Examination
SSY130 Applied Signal Processing
Suggested Solutions

14:00-18:00, January 13, 2016

Problems

1. A signal is known to contain 3 real valued sinusoids and is sampled at the rate of 12 kHz resulting in a total of 20 signal samples. Two of the sinusoids have equal amplitude and are large while the third sinusoid has an amplitude which is 5 times smaller. Figure 1 shows the magnitude of the DFT of the zero padded sampled signal with a total length of 100 samples. The locations of the 5 largest peaks are indicated in the graph. The location is given as the value of the DFT index k.

   (a) Assume all sinusoids have a frequency less than half the sampling frequency. Give an estimate of the frequencies (in Hz) of the 3 sinusoids. (4pt)

   (b) Explain why the magnitude of the DFT shown in Figure 1 have more than 3 peaks. (2pt)

   Solution:

   (a) Since all signals are real valued and have frequencies below the Nyquist frequency the peak locations below relative frequency 0.5 will give a good estimate of the freqency. Since two signals are larger than the third we expect to see two dominant peaks and one peak which is 5 times lower then the dominant ones. Peaks at locations $k = 4$, 11 and 31 är then suitable candidates. The frequencies are given as the index divided with the number of total signal samples including the zero-padding multiplied with the sampling frequency (here 100). Estimated frequencies are then $4/100 * 12$ kHz=0.48 kHz,$11/100 * 12$ kHz=1.3231 and 31/100*12 kHz=3.72 kHz.

   (b) Since the signal is real valued the DFT will also have three peaks corresponding to the negative frequencies. In the graph these are located above the Nyquist frequencies (since the DFT is periodic). The other peaks are the results from the sidelobes of the rectangular window. Clearly if the level of the third sinusoid would be even lower it would be difficult to make a correct estimation from the graph.

   ■

2. Consider the following linear *time varying* system given by

$$y(n) = -a(n)y(n-1) + bx(n)$$

where $y(n)$ is the noise free output of the system and $x(n)$ is the input signal. The output is measured by a sensor which yields the measured signal

$$y_m(n) = y(n) + e(n)$$

where $e(n)$ is the measurement noise which has zero mean and is independent of $x(n)$. Given that the input signal $x(n)$ is known we want to track the time varying parameter $a(n)$ based on the measured signal $y_m(n)$.

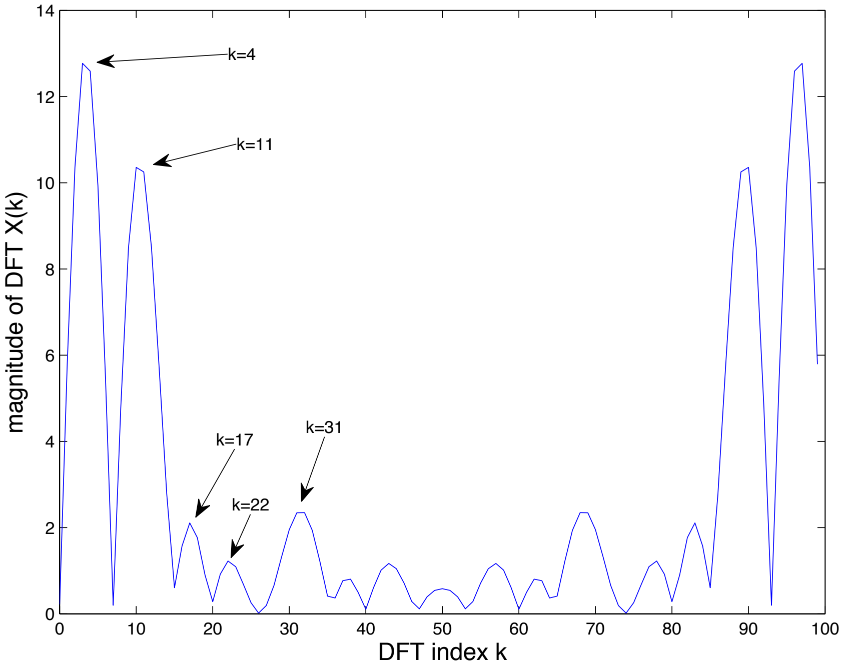


Figure 1: Magnitude of DFT.

(a) Assume first that $a(n) = a_1$ for all n, i.e. there is no time variation and that $e(n) = 0$. Show that, (for specific values of α_0 and α_1) a FIR filter with structure

$$\hat{x}(n) = \alpha_0 y_m(n) + \alpha_1 y_m(n-1)$$

yields $\hat{x}(n) = x(n)$. (5pt)

(b) Suggest an adaptive filter solution which will provide a method to track the time variability of the IIR filter coefficient $a(n)$. (7pt)

(c) Assume two scenarios:

i. $a(n) = 0.85 - 0.1\cos(0.01\pi n)$

ii. $a(n) = 0.85 - 0.1\cos(0.001\pi n)$

In which one of the two scenarios will the adaptive filter have the possibility to provide a more accurate tracking estimate. (3pt)

Solution:

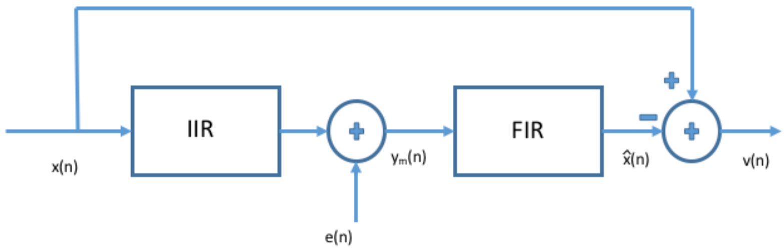
(a) Start by inserting the right hand of $y(n) = -a(n)y(n-1) + bx(n)$ into the definition of $\hat{x}(n)$. Since $e(n) = 0$ we also have $y_m(n) = y(n)$. This yields

$$\hat{x}(n) = \alpha_0(-a_1 y(n-1) + bx(n)) + \alpha_1 y(n-1)$$

If we select $\alpha_0 = 1/b$ and $\alpha_1 = a_1/b$ we obtain the desired result $\hat{x}(n) = x(n)$. The FIR filter is hence the inverse to the original IIR system. We can also obtain the same result by considering the the product of the two frequency functions:

$$\hat{X}(\omega) = \frac{\alpha_0 + \alpha_1 e^{-j\omega}}{1} \times \frac{b}{1 + a_1 e^{-j\omega}} X(\omega)$$

(b) In order to track the time varying coefficient $a(n)$ of the IIR filter we can use a FIR structure with two coefficients as in subproblem (a) and use the inverse modeling setup to estimate the two coefficents α_0 and α_1 according to the system setup shown below and where the adaptation rule is designed to minimize magnitude of the error signal $v(n)$. By using for example the LMS adaptive algorithm with a fixed step length the adaptive system will track the changes in the IIR system. As the noise is independent of $x(n)$ it will not influence the estimatin on average but will add to the variance of the estimation.



Denote by $\hat{\alpha}_0(n)$ and $\hat{\alpha}_0(n)$ the estimated coefficents obtained by the adaptive filter at sample n. Then the estimate of the time varying coefficent $a(n)$ is given by $\hat{a}(n) = \hat{\alpha}_1(n)/\hat{\alpha}_0(n)$.

(c) Since the inverse modeling adaptive filtering setup is based on *time-invariant* theory we can expect the methodology to work better if the paramter variation is slow. Comparing the two scenarios we see that the frequency of scenario ii is the lowest of the two. Hence we can expect scenario ii to yield the highest quality of the time varying coefficent $a(n)$.

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3. Assume the MATLAB vectors `x` and `h` are real valued and the length of the vector `x` is an even number. Consider the following MATLAB code:

```
Nx = length(x);
Nh = length(h);
xx = x(1:Nx/2)+i*x(Nx/2+1:Nx);
Nyy = Nx/2+Nh-1;
yy = ifft(fft(xx,Nyy).*fft(h,Nyy),Nyy);
y = [real(yy); zeros(Nx/2,1)] + [zeros(Nx/2,1); imag(yy)];
```

Explain what the code calculates. Recall that $\mathtt{i} = \sqrt{-1}$ (10pt) **Solution:** The MATLAB code combines two ideas to reduce the computational complexity.

(a) The first idea concerns using the DFT and IDFT to efficiently calulate the filtered output (convolution) for two finite real-valued signals `x` and `x2` of the same length using the FIR filter with impulse response `h`. We can calulate the first output according to

```
Ny = length(x)+lenght(h)-1;
y = ifft(fft(x,Ny).*fft(h,Ny))
```

Note that since both x and h are real valued, the result y is also real valued. Since both fft and ifft are linear operations we note that `fft(i*x2,Ny) = i*fft(x2,Ny)` and hence `ifft(fft(i*x2,Ny).*fft(h,Ny))` will yield a completly imaginary result where the imaginary part is equal to the desired result `y2=ifft(fft(x2,Ny).*fft(h,Ny))` i.e the result obtained when convolving x2 with h. Now using the linearity of the transform we can create a complex signal where the real part is signal x and the imaginary part is signal x2 and convolve this signal with h using the DFT method and then read out the resulting signals y and y2 from the real and imaginary parts of the complex result. In code we obtain

```
Ny = length(x)+lenght(h)-1;
yc = ifft(fft(x+i*x2,Ny).*fft(h,Ny));
y =  real(yc);
y2 = imag(yc);
```

and hence we have performed two real valued convolutions by only calulating one complex valued convolution using FFT.

(b) The second part used in the code uses the so-called overlap-and-add fft technique to calulate the convolution of a finite signal. The technique involves 1) dividing the input signal into two equal parts 2) convolve each part with the filter (using the DFT method) 3) add the results together to create the final result. In the third step care must be taken so the transient tail of the first signal part, which has a length one less then the filter length, is added to the beginning of output of the second signal part.

Hence, the listed MATLAB code calculates the convolution between x and the impulse response h by using DFT and IDFT of length `Nyy = Nx/2+Nh-1`. ■

4. Consider a deterministic Wiener filter problem where the filtered output is given by

$$\hat{x}(n) = \sum_{k=0}^{2} h(k)y(n-k)$$

and we seek the filter coefficients $h(k)$, $k = 0, 1, 2$ which minimize

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} (x(n) - \hat{x}(n))^2.$$

The input $y(n)$ is given by

$$y(n) = \cos(\frac{\pi}{2}n).$$

Let $r_{yx}(k)$ denote the sample based cross-correlation limit according to

$$r_{yx}(k) \triangleq \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} y(n)x(n+k).$$

The relation between $y(n)$ and $x(n)$ is captured by the sample based cross-correlations with numerical values as $r_{yx}(0) = -1$, $r_{yx}(1) = 2$ and $r_{yx}(2) = 1$.

(a) Show that the sample based auto-correlation matrix R_{yy} is given by

$$R_{yy} \triangleq \begin{bmatrix} r_{yy}(0) & r_{yy}(1) & r_{yy}(2) \\ r_{yy}(1) & r_{yy}(0) & r_{yy}(1) \\ r_{yy}(2) & r_{yy}(1) & r_{yy}(0) \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 1/2 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

(5pt)

(b) Consider the three filters given below. Which ones of them are optimal?

	A	B	C
$h(0)$	-2	0	-3
$h(1)$	4	4	4
$h(2)$	0	-2	-1

(5pt)

(c) Derive an explicit expression for all optimal solutions. (5pt)

Solution:

(a) The frequency of the sinusoidal input $y(n)$ yields for any integer k:

$$y(n+4k) = \begin{cases} 1, & n=0 \\ 0, & n=1 \\ -1 & n=2 \\ 0 & n=3 \end{cases}$$

This directly tell us that $r_{yy}(k) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} y(n)y(n+k)$ has the result

$$r_{yy}(n) = \begin{cases} 1/2, & n=0 \\ 0, & n=1 \\ -1/2 & n=2 \\ 0 & n=3 \end{cases}$$

which are the desired values.

(b) We can write the criterion to minimize as

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} (x(n)-\hat{x}(n))^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} (x(n)-\mathbf{y}^T(n)\mathbf{h})^2 = \mathbf{h}^T R_{yy}\mathbf{h} - 2R_{yx}\mathbf{h} + r_{xx}(0)$$

where $R_{yx}^T = \begin{bmatrix} r_{yx}(0) & r_{yx}(1) & r_{yx}(2) \end{bmatrix} = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}$ The criterion take its minimum value when the gradient is w.r.t $\mathbf{h}$ is zero. Hence an optimal solution $\mathbf{h}_*$ is a solution to

$$R_{yy}\mathbf{h} = R_{yx}$$

We can now check which of the three solutions satisfy the equation above. Filters A and C solves the equation and are hence optimal.

(c) The reason we found more than one optimal filter in (b) is that the sinusoidal input signal results in a matrix R_{yy} which is singular. We can see this for example by noting that the first and third column are linearly dependent. The linear dependency of the two columns imply that the rank of the matrix is 2. The dimension of the nullspace of the matrix is then $3-2=1$. The vector $\mathbf{v} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$ span this nullspace: For any scalar α, $R_{yy}(\alpha\mathbf{v}) = 0$.

If $\mathbf{h}_*$ is one optimal filter (e.g. A or C) then all optimal filters are given by $\mathbf{h}_* + \alpha\mathbf{v}$ where α is any arbitrary real scalar.

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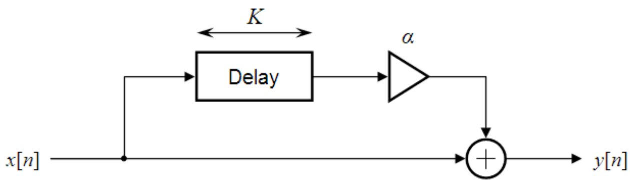
5. The comb filter is a filter with the general structure as illustrated below



which means that the filtered output is the present input added with the input delayed K samples, where K is a positive integer, and scaled with α.

(a) Derive the frequency function of the comb filter (3pt)

(b) Assume $\alpha = -1$. At which frequencies will the frequency function have magnitude zero? *Hint:* Recall the complex solutions to the algebraic equation $x^K = 1$ (3pt)

Solution:

(a) $y(n) = x(n) + \alpha x(n - K)$ which leads to the frequency function $H(\omega) = 1 + \alpha e^{-j\omega K}$

(b) solutions to $H(\omega) = 0$ then translates to solutions to $1 = e^{j\omega K}$. All solutions to this equation are $\omega = \frac{2\pi}{K} l$ where l is an arbitrary integer.

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END - Good Luck!