CHALMERS UNIVERSITY OF TECHNOLOGY
Department of Signals and Systems
Division of Automatic Control, Automation and Mechatronics

EXAMINATION IN NONLINEAR AND ADAPTIVE CONTROL

(Course ESS076)

Monday October 22, 2012

Time and place:	14:00 - 18:00 at Hörsalar
Teacher:	Torsten Wik (5146 or 0739 870570)

The following items are allowed (controlled by teacher):

1. *Control Theory* (Glad Ljung) or *Applied Nonlinear Control* (Slotine/Li)
2. ESS076 Supplement
3. Mathematical handbooks of tables such as Beta Mathematics Handbook.
4. Course summary from Lecture 18

Notes, calculator, mobile telephones, laptops or palmtops, are not allowed! Reasonable notes in the textbook are allowed but no solved problems.

The total points achievable are 30 with the following scales for grading

Grade 3:	at least 12 points
Grade 4:	at least 18 points
Grade 5:	at least 24 points

Incorrect solutions with significant errors, unrealistic results or solutions that are difficult to follow result in 0 points.

Grading results are posted not later than November 5. Review of the grading is offered on Tuesday October 6 at 12:30-13:30. If you cannot attend on this occasion, any objections concerning the grading must be filed in written form not later than two weeks after the regular review occasion.

Good Luck!

1. Consider the following two nonlinear systems:

$$S_1 \quad : \quad \begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + x_1^3/3 - x_2 \end{cases}$$

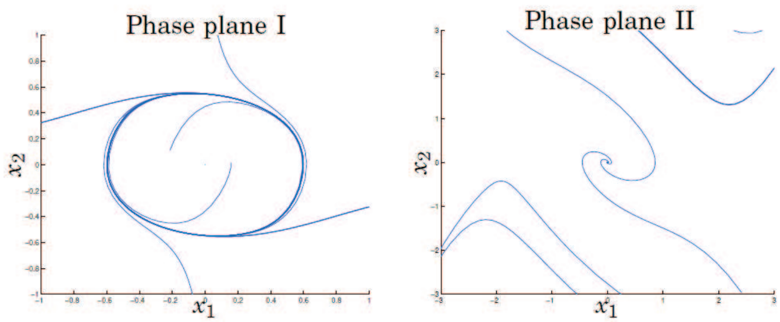
$$S_2 \quad : \quad \begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + 2x_2(1 - 2x_1^2 - 3.5x_2^2) \end{cases}$$

(a) Determine and classify all equilibrium points for the two systems.

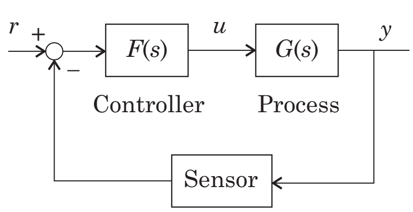
2 p.

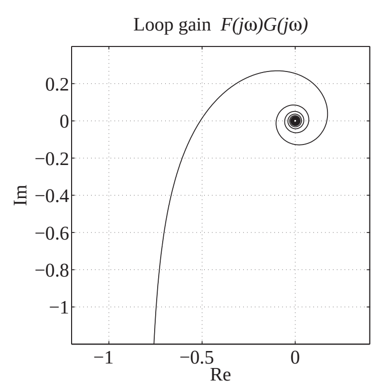
(b) Phase portraits of the two systems are shown in figure below. Copy the portraits, add arrow heads and determine which portrait belongs to which system.

2 p.



2. A time delay process $G(s)$ is controlled by an integral controller $F(s) = K/s$. In the figure below is the blockscheme and the Nyquist plot for FG shown.





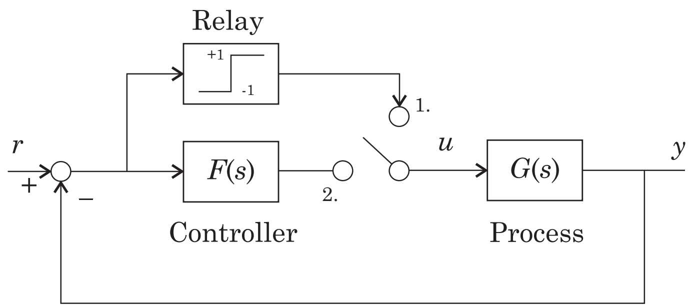
The process output is difficult to measure and the provider of the equipment can neither guarantee linearity nor a higher accuracy than ±20% (i.e. the maximum error is less than 20%). Assuming no dynamics in the sensor, will the accuracy be enough to ensure stability with the current controller setting?

3 p.

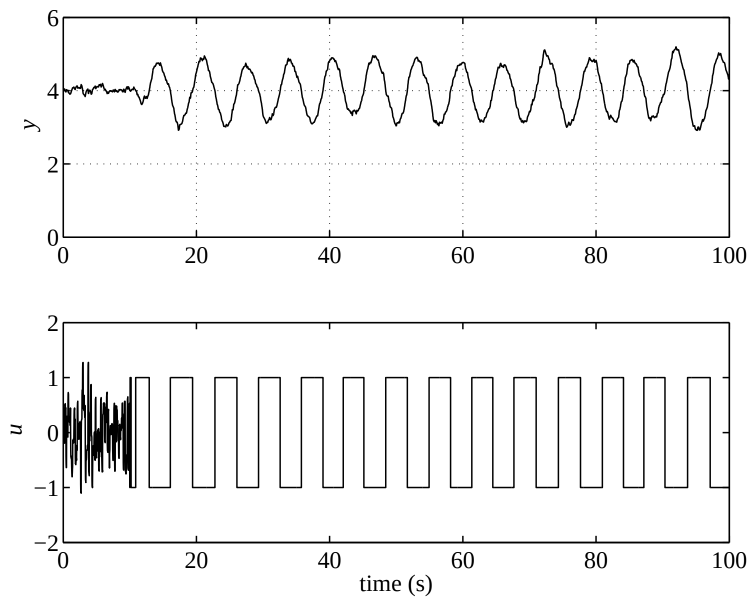
3. A retuning of a PID controller

$$F(s) = K(1 + \frac{1}{sT_i} + sT_d), \quad T_i = 4T_d$$

for a low pass process $G(s)$ is to be carried out in closed loop using an ideal relay, i.e. no hysteresis and unit amplitude (see figure).



The system was put in relay mode (position 1) at $t = 10$ s and the following oscillating output was observed:



What parameter values should the PID-controller have to give an approximate phase margin $\varphi_m \approx 45°$?

4 p.

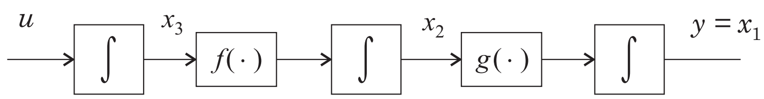
4. Consider the system

$$
\begin{aligned}
\dot{x}_1 &= 5x_1x_2 \\
\dot{x}_2 &= 2x_1^5 + 3u
\end{aligned}
$$

Use Lyapunov based methods to find a feedback control law for u such that it can be concluded that the origin is globally asymptotically stable.

5 p.

5. Consider the nonlinear system in the block diagram below, where u is the control signal, y is the output, $f(\cdot)$ and $g(\cdot)$ are nonlinear smooth functions, and the signals between the blocks are available for control.



How can this system be transformed into a linear system from a reference r to y by internal feedback of the available signals to u.

4 p.

6. In 1696 Bernoulli stated a challenge in solving the so-called *Brachystrone problem*, which can be said to later on lead to the Maximum principle in optimal control. The problem concerns finding the fastest path of a point mass, only exposed to gravity, from A to B in the vertical plane.

Applying the Maximum principle to the *Brachystrone problem* leads to the following equations being satisfied by an extremal:

$$
\begin{aligned}
\dot{x}(t) &= \frac{v(t)}{|\lambda(t)|}\lambda(t); \quad x(0) = 0, \quad x(t_f) = (x_B, y_B) \\
\dot{\lambda}(t) &= \begin{bmatrix} 0 \\ g|\lambda(t)|/v(t) \end{bmatrix}; \quad \lambda(t_f) = \mu \\
H(x, u, \lambda, n_0) &= n_0 - v(t)|\lambda(t)| \equiv 0
\end{aligned}
$$

where $v(t) = \sqrt{2gx_2(t) + v_0^2}$ and g, v_0, x_B and y_B are positive constants, and the notation agree with the notation in the text book.

Show that the optimization problem is normal.

2 p.

7. Determine the control law $u = L(x)$ that minimizes the cost function

$$V = \int_0^\infty (u(t)^2 + x^4(t))dt$$

for the system $\dot{x}(t) = -x(t) + u(t)$.

4 p.

8. Design an MRAS based on stability theory for the process

$$G(p) = \frac{1}{(p+a)^2}$$

and the reference model

$$G_m(p) = \frac{1}{1 + 2p + p^2}$$

4 p.

1a) **System 1**

$\dot{x}_1 = x_2$

$\dot{x}_2 = -x_1 + \frac{x_1^3}{3} - x_2 = x_1(x_1^2/3 - 1) - x_2$

Equilibrium points ($\dot{x}_1 = 0$, $\dot{x}_2 = 0$)

$\Rightarrow x_2 = 0$ and $x_1 = 0$ or $x_1 = \pm\sqrt{3}$

Linearization

$$A = \begin{bmatrix} 0 & 1 \\ -1+x_1^2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \quad \text{for } x = (0,0)$$

$$\det(\lambda I - A) = \lambda(\lambda+1) + 1 = 0$$

$$\lambda^2 + \lambda = -1$$

$$(\lambda + 0.5)^2 = -0.75$$

$$\lambda = -0.5 \pm i\frac{\sqrt{3}}{2} \quad \underline{\text{stable focus}}$$

$$A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \quad \text{for } x = (0, \sqrt{3}) \text{ and } (0, -\sqrt{3})$$

$$\det(\lambda - A) = \lambda(\lambda+1) - 2 = 0$$

$$(\lambda + 0.5)^2 = 2.25$$

$$\lambda = -0.5 \pm 1.5 \quad \underline{\text{saddle}}$$

System 2

Only one equilibrium point: $x = (0,0)$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \Rightarrow \lambda(\lambda - 2) + 1 = 0$$

$$\lambda_{1,2} = 1$$

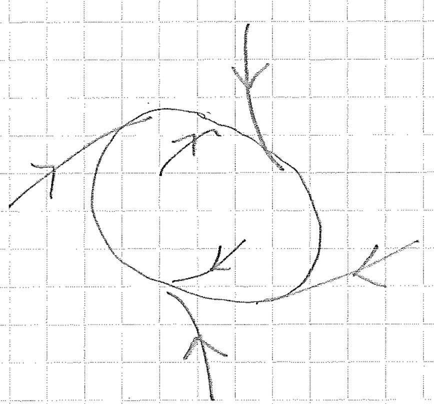
$\therefore$ Star node or tangent node

$$Av = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

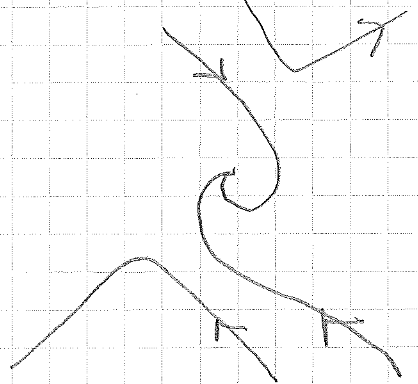
$$\Rightarrow -v_1 + 2v_2 = v_2 \Rightarrow v_1 = v_2$$

Only one eigen vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow$ Unstable tangent

1b) $f_1 = x_2$ for both makes it easy to determine directions



Only equilibrium in the origin $\Rightarrow S_2$



S_1

2) 20 % accuracy implies the sensor is a sector bounded static nonlinearity

$$0.8y \le f(y) \le 1.2y$$

Circle criterion ⇒ stable if FG to the right of the disc $\left(-\frac{5}{4}, -\frac{5}{6}\right)$

From diagram we have that FG is to the right of $-0.75 = -\frac{3}{4}$

Since $-\frac{3}{4} > -\frac{5}{6}$ the system is stable

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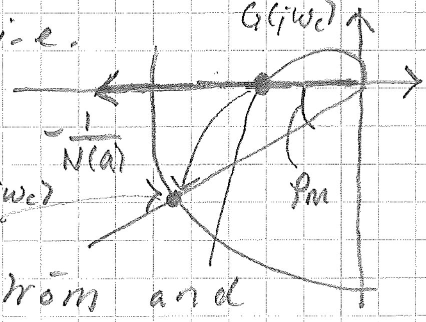
3) From plot we note amplitude $a \approx 1$ and period $T_p \approx 6.3$ s, i.e.

$$\omega_c = \frac{2\pi}{T_p} = 1 \text{ rad/s}$$



From the article by Åström and Hägglund $(\alpha = 4)$

$$T_d = \frac{\tan \varphi_m + \sqrt{1 + \tan^2 \varphi_m}}{2\omega_c} = \underline{\underline{\frac{1+\sqrt{2}}{2}}}$$

$$\Rightarrow T_i = \underline{\underline{2(1+\sqrt{2})}}$$

Descr. fcn $N(a) = \frac{4}{\pi a} \Rightarrow |G(j\omega_c)| = \frac{\pi}{4}$

$$K = \frac{\cos \varphi_m}{|G(j\omega_c)|} = \frac{\sqrt{2}/2}{\pi/4} = \frac{2\sqrt{2}}{\pi}$$ ///

4) $\dot{x}_1 = 5x_1 x_2$ (1)

$\dot{x}_2 = 2x_1^5 + 3u$ (2)

L-candidate

$V = \frac{1}{2}(x_1^2 + x_2^2)$

$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2$

$= x_2(5x_1^2 + 2x_1^5 + 3u)$

$u = \frac{1}{3}(-5x_1^2 - 2x_1^5 - x_2) \Rightarrow \dot{V} = -x_2^2 \leq 0$

$\dot{V} = 0 \Rightarrow x_2 \equiv 0$

Inserting u in (2):

$\dot{x}_2 = 2x_1^5 - 5x_1^2 - 2x_1^5 - x_2 = -5x_1^2 - x_2$

$x_2 \equiv 0 \Rightarrow \dot{x}_2 = 0 \Rightarrow -5x_1 = 0 \Rightarrow x_1 = 0$

Hence, $\dot{V} < 0 \quad \forall x \neq 0$

and $V \geq 0$

$V \to \infty, \ |x| \to \infty$

$V(0) = 0$

La Salle $\Rightarrow$ $x = 0$ globally as. stable

5)
$$\begin{cases} \dot{x}_1 = g(x_2) \\ \dot{x}_2 = f(x_3) \\ \dot{x}_3 = u \end{cases}$$

$$y = x_1 \quad \dot{y} = \dot{x}_1 = g(x_2)$$

$$\ddot{y} = g'(x_2)\dot{x}_2 = g'(x_2)f(x_3)$$

$$\dddot{y} = g''(x_2)\dot{x}_2 f(x_3) + g'(x_2)f'(x_3)\dot{x}_3$$

$$= g''(x_2)f^2(x_3) + g'(x_2)f'(x_3)u$$

Assume $g'(x_2)f'(x_3) \neq 0$

$$u = \frac{1}{g'(x_2)f'(x_3)}\left(-g''(x_2)f^2(x_3) + r\right)$$

gives

$$\frac{d}{dt}\begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

6) $\dot{\lambda}_1 = 0$; $\lambda(t_f) = \mu \Rightarrow \lambda_1 = \mu_1$

$t = 0$: $\quad n_0 = v(0)\sqrt{\lambda_1^2 + \lambda_2^2} = v_0\sqrt{\mu_1^2 + \lambda_2^2}$

$t = t_f$: $\quad n_0 = \sqrt{2gx_2 + v_0^2}\sqrt{\mu_1^2 + \mu_2^2}$

$n_0 = 0$ (abnormal) $\Rightarrow \mu_1 = \mu_2 = 0$

$\Rightarrow n_0 \neq 0 \Rightarrow$ normal (can always be scaled to $n_0 = 1$)

7) $$V = \int_0^\infty (u^2 + x^4)\,dt \quad ; \quad \dot{x} = -x + u$$

Stationary solution since $t_f \to \infty \Rightarrow V_t = 0$

Hamilton - Jacobi - Bellman

$$0 = \min_u \left(V_x \dot{x} + u^2 + x^4 \right)$$

$$= \min_u \big(\underbrace{V_x(-x+u) + u^2 + x^4}_{g(u)} \big)$$

$$\frac{dg}{du} = V_x + 2u = 0 \Rightarrow \boxed{u^* = -\frac{1}{2} V_x}$$

In HJB:

$$V_x\left(-x - \frac{1}{2}V_x\right) + \frac{1}{4}V_x^2 + x^4 = 0$$

$$-\frac{1}{4}V_x^2 - V_x x = -x^4$$

$$(V_x + 2x)^2 = 4x^4 + 4x^2 = 4x^2(1+x^2)$$

$$V_x = -2x\left(1 \pm \sqrt{1+x^2}\right)$$

We have $V_x(0) = 0 \Rightarrow u = 0$ for $x = 0$

$\Rightarrow V(0) = 0$

Since $V > 0 \ \forall x \neq 0 \quad V_x(\varepsilon) > 0$ for $\varepsilon > 0$

$$\Rightarrow V_x = -2x\left(1 - \sqrt{1+x^2}\right)$$

$$\Rightarrow u^*(x) = x\left(1 - \sqrt{1+x^2}\right)$$

8) $$G(p) = \frac{1}{(p+a)^2} \Rightarrow n=2, \; m=0$$

Rel. deg. > 1 $\Rightarrow$ Augmented error

$$\deg(R) = n-1 \Rightarrow R = p + r_1$$
$$\deg(S) = n-1 \Rightarrow S = s_0 p + s_1$$
$$\deg(A_0) = n-m^+-1 \Rightarrow A_0 = p+\alpha$$
$$Q = Q_1 Q_2 = (p+q)(p+q)$$

$$\bar{T} = A_0 B_m = (p+\alpha)\cdot 1$$

1. Determine $e(t) = y(t) - y_m(t)$
where
$$y_m(t) = \frac{1}{1+2p+p^2} r(t)$$
and form the augmented error

$$\varepsilon(t) = e(t) - G(p)\left(\frac{1}{1+q} u(t) + \hat{\theta}(t)\varphi(t)\right)$$

where
$$G(p) = \frac{Q}{A_m A_0} = \frac{(p+q)^2}{(p^2+2p+1)(p+\alpha)}$$

has to be SPR. For example if $q=1$ we get
$$G(p) = \frac{1}{p+\alpha}$$

Here $\hat{\theta}(t) = [r_1 \; -q \; s_0 \; s_1 \; \beta]$

$$\varphi^T(t) = \left[\frac{1}{Q}u(t) \;\; \frac{p}{Q}y(t) \;\; \frac{1}{Q}y(t) \;\; -\frac{p+\alpha}{Q}r(t)\right]$$

See p. 58 in Supplement

2. Update $\hat{\theta}$ according to

$$\dot{\hat{\theta}} = \gamma \varphi(t) \varepsilon(t)$$

3 Compute the control signal

$$u(t) = -\hat{\theta}^T(t)(p+g)\varphi(t)$$