Examination
SSY130 Applied Signal Processing
Suggested Solutions

14:00-18:00, January 10, 2018

Instructions

- *Responsible teacher:* Tomas McKelvey, phone number 8061. Teacher will visit the site of examination at approximately 14:45 and 16:30.
- Score from the written examination will together with course score determine the final grade according to the Course PM.
- Your preliminary grade is reported to you via email.
- Exam grading review will be held at 12:00-12:50 on January 24 in room 7430 (Landahlsrummet).

Allowed aids at exam:

- L. Råde and B. Westergren, Mathematics Handbook (any edition, including the old editions called Beta or copied sections from it), Formulaires et tables Mathématiques, Physique, Chimie, or similar.
- Any calculator
- One A4 size single sheet of paper with *handwritten* notes on both sides.

Note:

- The exam consists of 4 numbered problems.
- The ordering of the questions is arbitrary.
- All solutions should be well motivated and clearly presented in order to render a full score. Unclear presentation or adding, for the problem in questing, irrelevant information render a reduction of the score.
- Write solutions to each problem on a *separate* sheet of paper.
- The maximum score is 52 points.

Problems

1. Multiple Choice Questions. Select one option per subquestion. No motivation is needed.

   (a) A DSP system consists of a sampling unit which samples at a rate of 10 kHz, a processing unit which filters the signal using a FIR filter and a digital to analog converter (DAC) which operates in a zero-order hold (ZOH) mode. The input to the system is a single sinusoidal signal with a frequency of 4 kHz. Which statement below is *incorrect*: (2pt)

      i. The sample rate is high enough to ensure that no aliasing occur during the sampling process.
      ii. The output of the system is composed of infinitely many sinusoidal signals.
      iii. Since no aliasing occur during the sampling the output is a single sinusoidal with the original frequency of 4 kHz.
      iv. The FIR filter will change the amplitude and phase of the sampled sinusoidal signal.

   (b) A zero mean stationary white noise process with variance 1 is filtered through a FIR filter of length 2 with filter coefficients $h(0) = 1$ and $h(1) = 2$. The autocorrelation sequence $\phi_{yy}(k) = \mathbf{E}\, y(n)y(n+k)$ of the output $y(n)$ is: (4pt)

      i. $\phi_{yy}(0) = 5 \quad \phi_{yy}(1) = 2$
      ii. $\phi_{yy}(0) = 2 \quad \phi_{yy}(1) = 5$
      iii. $\phi_{yy}(0) = 2 \quad \phi_{yy}(1) = 3$
      iv. $\phi_{yy}(0) = 4 \quad \phi_{yy}(1) = 1$

   (c) Two real valued stationary stochastic processes x and y has a cross-correlation function $\phi_{xy}(k) = \mathbf{E}\, x(n)y(n+k)$ and power spectral density $S_{xy}(\omega) = \sum_{n=-\infty}^{\infty} \phi_{xy}(n)e^{-j\omega n\Delta t}$. Which statement below is *correct*: (3pt)

      i. $\phi_{yx}(n)$ has no relation to $\phi_{x,y}(n)$
      ii. $\phi_{yx}(n) = -\phi_{xy}(n)$
      iii. $\phi_{yx}(n) = \phi_{xy}(-n)$
      iv. $S_{yx}(\omega) = S_{xy}(\omega)$

   (d) A noise free single sinusoidal signal is sampled. The frequency of the signal is less than half the sampling frequency f_s. A total of N samples are available for analysis using DFT. The frequency estimate is taken as the frequency location where the DFT has its maximum magnitude. Which statement below is *incorrect.* (3pt)

      i. Using only the N signal samples available the best possible estimate will have a maximum error of $f_s/(2N)$
      ii. By zero padding the sampled signal with Z zeros we will effectively sample the DTFT of the signal at $Z+N$ points and hence the maximum error will decrease as Z increases.
      iii. Using only a finite number of signal samples the frequency can never be estimated correctly.
      iv. The DFT operation will provide equidistant samples (in frequency) of the DTFT of the signal.

   (e) An LMS filter can be described by the equations:

$$\hat{x}(k) = \mathbf{h}^T\mathbf{y}(k)$$
$$e(k) = x(k) - \hat{x}(k)$$
$$\mathbf{h} = \mathbf{h} + 2\mu\,\mathbf{y}(k)e(k)$$

   where $\mathbf{h}$ is a M-length vector with the FIR filter coefficients and $\mathbf{y}(k)$ is a vector with signal samples $y(k)$ and $M-1$ past signal samples. Which statement below is *incorrect*? (2pt)

i. A small step-length gives a fast converging filter
ii. For stability reasons the step-length must be small enough.
iii. The step-length should be chosen based on of the variance of the filter input $y(k)$
iv. For stability reasons the step-length must be a positive number

(f) The Least-Mean-Square (LMS) algorithm and the Recursive Least-Squares (RLS) algorithms can both be used for adaptive filtering. Which statement below is *incorrect:* (2pt)

i. The RLS algorithm has a more advanced step length adjustment and generally converges faster than LMS.
ii. RLS automatically selects the optimal step length.
iii. LMS is computationally more complex than the RLS algorithm.
iv. RLS with a forgetting factor is suitable when the system is slowly time-varying.

Solution:

a	iii	2pt
b	i	4pt
c	iii	3pt
d	iii	3pt
e	i	2pt
f	iii	2pt

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2. A sampled signal of length N, referred to as a *block*, is to be processed by an FIR filter of length M.

(a) Directly solving the convolution sum requires multiplication and addition. How many multiplications are needed to filter the block of data? (3pt)

(b) Describe a method which uses the FFT to solve the filtering problem. (5pt)

(c) How does the number of multiplications required to solve an FFT scale with the length of the signal to be transformed? (2pt)

Solution:

(a) Filtering results in an output of length $N + M - 1$.

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k), \quad n = 0, \dots, N + M - 1$$

The number of non-zero multiplications needed to produce one output sample is M when $M - 1 \leq n \leq N - 1$, i.e $M(N - M + 1)$. For the two edges a total of $2 * \sum_{k=0}^{M-1} k = M(M-1)$ multiplications are needed. And hence MN in total.

(b) Filtering with FFT can be applied. First the length of the FFT is determined (N_{FFT}) as the number which is a power of 2 such that $N_{\text{FFT}} >= N + M - 1$ (assuming we use the standard radix-2 method). Zero pad the input and the filter to the FFT length and calculate the FFT of both signals. Take the product between them and then apply the inverse FFT to calculate the output of the filter.

(c) In each FFT we need in the order of $N_{\text{FFT}} \log_2 N_{\text{FFT}}$ plus N_{FFT} multiplications.

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3. Consider the stochastic processes $x(k)$ and $y(x)$. The optimal Wiener filter $h(k)$ minimizes the variance of $e(n)$ where

$$e(n) = x(n) - \sum_{k=0}^{M-1} h(k)y(n-k).$$

The filter input $y(n)$ and desired signal $x(n)$ are both zero mean and have auto- and cross-correlation functions

$$\mathbf{E}\{y(0)y(n)\} = \phi_{yy}(n) = \begin{cases} \sigma_y^2, & n = 0 \\ 0, & n \neq 0 \end{cases}, \quad \mathbf{E}\{y(0)x(n)\} = \phi_{yx}(n) = \begin{cases} \alpha_1, & n = 0 \\ \alpha_2, & n = 1 \\ 0, & n < 0, n > 1 \end{cases}$$

$$\mathbf{E}\{x^2(n)\} = \sigma_x^2$$

- Derive an explicit expression for $\mathbf{E}\, e^2(n)$ (4pt)
- What is the length of the optimal filter? (2pt)
- Derive an expression for the coefficients of the optimal filter? (2pt)

Solution:

$$\mathbf{E}\, e^2 = \sigma_x^2 + h_0^2\sigma_y^2 + h_1\sigma_y^2 - 2\alpha_1 h_0 - 2\alpha_2 h_1 + \sum_{k=2}^{M-1} h_k^2 \sigma_y^2$$

Length is obviously 2 since setting $h_k = 0$ for $k > 1$ will minimize $\mathbf{E}\, e^2$. Setting gradient of $\mathbf{E}\, e^2$ w.r.t. h to zero yields

$$\begin{aligned} 2h_0\sigma_y^2 - \alpha_1 \\ 2h_1\sigma_y^2 - \alpha_2 \\ \Rightarrow \\ h_0 = \alpha_1/\sigma_y^2 \quad h_1 = \alpha_2/\sigma_y^2 \end{aligned}$$

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4. In this problem we will consider thea case when a measurement setup is influenced by disturbance originating from the mains power supply system. The sensor signal is sampled by a DSP system and we want to devise a suitable signal processing method to reduce the disturbance from the sampled sensor signal. Mathematically we can describe the sampled sensor signal as

$$y(n) = s(n) + m(n) \tag{1}$$

where $s(n)$ is the sampled desired sensor signal and the mains disturbance with a 50 Hz disturbance frequency, which is what is used in Europe, is given by

$$m(n) = \alpha \cos(2\pi 50 n\Delta t + \varphi) \tag{2}$$

where Δt is the sampling frequency. We assume $s(n)$ is a zero mean stochastic process independent of $m(n)$.

(a) A very simple approach to mitigating the 50 Hz disturbance is to filter the sampled signal $y(n)$ with a filter which has frequency response which is zero at +50 Hz and -50 Hz. Ideally, the frequency response should be 1 for all other frequencies to minimize the distortion on the desired signal $s(n)$. Practically this is not possible so we focus on placing zeros in the frequency response at +50 Hz and -50 Hz and have unit gain at the zero frequency. This can be accomplished with a FIR filter with 3 coefficients and is called a notch filter. Derive an expressions for the 3 filter coefficients such that the frequency response satisfy $H(0) = 1$, $H(2\pi 50) = 0$ and $H(-2\pi 50) = 0$.
Hint: The frequency response of a FIR filter of length 3 can be written

$$H(\omega) = \alpha(\beta_1 - e^{-j\omega\Delta t})(\beta_2 - e^{-j\omega\Delta t})$$

(5pt)

(b) A more clever way would be to use the fact that the frequency (50 Hz) of the mains disturbance is known. Describe an approach similar to Project 2 to mitigate the mains disturbance $m(n)$ (i.e. assume the disturbance signal is known, but with incorrect phase and magnitude). Give your answer as a signal diagram and discuss a suitable length of the FIR filter. (5pt)

(c) The disturbance signal can instead of (2) alternatively be described as

$$m(n) = a\cos(2\pi 50 n\Delta t) + b\sin(2\pi 50 n\Delta t) \quad (3)$$

Derive the relations between α, φ in (2) and a, b in (3). (3pt)

(d) The alternative representation of the mains disturbance given by (3) gives yet another alternative to remove the disturbance $m(n)$ from the measured signal. If we can estimate a and b from the signal $y(n)$ we can subsequently remove $m(n)$ by subtraction. The estimation of a and b can be (among other methods) be accomplished by Kalman filtering where we use the parameters a and b as the states. In this case the measurement equation becomes time varying (but known). Derive the state and measurement equations suitable for solving the problem using the time varying Kalman filter. (5pt)

Solution:

(a) Let $\omega_0 = 2\pi 50\Delta t$. Selecting $\beta_1 = e^{-j\omega_0}$ and $\beta_2 = \bar{\beta}_1$ directly ensure that the frequency response is zero at ± 50 Hz. Since $1 = H(0) = \alpha(\beta_1 - 1)(\beta_2 - 1) = \alpha(2 - 2\cos\omega_0)$ we get $\alpha = \frac{1}{2-2\cos\omega_0}$. The frequency response can also be written $H(\omega) = \alpha(1 - (\beta_1 + \beta_2)e^{-j\omega\Delta t} + e^{-2j\omega\Delta t}) = \alpha(1 - 2\cos\omega_0 e^{-j\omega\Delta t} + e^{-2j\omega\Delta t})$ which gives the impulse response coefficients $h_0 = h_2 = \alpha$ and $h_1 = -\alpha 2\cos(\omega_0)$.

(b) If we assume the processing unit also (for power reasons) is also connected to the power supply we can sample the mains voltage signal and use this as a disturbance source $m_s(n)$. Alternatively we can directly in the processing unit create a 50 Hz sinusoidal signal and use it as the disturbance source. We then filter the disturbance source through an FIR filter $\hat{h}$ to match the amplitude and phase of the disturbance $m(n)$, i.e. $\hat{m}(n) = \hat{h}(n) \star m_s(n)$. Finally this signal is subtracted from the samples of $y(n)$, i.e. $e(n) = y(n) - \hat{m}(n)$ is the noise reduced estimate of the desired signal. We use the LMS technique to tune the FIR filter by sequentially take small negative gradient steps of the function $e^2(n)$. To remove the sinusoidal signal we need to match the phase and amplitude of the disturbance $m(n)$. For a linear filter this is given by the complex value of the frequency function evaluated at the specific frequency $2\pi 50$. Since the frequency response for a FIR filter of length 2 at frequency $2\pi 50$ is given by $h_0 + h_1 e^{-j2\pi 50\Delta t}$ we note that the complex value can be selected arbitrary by a proper choice of real valued coefficients h_0 and h_1. A minimum length filter is hence 2. Particularly if $1/\Delta t$ is large compered to 50 Hz ill-conditioning will occur when using few filter coefficients. A faster and more robust performance is then obtained with more coefficients.

(c) Set $\omega_0 = 2\pi 50\Delta t$. Then

$$\begin{aligned} m(n) &= \alpha\cos(\omega_0 n + \varphi) = \mathrm{Re}(\alpha e^{j(\omega_0 n + \varphi)}) = \mathrm{Re}(\alpha e^{j\varphi} e^{j\omega_0 n}) \\ &= \mathrm{Re}(\alpha(\cos\varphi + j\sin\varphi)e^{j\omega_0 n}) = \alpha(\cos\varphi\cos(\omega_0 n) - \sin\varphi\sin(\omega_0 n)) \end{aligned}$$

and we obtain the identities $a = \alpha\cos\varphi$ and $b = -\alpha\sin\varphi$

(d) Set $\omega_0 = 2\pi 50\Delta t$. We can interpret $y(n) = a\cos(\omega_0 n) + b\sin(\omega_0 n) + s(n)$ as the measurement equation in the Kalman filter and hence let a and b be the unknown constants we want to estimate (or track). Here we treat $s(n)$ as the measurement error. If we asume a and b as constants (or slowly time varying) it is fair to assume that $a(n+1) \approx a(n)$ and the same for $b(n)$. With the state $x(n) = [a(n),\, b(n)]^T$ we let the state equation be

$$
\begin{aligned}
x(n+1) &= \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A} x(n) + w(n) \\
y(n) &= \underbrace{\begin{bmatrix} \cos(\omega_0 n) & \sin(\omega_0 n) \end{bmatrix}}_{C_n} x(n) + s(n)
\end{aligned} \tag{4}
$$

where $w(n)$ and $s(n)$ corresponds to the state and measurements noise respectively. This model makes sence as long as $s(n)$ is zero mean and independent from $m(n)$. The model also implicitly assumes that $s(n)$ is approximately a white noise signal, i.e. the power spectral density is approximately constant over frequencies.

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END - Good Luck!