Suggested solutions to examination
SSY130 Applied Signal Processing

14:00-18:00, December 16, 2008

Instructions

- *Responsible teacher:* Tomas McKelvey, ph 8061. Teacher will visit the site of examination at 15 and 17.
- Score from the written examination will together with course score determine the final grade according to the Course PM.
- Solutions are published on the course home-page latest 12 noon December 17.
- Your preliminary grade is reported to you via email.
- Exam grading review will be held in the "Blue Room" on level 6 at 12:15-13:00 on January 15 and at 12:15-13:00 on January 22.

Allowed aids at exam:

- L. Råde and B. Westergren, Mathematics Handbook (any edition, including the old editions called Beta).
- Any calculator
- One a4 size single page with written notes

Other issues:

- All solutions should be well motivated and clearly presented in order to render a full score.
- The maximum score is 52 points.

Problems

1. A sampled signal $x(n)$ has a DTFT of the following shape

$$|X(\omega)| = 1 - \frac{|\omega|}{\pi}, \quad |\omega| < \pi$$

where ω is relative with respect to the sample rate (i.e. rad/sample).

(a) Sketch the magnitude of the DTFT of $x(n)$. (1pt)

(b) A new signal $y(n)$ is created by up-sampling $x(n)$ a factor 3. Show exactly how $y(n)$ is defined in terms of $x(n)$. (2pt)

(c) Sketch the magnitude of the DTFT of $y(n)$. Use a frequency scale relative to the new sample rate. (3pt)

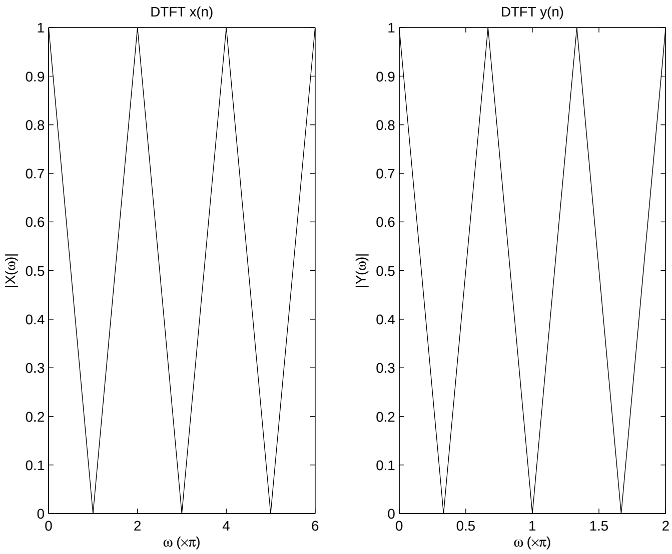
Solution:

(a) See left graph below.

(b)

$$y(n) = \begin{cases} x(n/3) & n = 0, \pm 3 \pm 6, \ldots \\ 0 & otherwise \end{cases}$$

(c) Direct application of the DTFT of $y(n)$ yields $Y(\omega) = X(3\omega)$ see, right graph below.



□

2. Describe the key steps involved when designing a digital Low-pass IIR filter with the 3dB cut-off frequency located at 8 kHz using the bilinear transformation method. The sampling frequency of the signal processing system is 48 kHz. (4pt)

Solution: The first step is to convert the discrete time frequency specification to the continuous time. Here $f = 8/48 = 1/6$, $\Omega = 2\tan(2\pi f/2)$ Second step involves designing an analog filter $H(s)$ with the desired amplitude requirements (i.e. select filter type and order) and with cut off frequency at Ω. In the third step the discrete time filter is obtained by the bilinear transform as $H_d(z) = H(s)|_{s=2\frac{1-z^{-1}}{1+z^{-1}}}$. □

3. A discrete time causal linear system has a periodic input with a period of P samples. Assume one period of the output is known. Describe a computational procedure which calculates the input to the system for the following two cases:

   (a) The linear system is FIR and has a known impulse response $h(n)$ with a length less than P. (4pt)

   (b) The linear system is FIR and has a known impulse response $h(n)$ with a length larger than than $P+1$. (6pt)

   Solution: Let $Y(k)$ and $X(k)$ denote the length P DFT of the output and input respectively. Then since the DFT of one period of a periodic signal equals the Fourier series coefficients we directly have $Y(k) = H(\omega_k)X(k)$ where $\omega_k = 2\pi k/P$.

   (a) Since length of the impulse response is less then P we have $H(\omega_k) = H(k)$ where $H(k)$ is the length P DFT of the impulse response of the system zero padded to a total length of P. Consequently we can calculate the input as $x(n) = \text{IDFT}[Y(k)/H(k)]$

   (b) For this case the impulse response length is longer than the period time and we can thus not directly use the length P DFT to get $H(\omega_k)$. Let N_h denote the length of the impulse response. We can use the definition of the DTFT to calculate $H(\omega_k) = \sum_{n=0}^{N_h-1} h(n)e^{-j2\pi nk/P}$ and then calculate the input as in (a) □

4. Figure 1 shows part of the time evolution of the error signal $e(n)$ for two different standard LMS filters (like the one used in Project 2). Both filters are fed with the same $y(n)$ and $x(n)$ signals and the filtering is started at sample 0. The length of the FIR-filter which is updated by the LMS algorithm are the same for both filters.

   (a) Explain which algorithm feature that differ between the two filters. (3pt)

   (b) A numerical parameter control the feature discussed in (a). Which of the two filters has the largest numerical value of this parameter? (3pt)

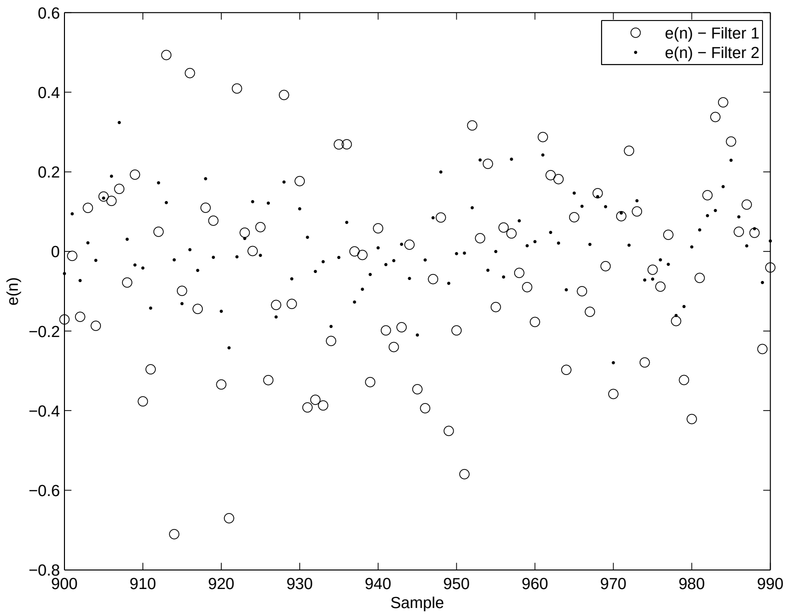


Figure 1: Error signal for two different LMS filters

Solution:

(a) Since the input signals as well as the filter length are the same the only thing left to be different is the step length μ. The value of the step length influence the speed of convergence as well as the size of the residual variance.

(b) Since the illustrated time samples of $e(n)$ show samples 900-990 we can expect that the differences are mainly due to the residual variance. Since Filter 1 has a larger deviation from 0 (on average) we can conclude that the residual variance for Filter 1 is larger then Filter 2. Since a larger step length yields a higher residual variance Filter 1 has a larger step length than Filter 2.

□

5. Consider the adaptive filtering problem

$$\hat{x}(n) = \mathbf{h}(n)^T \mathbf{y}(n)$$
$$e(n) = x(n) - \hat{x}(n)$$

The recursive least-squares algorithm (RLS) can be formulated to minimize (with respect to $\mathbf{h}(n)$) the following functional

$$L = \sum_{k=0}^{n} \alpha^{n-k} e(k)^2$$

(a) What it the name of the parameter α? (1pt)

(b) In what numerical range should α be selected to make the algorithm meaningful? (2pt)

(c) Two RLS algorithms are run on the same input data and with the same length of $\mathbf{h}(n)$. Both algorithms are started at sample $n = 0$. The only difference is the parameter α. Figure 2 shows the histogram for $e(n)$ calculated from samples $800 \leq n < 990$. Which algorithm has the largest α? (3pt)

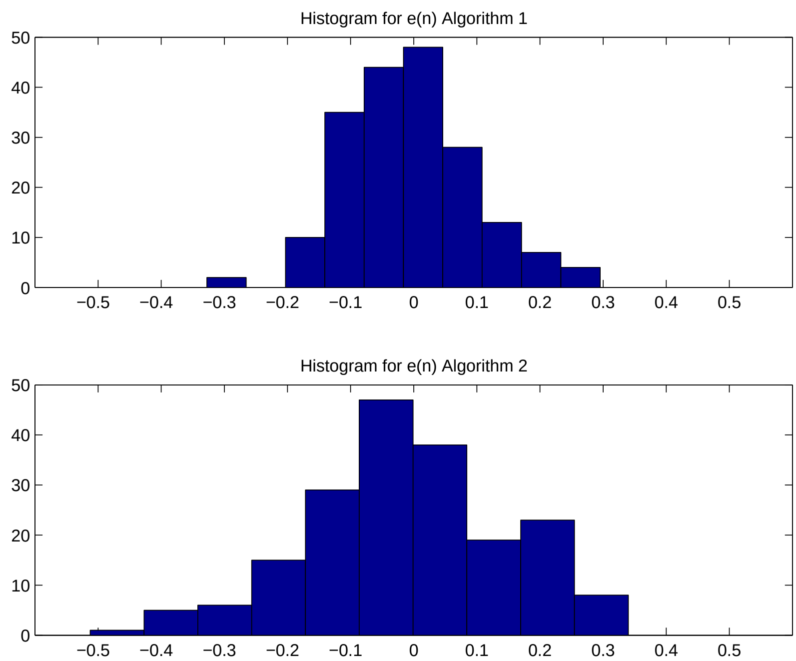


Figure 2: Histogram for the error signal for two different RLS filters

(d) Give an example of a case where $\alpha = 1$ is the right choice. (2pt)

Solution:

(a) The parameter α is called the forgetting factor.

(b) The purpose of the forgetting factor is to make the RLS algorithm track time varying applications. This means that the algorithm should pay more attention to recent values of the error rather than past values. If α is chosen in the range $0 < \alpha < 1$, the functional provide an exponentially decreasing weight for past values. A large value of α provides a slowly decreasing weight and hence the filter will slowly track changes while a smaller value of of α will yield a filter which quickly track changes.

(c) Analysis of the histograms show that Algorithm 1 has a smaller residual variance of the error. Thus, Algorithm 1 track more slowly than Algorithm 2 and consequently α for Algorithm 1 is larger than for Algorithm 2.

(d) If $\alpha = 1$ then the Algorithm will consider all filter errors with equal weight. This is desirable when it is known that the solution of the optimal filtering problem is time-invariant and tracking is not necessary. □

6. Linear convolution can be calculated using DFT.

(a) Explain the main reason why this approach can be beneficial. (2pt)

(b) Zero padding plays a key role in this approach. What is zero padding? (2pt)

(c) Explain in detail how to calculate, using DFT, the result of a linear convolution between a filter with impulse response of length P and a signal of length N. (4pt)

Solution:

(a) DFT can be efficiently calculated by the Fast Fourier Transform if the length of the signal is a power of 2. When employing the FFT algorithm for calculating linear convolution the number arithmetic operation are considerably less than a time domain approach, particularly when the length of the impulse response is large.

(b) Zero padding is referred to the operation where a signal of say length N is extended to length $N + Z$ where the appended samples are all zero. This is particularly useful when more samples of the DTFT is desired than provided by the DFT of the original signal.

(c) Let N and M be the length of the input signal $x(n)$ and impulse response $h(n)$ of the linear filter respectively. A direct application of the time domain convolution yields that the output $y(n)$ can be non-zero only for indices from 0 to $N + M - 2$, i.e. the output has a length of maximally $N + M - 1$. Select now the smallest integer p such that $N + M - 1 \leq 2^p$. Zero pad the input signal as well as the impulse response to the length 2^p. Calculate the FFT of both signals and multiply them together as $Y(k) = H(k)X(k)$. The output of the convolution is now provided by the inverse FFT of $Y(k)$. □

7. In an application it is required to sample and store (on a hard drive) the key information in a real valued continuous time source signal. The key information in the source signal has all its power located between 180 kHz to 182 kHz. The source signal also includes two disturbance signals with power of the same magnitude as the key signal. The first disturbance is located at 202-220 kHz and the second one is located at 140-162 kHz. The source signal is also contaminated with noise with a flat spectrum from 0-220kHz with a power 60 dB below the signal components. Discuss a signal processing solution which store the key information in a manner such that it can be recovered with high fidelity but still use as little hard drive memory as possible per second of sampled signal. Your discussion on the solution should include:

- a description of the overall architecture
- the sample rate/rates used
- the number of bits the AD converter use
- the type of filters needed and their specifications
- the number of bits per second required to store the key information

Hint: Select the number of bits in the AD converter such that the quantization noise is just below the magnitude of the noise in the continuous time source signal. (10pt)

Solution: The signal is composed of mainly two parts. Two large interfering disturbances which need to be removed by bandpass filtering is one part and the second part is white noise measurement noise and quantization noise. Since the signal to be recorded is real and band-limited to 2 kHz it thus desirable to down-sample to a rate of at least 4 kHz.

A possible system design is as follows:

- Since the noise level is -60dB and the signal to quantization noise level is $1.76+6\times$number of bits, a selection of 10 bits for the AD converter gives a balance between signal noise level and the quantization noise.
- In front of the AD converter we put a simple passive RC filter with the 3dB crossover frequency at 200 kHz. This provides adequate attenuation of possible other high frequency components.
- The sampling frequency should be selected high enough to reduce aliasing effects. The dominating part of the aliasing will originate from the disturbance at 200-220 kHz. If we select a sample rate at $2 \times 210 = 420$ kHz aliasing from the 200-220 kHz component will occur between 190-210 kHz in the sampled signal and thus there is an 8 kHz guard to the key signal component.
- The sampled signal is then bandpass filtered with a filter with passband between 180-182 kHz and stop band frequencies at 190 kHz and 162 kHz. The stop-band attenuation should be around 70-80 dB to reduce the aliasing when down-sampling.
- After bandpass filtering the signal only comprises the key signal with a signal spectrum between 180-182 kHz. A factor $420/4 = 105$ decimation of this signal would then effectively move the spectrum from 180-182 kHz down to 0-2 kHz and the new sample rate is 4 kHz.
- The resulting bit rate to store on the hard drive will then be 4000 samples $\times$ 10 bits / second = 40 kbits/s.

END

Good Luck!