Suggested solutions to Examination SSY130 Applied Signal Processing

14:00-18:00, August 19, 2010

1. (a) A zero mean white noise signal with variance σ^2 is filtered through an FIR filter with impulse response $h(k)$, $k = 0, \ldots, m$. What is the variance of the filtered signal? (4pt)

   (b) Describe the difference between the power spectral density (PSD) of a signal and the probability density function (pdf) of a signal. (4pt)

   (c) A 12 bit A/D converter is operating in the range from -10 V to +10 V. What is a suitable pdf for the quantization noise and derive the variance. (4pt)

   Solution:

   (a) $\operatorname{var} y(n) = \mathbf{E}(y(n) - \mathbf{E}\, y(n))^2 = \mathbf{E}\, y^2(n) = \mathbf{E}\left(\sum_{k=0}^{m} h(k)e(n-k)\right)^2 = \sigma^2 \sum_{k=0}^{m} h^2(k)$

   (b) The power spectral density is defined as the DTFT of the autocorrelation function of a stochastic process while the probability density function $p(x)$ of a signal provide us with the probability that a signal sample takes a value within a specified interval $\mathcal{I}$, i.e.

   $$P(x(n) \in \mathcal{I}) = \int_{\mathcal{I}} p(x)\, dx$$

   (c) The quantization interval is $\Delta x = (10 - (-10))/2^{12} = 0.0049\, V$ and the maximum magnitude of the quantization error is half of this. A natural model for the quantization error is then a zero mean random variable uniformly distributed in the interval $[-\Delta x/2, \Delta x/2]$. The variance of such a variable is $\int_{-\Delta x/2}^{\Delta x/2} x^2 \frac{dx}{\Delta x} = \Delta x^2/12 = 2.0 \times 10^{-6}\, V^2$.

   ■

2. It is desired to model an unknown discrete time system using a FIR model structure

   $$y(n) = \sum_{k=0}^{m} h(k)x(n-k)$$

   where $x(n)$ is the input signal and $y(n)$ is the (noise free) output signal and $h(k)$ is the impulse response of the model. Assume the MATLAB column vectors `x` and `y` contain N samples of the input and output signals respectively. Write a MATLAB script which estimates the impulse response of the filter from the input and output samples using the least-squares method. What is the necessary minimum number of samples N for a unique solution? What is sufficient to guarantee a unique solution? (10pt)

   Solution: A MATLAB script solution

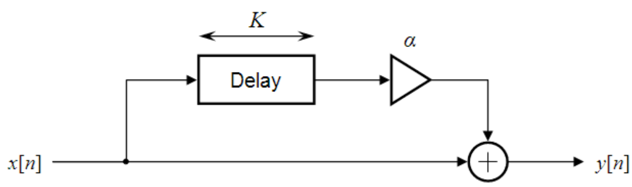
```
% First create example data
h = [1 2 3 4]; % An example filter
n= m+1; % Number of filter coefficients
```

```
x = randn(100,1); % Input
yy = conv(x,h);  % Filter
y = yy(1:length(x)); %remove the tail in the conv output

% Solution to the FIR estimation problem
H = toeplitz(x(n:end),x(n:-1:1)); % Create regressor matrix
Y = y(m:end); % Select valid outputs from sample n
hest = H\Y    % Solve Least-squares problem
```

The least squares problem has a unique solution if the matrix `H` has rank `n`. A necessary requirement for this to hold is that the number of rows in `H` is at least `n` which then translates to $N \geq 2(m+1) - 1 = 2m + 1$. ■

3. The comb filter is a filter with the general structure as illustrated below



which means that the filtered output is the present input added with the input delayed K samples, where K is a positive integer, and scaled with α.

(a) Explain why the filter is particularly inexpensive to implement if $\alpha = \pm 1$ (2pt)

(b) Derive the frequency function for the comb filter (3pt)

(c) Which choice $\alpha = +1$ or $\alpha = -1$ would yield a filter where the lower frequencies will pass through the filter? (2pt)

(d) Assume $\alpha = -1$. At which frequency locations will the frequency function have magnitude zero? Hint recall the complex solutions to the algebraic equation $x^K = 1$ (3pt)

Solution:

(a) In this case the filter does not need any multiplier which hence saves speed or circuit space depending on the implementation platform.

(b) $y(n) = x(n) + \alpha x(n-K)$ which leads to the frequency function $H(\omega) = 1 + \alpha e^{-j\omega K}$

(c) For $\alpha = 1$ we obtain $H(0) = 2$ and for For $\alpha = -1$ we obtain $H(0) = 0$. Hence, $\alpha = 1$ is the correct choice.

(d) solutions to $H(\omega) = 0$ then translates to solutions to $1 = e^{j\omega K}$. All solutions to this equation are $\omega = \frac{2\pi}{K} l$ where l is an arbitrary integer.

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4. A sampled signal has a sample rate of 22 kHz. Discuss a sample rate converter design which converts the rate of the sampled signal to 40 kHz. Motivate all components in the design and explain how specifications should be selected and the associated trade-offs between cost and quality. (10pt)

Solution: A sample rate converter consists of filters as well as down and up sampling operations. The ratios between new and old rates are $\frac{40}{22} = \frac{20}{11}$ where 11 the least common integer denominator of the ratio. The rate change can hence be realized by a factor 20 interpolator followed by a factor 11 decimator. The interpolator consists of the up-sampling operation and the associated filter which suppresses the spectral images created in the up-sampling operation which if not removed would lead to alias effects during the downsampling step. The new rate is 40 KHz and hence the filter should be of LP-type with the stop band starting at 20 kHz. After downsampling to 40 kHz rate the remainder of the images in the frequency band between 11 kHz and 20 kHz must be removed by a second LP-filter with the stop-band starting at 11 kHz. The reason for doing filtering both before and after the down-sampling step is that it is computationally more efficient to remove the spectral images near the original signal at the lower rate since the filters transition region can then be allowed larger when considering relative frequencies. Also the factor 20 up-sampling should be factored into several interpolation steps to save computations for a given signal quality. Finally by using the polyphase decomposition of the filters further computational savings can be gained. ■

5. The standard LMS adaptive filter algorithm can be described by the equations

$$e(n) = x(n) - \mathbf{h}^T(n-1)\mathbf{y}(n-1)$$
$$\mathbf{h}(n) = \mathbf{h}(n-1) + \mu\mathbf{y}(n-1)e(n).$$

Small changes to the filter coefficient update equation leads to alternative algorithms. The *signed LMS* algorithm has the update equation

$$\mathbf{h}(n) = \mathbf{h}(n-1) + \mu\mathbf{y}(n-1)\,\mathrm{sign}(e(n))$$

where the sign operator is defined as

$$\begin{cases} sign(x) = 1 & x > 0 \\ sign(x) = -1 & x \leq 0. \end{cases}$$

The *normalized LMS* algorithm has an update equation defined as

$$\mathbf{h}(n) = \mathbf{h}(n-1) + \frac{\mu}{\|\mathbf{y}(n-1)\|^2}\mathbf{y}(n-1)e(n)$$

Discuss the three different update equations regarding

- computational complexity / cost of implementation
- accuracy of the filter including convergence speed
- robustness to amplitude variations in the input signal $y(n)$ (10pt)

Solution: The standard LMS filter updating equation moves the coefficients in the approximate stochastic negative gradient direction of the MSE criterion. The signed LMS algorithm do not consider the size of the error but only the sign. Hence the update vill still be in the same direction but with a size not dependent on the error magnitude. Hence, on average the filter will converge in a similar fashion as the standard algorithm but as the error decreases the signed algorithm will keep taking larger steps than the standard algorithm. This will lead to an increase of the MSE for a “converged” filter. Clearly the signed filter is less expensive to implement since the n multiplications between the error and the input vector are not necessary. The standard algorithm and the signed algorithm both have a fixed step length controlled by μ. If the input has a varying dynamic range it can be challenging to find a good trade-off between convergence speed and robustness against divergence. The normalized algorithm solves this issue since the step length of the update equation is in this case independent of the amplitude of the input to the filter. However, this benefit is associated with a significantly increased computational cost involving n multiplications and one division. ■