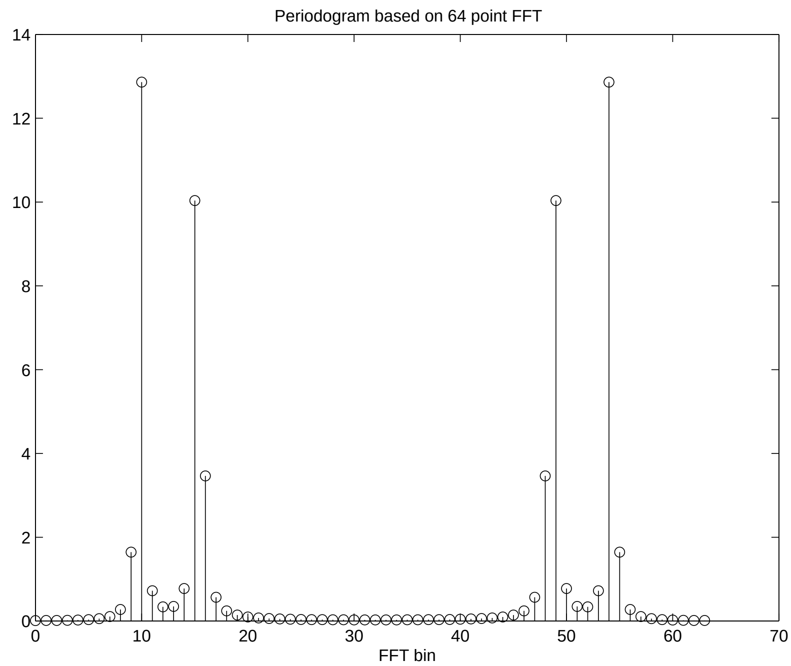
Suggested solutions to examination SSY130 Applied Signal Processing

14:00-18:00, December 18, 2013

1. A signal consists of two sinusoidal components with frequencies, f_1 and f_2. It is known that $1 < f_1 < 2$ [Hz] and $7 < f_2 < 8$ [Hz]. The signal is sampled at the rate 10 Hz, without using any anti-aliasing filter. A 64-point FFT is computed from the digital signal, and based on this the periodogram is formed. The result is shown in the figure below where FFT bincorresponds to the DFT index k.



Based on the plot, determine (if possible) approximate values of f_1 and f_2 (in Hz). (4p)

Solution: The sampling leads to a discrete time Fourier transform $X_d(2\pi f) = f_s \sum_{k=-\infty}^{\infty} X(2\pi f + 2\pi f_s k)$. The second frequency f_2 will be folded to a frequency below $f_s/2$ the Nyquist frequency. The location of the folded frequency will then be between $10-8=2$ and $10-7=3$ [Hz]. The two leftmost peaks are at indices 10 and 15 which then imply the frequency $f_1 = 10/64 * 10 = 1.6$ [Hz] and frequency $15/64 * 10 = 2.3$ [Hz]. Hence the second peak is the folded frequency and consequently $f_2 = 10 - 2.3 = 7.7$ [Hz]. ■

2. Recall the setup of the communication system from Project 1 with number of symbols $N = 128$, sample rate $f_s = 22050$ Hz and modulation frequency of 4000 Hz. Consider three cases when the interpolation/decimation factors are 5,10 and 100 respectively and answer the following for each one of the cases.

(a) What is the duration in seconds for sending N = 128 symbols (excluding the cyclic prefix and training symbols)? (2pt)

(b) What is bandwidth utilized to transmit the symbols assuming ideal filters in the interpolation and decimation stages? (2pt)

(c) The impulse response of the plastic tube (including the electronics) has a duration of approximately 3.6 ms. How many samples does this correspond to in the equivalent base-band channel? Do not count the impulse responses of the LP filter in the interpolation and decimation stages. (3pt)

(d) Describe how the cyclic prefix should be chosen based on the interpolation/decimation factors. (2pt)

Solution:

(a) The duration of the OFDM signal is $N * R/f_s$ where R is the used up-sample/down-sample rate in the interpolation/decimation stages. Hence the durations are

R	5	10	100
Duration [ms]	29.0	58.0	580

(b) The bandwidth (counting positive frequencies) is equal to $B = f_s/R$ resulting in

R	5	10	100
B [Hz]	4410	2205	220.5

(c) The length of the impulse response of the channel, as seen from the baseband, will be scaled by the choice of R. Hence, $N_h = \text{ceil}(3.6 \times 10^{-3} f_s/R))$ [-], where $\text{ceil}(\cdot)$ rounds upwards to nearest integer.

R	5	10	100
N_h	16	8	1

(d) The length of the cyclic prefix N_{cp} should be equal or larger than the length impulse response of the Channel to avoid inter-symbol interference. Hence,

R	5	10	100
N_{cp}	≥ 16	≥ 8	≥ 1

On top of this length we also need to compensate for the effect of the LP-filters in the interpolation and decimation stages, which will increase the necessary cyclic prefix length.

■

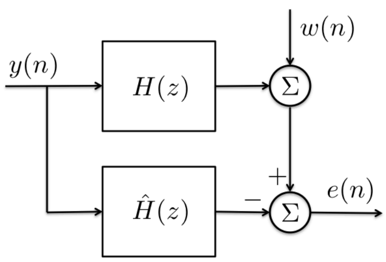


Figure 1: Adaptive filter problem.

3. Consider the adaptive filter problem illustrated in Figure 1 where the filter $\hat{H}$ can be described as

$$\hat{H}(z) = \sum_{k=0}^{M-1} h_k z^{-k}$$

where h_k are real valued scalars, $k = 0, \ldots, M-1$ and z^{-1} is the one step delay operator. The adaptation is aimed to minimize the variance of the error $e(n)$ by modifying the filter $\hat{H}$.

(a) Assume $w(n)$ and $y(n)$ are uncorrelated stochastic processes with zero mean value. Show that the optimal filter is given by the minimizing argument to

$$\min_{\hat{h}_0, \hat{h}_1 \ldots \hat{h}_{M-1}} \frac{1}{2\pi} \int_0^{2\pi} \left| H(\omega) - \hat{H}(\omega) \right|^2 S_{yy}(\omega)\, d\omega \tag{1}$$

Hint: $\mathbf{E}\, e(n)^2 = \frac{1}{2\pi} S_{ee}(\omega)\, d\omega$. (5pt)

(b) Assume that the signal $w(n)$ is given by

$$w(n) = G(z)y(n) + v(n)$$

where $v(n)$ is independent of $y(n)$ and are of zero mean. Derive the expression of the optimal solution, i.e. how does the expression in equation (1) change. (5pt)

Solution:

(a) The error is given by $e = h * y + w - \hat{y} * y$. Since y and w are uncorrelated we obtain $S_{ee}(\omega) = |H(\omega) - \hat{H}(\omega)|^2 S_{yy}(\omega) + S_{ww}(\omega)$. Minimizing the variance of the error is equivalent to minimizing the integral in (1) and the result follows.

(b) Here, $e = h*y+w-\hat{y}*y = h*y+g*y+v-\hat{y}*y$. Since y and v are independent they are also uncorrelated and we obtain $S_{ee}(\omega) = |H(\omega) + G(\omega) - \hat{H}(\omega)|^2 \S_{yy}(\omega) + S_{vv}(\omega)$. Finally we arrive at:

$$\min_{\hat{h}_0, \hat{h}_1 \ldots \hat{h}_{M-1}} \frac{1}{2\pi} \int_0^{2\pi} \left| H(\omega) + G(\omega) - \hat{H}(\omega) \right|^2 S_{yy}(\omega)\, d\omega$$

The optimal filter will now try to approximate the frequency function to match the sum of the frequency functions of H and G with the Power spectrum of the input as a frequency dependent weight.

■

4. Frequency selective filters are specified by the following properties

- Stopband rejection
- Passband ripple
- Passband edge frequency
- Stopband edge frequency
- Width of transition region

(a) Draw a frequency function of a high-pass filter and illustrate in the graph how the specifications above are defined. (4pt)

(b) Filter design involves a fundamental tradeoff which limits the performance of a filter of a given length. Describe this tradeoff. (2pt)

Solution:

(a) The definitions for a LP filter are as given in Figure 2. For a HP filter simply mirror the image. Stopband rejection is in the figure denoted by stopband attenuation.

(b) For a given filter length the main trade-off is between on one hand having a narrow transition region and on the other hand have a small passband ripple and high attenuation in the stopband.

■

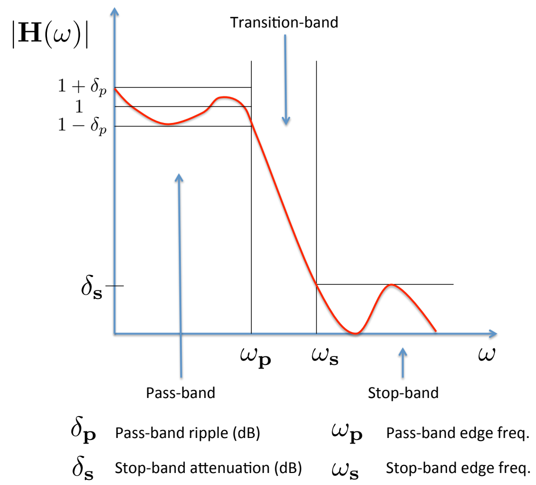


Figure 2: Illustration of filter specifications for a frequency selective filter.

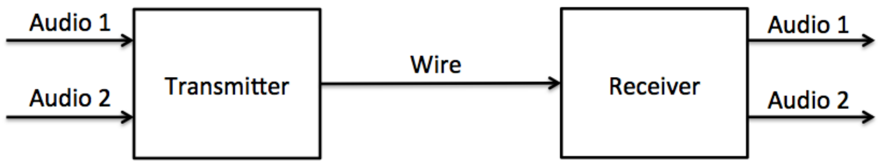


Figure 3: Signal diagram for frequency division transmission application.

5. In this problem you will design a signal transmission system which can carry two independent sampled audio signals over a single channel in the form of standard telephone 2-lead wire using frequency division multiplexing (FDM)[1]. In Figure 3 a signal diagram illustrates the function of the system. The transmission channel (the wire) has useful gain in the frequency range from 0 to 8 kHz, i.e. the bandwidth is 8 kHz. On the transmitter side the system receives the two real valued independent sampled audio signals, each of them with a sampling frequency of 8 kHz. On the receiver side the system should deliver the corresponding discrete time audio signal as outputs, again sampled at a rate of 8 kHz. Design the transmitter and receiver such that the two audio signals can be transmitted over the single wire. Use a design which only employ the following signal processing blocks

   - Filters
   - Upsample and downsample circuits
   - Digital to analog converter (DAC)
   - Analog to digital converter (ADC)

   (a) First assume the DAC and ADC are ideal in the sense that the ADC has an ideal anti-aliasing filter and the DAC performs perfect Nyquist reconstruction of the analogue signal from the sampled signal. In your answer you should create a block diagram illustrating the function blocks in the transmitter and in the receiver respectively. For filters used in your design you should specify the frequencies (normalized to the sample

[1]Frequency division multiplexing means that the spectrum of the channel is divided into non-overlapping frequency bands, wherein the information in each sub-channel is transmitted.

rate) defining the transition between pass-band and stop-band. For up-and downsamplers the factor of rate change should be specified. Draw a graph illustrating where the frequency bands of Audio 1 and Audio 2 are located in the spectrum of the signal transmitted over the wire. Hint: Use the up-sampler to modulate one of the channels to a higher frequency band and down-sampler to demodulate. (10pt)

(b) Now extend the solution and suggest a more realistic design of the ADC and DAC. Use a 4 times oversampling solution for both the ADC and DAC. We assume the DAC uses a ZOH reconstruction. Specify the sample rates (in Hz) and the frequency specifications for necessary filters in the same way as in subproblem (a) (5pt)

(c) In this application the required SNR is 60 dB. How many bits are then necessary for the ADC if we assume the quantization noise is the dominant source of the noise at the receiver end. (3pt)

Solution:

(a) The two audio channels are sampled at a rate of 8 kHz and are real valued. The bandwidth of the information is maximally 4 kHz per channel. If we upsample both channels to 16 kHz and use a LP filter with a cut off frequency (location of filters transition from passband to stop band) of 0.25 (normalized to 16 kHz) for the first channel and use a HP filter with the same cut off frequency for channel 2 we now have two signals with non-overlapping spectra. We now add the two signals together and create an analogue signal using the ideal DAC. Since the DAC is ideal the received analogue signal can be converted by the ADC undisturbed. The receiver then proceeds by creating two channels again by LP filtering and HP filtering the received signal using the same filters as in the transmitter. After filtering, each signal is downsampled a factor 2 which forms the final result for each channel respectively. The downsampling is alias free since the two filters limit the bandwidth to the required 4 kHz. Figure 4 illustrates how the signals power spectra are located in the power spectrum of the signal on the wire.

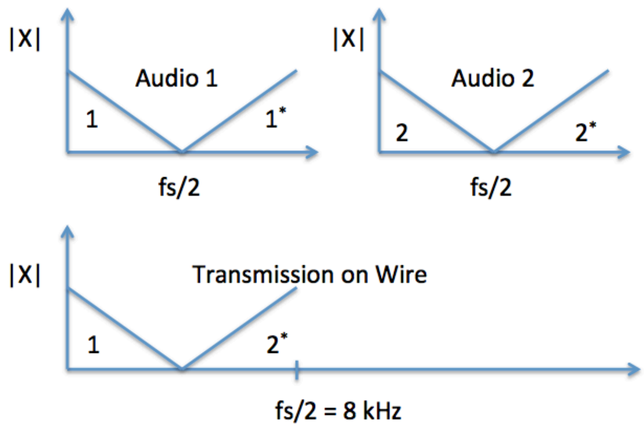


Figure 4: Power spectra for original and signal transmitted on the wire

(b) After adding the two signals as in (a) we now employ 4 times interpolation to increase the sample rate to 16*4 = 64 kHz. The required LP filter should have the cut off frequency at 8 kHz which in normalized frequencies is 8/64 = 0.125. The final stage in the transmitter is the reconstruction step with a ZOH DAC. We also assume there is an analog LP filter which limits the energy in the higher frequencies due to the non-ideal reconstruction. On the receiver end we employ an ADC to sample (and quantize) the signal. A LP filter is then applied (with cut off at 0.125) next to avoid aliasing when the signal is finally downsampled a factor 4. The rest of the processing is as in (a).

(c) An ADC will introduce a quantization noise which yields a SNR of $1.76 + 6.02B$ [dB] where B is the number of bits. Hence we need 10 bits in the ADC. However as we use oversampling the quantization noise is reduced. Since we use a a factor 4 oversampling we will gain $\log_2(4)/2 = 1$ bit. The required number of bits in the ADC is 9.

■

END