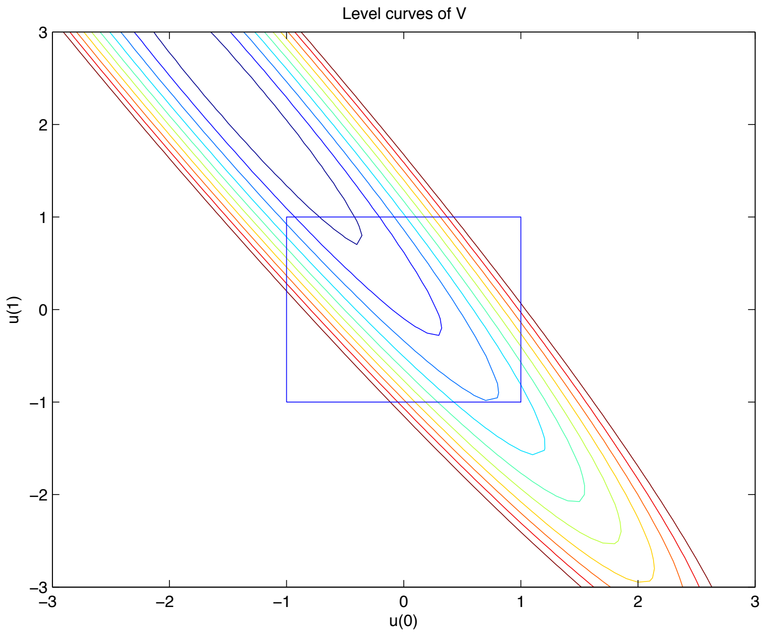
where x is the current state and $\mathcal{U} = (u(0)\ u(1))^T$. Using the assumption that the control signal is equal to 0 beyond the control horizon (i.e. in contrast to the usual assumption that the control signal stays constant beyond the control horizon), give expressions for Φ and Ψ in terms of the system matrices A, B and C. Do not bother about numerical values at this stage! (2 p)

b. Calculate the *unconstrained* optimal control law *numerically.* (2 p)
Hint: Here are some numerical expressions that *may be* useful:

$$\Psi = \begin{bmatrix} 1.00 & 0.14 \\ 1.00 & 0.26 \\ 1.00 & 0.36 \\ 1.00 & 0.45 \end{bmatrix} \qquad \Phi = \begin{bmatrix} 0.21 & 0 \\ 0.60 & 0.21 \\ 0.94 & 0.60 \\ 1.22 & 0.94 \end{bmatrix}$$

$$\Phi^T \Phi = \begin{bmatrix} 2.78 & 1.84 \\ 1.84 & 1.29 \end{bmatrix} \qquad \Phi^T \Psi = \begin{bmatrix} 2.98 & 1.08 \\ 1.75 & 0.70 \end{bmatrix}$$

c. We are now interested in the case with *control constraints.* In this case, the minimization of V should be carried out with the constraints $|u(0)| \leq 1$ and $|u(1)| \leq 1$. In the figure below, the level curves of V are shown for a specific value of the state x, namely $x = (1\ \ -3)^T$. The box indicates the control constraints.
Which value of the control signal is computed by the constrained MPC? How does this compare with the saturated ($-1 \leq u \leq 1$) control signal from the unconstrained control law? (2 p)



Solution:

a. *The matrices are found by iterating the state equations (see chapter 5 of the Lecture notes) and using the fact that $u(2) = u(3) = 0$:*

$$\Psi = \begin{bmatrix} CA \\ CA^2 \\ CA^3 \\ CA^4 \end{bmatrix} \qquad \Phi = \begin{bmatrix} CB & 0 \\ CAB & CB \\ CA^2B & CAB \\ CA^3B & CA^2B \end{bmatrix}$$

b. *Minimizing*

$$V = (\Psi x + \Phi \mathcal{U})^T (\Psi x + \Phi \mathcal{U}) = \mathcal{U}^T \Phi^T \Phi \mathcal{U} + 2x^T \Psi^T \Phi \mathcal{U} + x^T \Psi^T \Psi x$$

gives the unconstrained minimum

$$\mathcal{U} = -(\Phi^T \Phi)^{-1} \Phi^T \Psi x \approx \begin{bmatrix} -3.11 & -0.52 \\ 3.08 & 0.21 \end{bmatrix} x$$

Only the first control signal is used and gives the control law

$$u = \begin{bmatrix} -3.11 & -0.52 \end{bmatrix} x$$

c. *The unconstrained control law gives*

$$u = \begin{bmatrix} -3.11 & -0.52 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = -1.55,$$

which after saturation gives $u = -1$.
However, from the diagram of the level curves, it is seen that the minimum value inside the box is obtained at the upper boundary for $u(1) = 1$ *and approximately* $u(0) = -0.6$. *The latter is the control signal delivered by the constrained MPC.*

THE END!