# MVE041, Lösningar tentamen 29/5 -17

1. $f(x,y) = 2x^4 + y^2 + x$, $(1,1,4)$

(a) $\frac{\partial f}{\partial x} = 8x^3 + 1 \Rightarrow \frac{\partial f}{\partial x}(1,1) = 9$

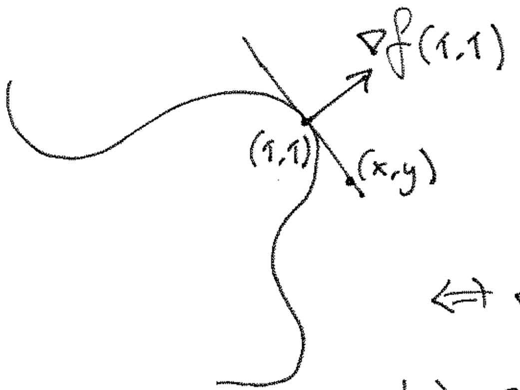
$\frac{\partial f}{\partial y} = 2y \Rightarrow \frac{\partial f}{\partial y}(1,1) = 2$

$$z = f(1,1) + \frac{\partial f}{\partial x}(1,1)(x-1) + \frac{\partial f}{\partial y}(1,1)(y-1) \Leftrightarrow$$

$$\Leftrightarrow z = 4 + 9(x-1) + 2(y-1) \Leftrightarrow$$

$$\Leftrightarrow \underline{\underline{9x + 2y - z = 7}}$$

(b)




$\nabla f(1,1) = (9,2)$

$\nabla f(1,1) \perp (x-1, y-1) \Leftrightarrow$

$\Leftrightarrow \nabla f(1,1) \cdot (x-1, y-1) = 0 \Leftrightarrow$

$\Leftrightarrow 9(x-1) + 2(y-1) = 0 \Leftrightarrow$

$\Leftrightarrow \underline{\underline{9x + 2y = 11}}$

(c) $\nabla f = 0 \Leftrightarrow \begin{cases} 8x^3 + 1 = 0 \Leftrightarrow x^3 = -\frac{1}{8} \Leftrightarrow x = -\frac{1}{2} \\ 2y = 0 \Leftrightarrow y = 0 \end{cases}$

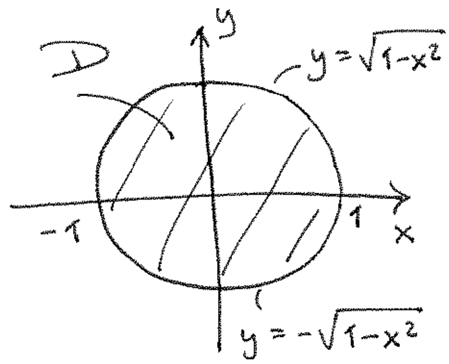
$\therefore \left(-\frac{1}{2}, 0\right)$

$$\mathcal{H} = \begin{pmatrix} 24x^2 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow \mathcal{H}\left(-\tfrac{1}{2}, 0\right) = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \lambda_1 = 6 > 0 \\ \lambda_2 = 2 > 0 \end{cases}$$

$\therefore \left(-\frac{1}{2}, 0\right)$ min. punkt

2. $\iint_D x^2 y^2 \, dA = \left\{ \begin{matrix} \text{Polära} \\ \text{koord.} \end{matrix} \right\} =$




$$= \int_0^{2\pi} \left( \int_0^1 r^2 \cos^2\theta \, r^2 \sin^2\theta \, r \, dr \right) d\theta =$$

$$= \int_0^{2\pi} \cos^2\theta \sin^2\theta \, d\theta \cdot \int_0^1 r^5 \, dr =$$

$$= \int_0^{2\pi} \frac{\sin^2(2\theta)}{4} \, d\theta \cdot \left[ \frac{r^6}{6} \right]_0^1 = \left\{ \begin{matrix} \cos(2\theta) = \cos^2\theta - \sin^2\theta = \\ = 1 - 2\sin^2\theta \end{matrix} \right\}$$

$$= \frac{1}{24} \int_0^{2\pi} \frac{1 - \cos(4\theta)}{2} \, d\theta = \frac{1}{48} \left[ \theta - \frac{\sin(4\theta)}{4} \right]_0^{2\pi} =$$

$$= \frac{1}{48} \cdot 2\pi = \frac{\pi}{24}$$

3. $\nabla f = (3x^2y \, , \, x^3)$ ← Inga singulära punkter

<u>Kritiska punkter:</u>

$$\nabla f = 0 \iff \begin{cases} 3x^2y = 0 \\ x^3 = 0 \end{cases} \Rightarrow (0,y) \text{ kritiska punkter}$$

$$f(0,y) = 0$$

<u>Randpunkter:</u>

Vill optimera $f(x,y) = x^3y$ under bivillkoret
$g(x,y) = x^2 + 2y^2 - 1 = 0$

$$\Rightarrow L(x,y,\lambda) = x^3y + \lambda(x^2 + 2y^2 - 1)$$

$$\frac{\partial L}{\partial x} = 3x^2y + 2x\lambda = 0 \iff x(3xy + 2\lambda) = 0$$

$$\frac{\partial L}{\partial y} = x^3 + 4y\lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = x^2 + 2y^2 - 1 = 0$$

Två fall: $x = 0$ eller $3xy + 2\lambda = 0$

<u>I. $x=0$:</u> $2y^2 - 1 = 0 \iff y^2 = \frac{1}{2} \iff y = \pm\frac{1}{\sqrt{2}}$

$$\Rightarrow \left(0, \frac{1}{\sqrt{2}}\right), \left(0, -\frac{1}{\sqrt{2}}\right)$$

<u>II. $x \neq 0$:</u> $\begin{cases} 3xy + \lambda \cdot 2 = 0 \iff 3xy = -2\lambda \\ x^3 + 4\lambda y = 0 \iff \frac{x^3}{2y} = -2\lambda \\ x^2 + 2y^2 - 1 = 0 \end{cases}$

$$\Rightarrow \quad 3xy = \frac{x^3}{2y} \iff 6xy^2 = x^3 \iff x(6y^2 - x^2) = 0$$

$$\Rightarrow \{x \neq 0\} \Rightarrow 6y^2 = x^2$$

$$x^2 + 2y^2 = 1 \iff 6y^2 + 2y^2 = 1 \iff 8y^2 = 1$$

$$\Rightarrow \quad y = \pm \frac{1}{\sqrt{8}} = \pm \frac{1}{2\sqrt{2}}$$

$\underline{y = \frac{1}{2\sqrt{2}}}$ : $\quad 6 \cdot \frac{1}{8} \cancel{8} = x^2 \Rightarrow x = \pm \frac{\sqrt{3}}{2}$

$$\Rightarrow \left(\frac{\sqrt{3}}{2}, \frac{1}{2\sqrt{2}}\right), \left(-\frac{\sqrt{3}}{2}, \frac{1}{2\sqrt{2}}\right)$$

$\underline{y = -\frac{1}{2\sqrt{2}}}$ : $\quad 6 \cdot \frac{1}{8} = x^2 \Rightarrow x = \pm \frac{\sqrt{3}}{2}$

$$\Rightarrow \left(\frac{\sqrt{3}}{2}, -\frac{1}{2\sqrt{2}}\right), \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2\sqrt{2}}\right)$$

$$f\left(\frac{\sqrt{3}}{2}, \frac{1}{2\sqrt{2}}\right) = f\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2\sqrt{2}}\right) = \frac{3\sqrt{3}}{16\sqrt{2}}$$

$$f\left(-\frac{\sqrt{3}}{2}, \frac{1}{2\sqrt{2}}\right) = f\left(\frac{\sqrt{3}}{2}, -\frac{1}{2\sqrt{2}}\right) = -\frac{3\sqrt{3}}{16\sqrt{2}}$$

$\therefore$ Max. värde: $\frac{3\sqrt{3}}{16\sqrt{2}}$

Min. värde: $-\frac{3\sqrt{3}}{16\sqrt{2}}$

4 (a) $\mathbb{F}: D \to \mathbb{R}^3$, $D \subset \mathbb{R}^3$ konservativt om:

(i) $D$ enkelt sammanhängande

(ii) $\nabla \times \mathbb{F} = \mathbb{O}$

(b) Vill bestämma $A, B \in \mathbb{R}$ så att $\nabla \times \mathbb{F} = \mathbb{O}$ då $\mathbb{F}$ definierad på hela $\mathbb{R}^3 \leftarrow$ enkelt sammanhängande

$$\nabla \times \mathbb{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Ax\sin(\pi y) & x^2\cos(\pi y) + Bye^{-z} & y^2 e^{-z} \end{vmatrix} =$$

$$= \left(2ye^{-z} + Bye^{-z},\ 0,\ 2x\cos(\pi y) - A\pi x\cos(\pi y)\right)$$

$$\nabla \times \mathbb{F} = \mathbb{O} \iff \begin{cases} A = \frac{2}{\pi} \\ B = -2 \end{cases}$$

$$\Rightarrow \mathbb{F} = \left(\frac{2}{\pi}x\sin(\pi y),\ x^2\cos(\pi y) - 2ye^{-z},\ y^2e^{-z}\right)$$

Om $\phi: \mathbb{R}^3 \to \mathbb{R}$ s.a. $\nabla\phi = \mathbb{F}$ så

$$\frac{\partial \phi}{\partial x} = \frac{2}{\pi}x\sin(\pi y) \Rightarrow \phi(x,y,z) = \frac{x^2}{\pi}\sin(\pi y) + h(y,z)$$

$$x^2\cos(\pi y) - 2ye^{-z} = F_2 = \frac{\partial\phi}{\partial y} = x^2\cos(\pi y) + \frac{\partial h}{\partial y}$$

$$\Rightarrow \frac{\partial h}{\partial y} = -2ye^{-z} \Rightarrow h(y,z) = -y^2e^{-z} + g(z) \Rightarrow$$

$$\Rightarrow \phi(x,y,z) = \frac{x^2}{\pi}\sin(\pi y) - y^2e^{-z} + g(z)$$

$$y^2 e^{-z} = F_3 = \frac{\partial \phi}{\partial z} = y^2 e^{-z} + g'(z) \iff g'(z) = 0 \Rightarrow g(z) = C$$

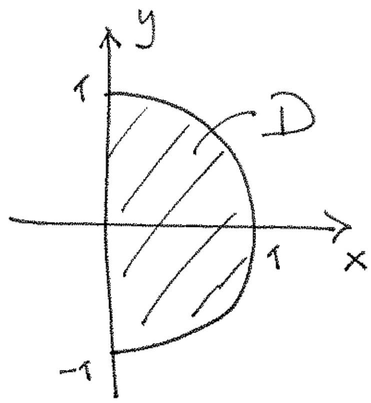
$$\therefore \quad \phi(x,y,z) = \frac{x^2}{\pi}\sin(\pi y) - y^2 e^{-z} + C$$

$$\int_{\mathcal{C}} \mathbb{F} \cdot d\mathbf{r} = \int_{\mathcal{C}} \nabla\phi \cdot d\mathbf{r} = \phi(\mathbf{r}_{slut}) - \phi(\mathbf{r}_{start}) =$$

$$= \phi\left(1, \tfrac{1}{2}, 2\right) - \phi(0,0,0) =$$

$$= \frac{1}{\pi}\sin\left(\frac{\pi}{2}\right) - \frac{1}{4}e^{-2} + \cancel{C} - \cancel{C} = \frac{1}{\pi} - \frac{e^{-2}}{4}$$

5. Parametrisering av $S$:




$$\mathbf{r}(x,y) = \left(x,\ y,\ \frac{x^2}{2} + \frac{y^2}{2}\right), \quad (x,y) \in D$$

$$\frac{\partial \mathbf{r}}{\partial x} = (1, 0, x), \quad \frac{\partial \mathbf{r}}{\partial y} = (0, 1, y)$$

$$\Rightarrow \left| \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} \right| = \left\| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & x \\ 0 & 1 & y \end{matrix} \right\| = |(-x, -y, 1)| =$$

$$= \sqrt{1 + x^2 + y^2}$$

$$\text{Area}(S) = \iint_S 1\, dS = \iint_D \sqrt{1+x^2+y^2}\, dA = \left\{ \begin{matrix} \text{Polära} \\ \text{koord.} \end{matrix} \right\} =$$

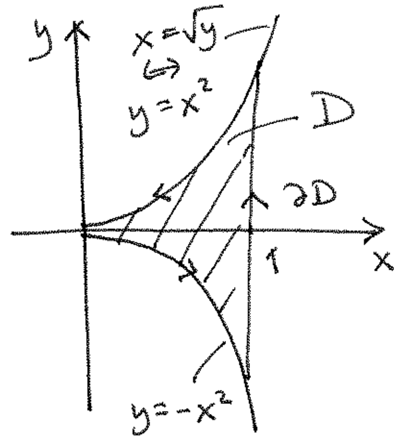
$$= \int_{-\pi/2}^{\pi/2} \left( \int_0^1 \sqrt{1+r^2}\, r\, dr \right) d\theta = \pi \cdot \frac{1}{2} \int_0^1 2r\sqrt{1+r^2}\, dr =$$

$$= \left\{ \begin{matrix} u = 1 + r^2, & r = 0 \Leftrightarrow u = 1 \\ du = 2r\,dr, & r = 1 \Leftrightarrow u = 2 \end{matrix} \right\} = \frac{\pi}{2} \int_1^2 \sqrt{u}\, du =$$

$$= \frac{\pi}{2} \left[ \frac{u^{3/2}}{3/2} \right]_1^2 = \frac{\pi}{3}\left( (\sqrt{2})^3 - (\sqrt{1})^3 \right) = \frac{\pi(2\sqrt{2} - 1)}{3} \text{ a.e.}$$

6. Vill beräkna

$$\oint_{\partial D} \mathbb{F} \cdot d\mathbb{r}$$




Använder Greens sats:

$$\oint_{\partial D} \mathbb{F} \cdot d\mathbb{r} = \begin{Bmatrix} \text{Greens} \\ \text{sats} \end{Bmatrix} =$$

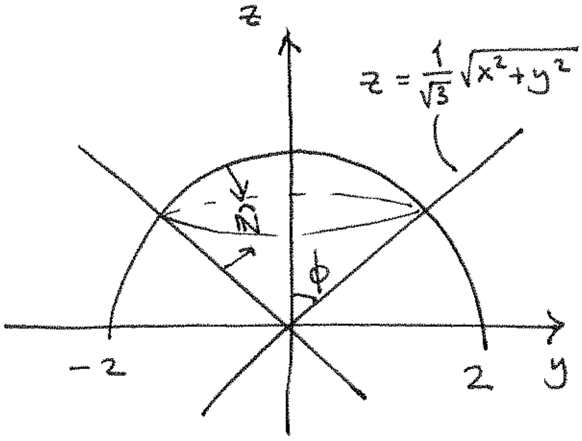
$$= \iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy = \iint_D \Big( 0 - (-\sin(x^3)) \Big)\, dx\,dy =$$

$$= \iint_D \sin(x^3)\, dx\,dy = \int_0^1 \left( \int_{-x^2}^{x^2} \sin(x^3)\, dy \right) dx =$$

$$= \int_0^1 \sin(x^3) \left[ y \right]_{-x^2}^{x^2} dx = 2 \int_0^1 x^2 \sin(x^3)\, dx =$$

$$= \frac{2}{3} \left[ -\cos(x^3) \right]_0^1 = \frac{2(1 - \cos(1))}{3}$$

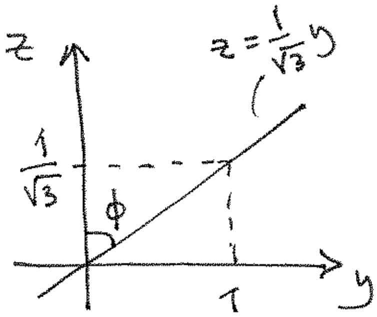
7. Vi använder Gauss sats:




$$\underset{\partial K}{\oiint} \mathbb{F}\cdot \overset{\leftarrow\ \text{inåt}}{\hat{\mathbb{N}}}\, dS = -\underset{\partial K}{\oiint} \mathbb{F}\cdot \overset{\swarrow\ \text{utåt}}{(-\hat{\mathbb{N}})}\, dS =$$

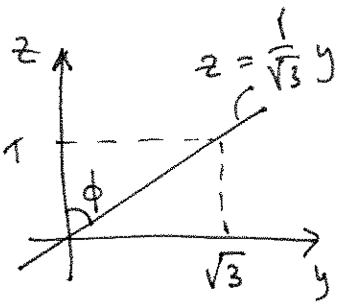
$$= \left\{ \begin{matrix}\text{Gauss}\\ \text{sats}\end{matrix} \right\} = -\iiint_K \nabla\cdot\mathbb{F}\, dV =$$

$$= -\iiint_K \left(-y^2 + 2z^2 - 3z^2 - x^2\right) dV =$$




$$= -\iiint_K \left(-x^2 - y^2 - z^2\right) dV =$$

$$= \iiint_K \left(x^2 + y^2 + z^2\right) dV$$




Sfäriska koord. lämpliga:

$$\begin{cases} 0 \le R \le 2 \\ 0 \le \phi \le \arctan\left(\frac{1}{1/\sqrt{3}}\right) = \frac{\pi}{3} \\ 0 \le \theta \le 2\pi \end{cases}, \quad dV = R^2 \sin\phi\, dR\, d\phi\, d\theta$$

$$\iiint_K (x^2+y^2+z^2)\, dV = \int_0^{2\pi}\left(\int_0^{\pi/3}\left(\int_0^2 R^2\cdot R^2 \sin\phi\, dR\right) d\phi\right) d\theta =$$

$$= 2\pi \int_0^{\pi/3} \sin\phi\, d\phi \cdot \int_0^2 R^4\, dR = 2\pi \left[-\cos\phi\right]_0^{\pi/3} \cdot \left[\frac{R^5}{5}\right]_0^2 =$$

$$= 2\pi\cdot\frac{32}{5}\cdot\Big(1 - \underbrace{\cos\left(\tfrac{\pi}{3}\right)}_{=1/2}\Big) = \frac{32\pi}{5}$$

8. $$\frac{\partial u}{\partial x} = \frac{z}{y}\cdot\frac{\partial}{\partial x} f\left(\frac{xy}{z^2}, \frac{y}{x}\right) = \frac{z}{y}\left(f_1\cdot\frac{y}{z^2} + f_2\cdot\left(-\frac{y}{x^2}\right)\right) =$$

$$= \frac{1}{z} f_1 - \frac{z}{x^2} f_2$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{z}{y} f\left(\frac{xy}{z^2}, \frac{y}{x}\right)\right) = -\frac{z}{y^2} f + \frac{z}{y}\frac{\partial}{\partial y} f\left(\frac{xy}{z^2}, \frac{y}{x}\right) =$$

$$= -\frac{z}{y^2} f + \frac{z}{y}\left(f_1\cdot\frac{x}{z^2} + f_2\cdot\frac{1}{x}\right) =$$

$$= -\frac{z}{y^2} f + \frac{x}{yz} f_1 + \frac{z}{xy} f_2$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\left(\frac{z}{y} f\left(\frac{xy}{z^2}, \frac{y}{x}\right)\right) = \frac{1}{y} f + \frac{z}{y}\frac{\partial}{\partial z} f\left(\frac{xy}{z^2}, \frac{y}{x}\right) =$$

$$= \frac{1}{y} f + \frac{z}{y}\left(f_1\cdot\left(-\frac{2xy}{z^3}\right) + f_2\cdot 0\right) = \frac{1}{y} f - \frac{2x}{z^2} f_1$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = x\left(\frac{1}{z} f_1 - \frac{z}{x^2} f_2\right) +$$

$$+ y\left(-\frac{z}{y^2} f + \frac{x}{yz} f_1 + \frac{z}{xy} f_2\right) + z\left(\frac{1}{y} f - \frac{2x}{z^2}\right) f_1 =$$

$$= \frac{x}{z} f_1 - \frac{z}{x} f_2 - \frac{z}{y} f + \frac{x}{z} f_1 + \frac{z}{x} f_2 + \frac{z}{y} f - \frac{2x}{z} f_1 = 0$$