CHALMERS UNIVERSITY OF TECHNOLOGY
Department of Signals and Systems
Division of Automatic Control, Automation and Mechatronics

EXAMINATION IN NONLINEAR AND ADAPTIVE CONTROL

(ESS076)

Wednesday January 11, 2012

Time and place: 14.00 – 18.00, V-building

Teacher: Claes Breitholtz, phone 3718

The following items are allowed:

1. *Control Theory* (Glad, Ljung) or *Applied Nonlinear Control* (Slotline/Li)
2. ESS076 Supplement
3. Mathematical handbooks of tables such as Beta Mathematics Handbook

Notes, calculator, mobile telephones, laptops or palmtops, are not allowed!
Reasonable notes in textbook are allowed but no solved problems.

The total points achievable are 30 with the following scales for grading:

Grade 3: at least 12 points
Grade 4: at least 18 points
Grade 5: at least 24 points

Incorrect solutions with significant errors, unrealistic results or solutions that are difficult to follow result in 0 points.

Grading results are posted not later than January 27. Review of the grading is offered Monday January 30 at 12.30-13.30. If you cannot attend this occasion, any objections concerning the grading must be filed in written form not later than two weeks after the regular review occasion.

Good luck!

1. Consider the following nonlinear system:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ (2-\cosh x_1)\cdot x_2 - \sinh x_1 \end{bmatrix}$$

Compute the stationary points of the system and formulate the corresponding linearized systems. What conclusion regarding local stability of the zero solution can be drawn from ***Lyapunov's linearization method*** (section 12.1 in the text-book)?

3 points

2. Consider the following nonlinear system:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 - x_1^3 \\ 3x_1 - 2x_2^5 \end{bmatrix}$$

Show that the above system is asymptotically stable. Is the stability global or local?

4 points

3. The linear system $y(t) = G(p)u(t)$ is given. The transfer function G is given by

$$G(p) = \frac{p^3 + 2p^2 + 3p + 8}{(p+1)^4}$$

Is it possible to write this system as a linear state space model, in such a way that two positively definite matrices, P and Q, according to the Kalman-Yakubovich Lemma is found? Motivate Your answer carefully!

4 points

4. The linear time invariant system

$$G(s) = \frac{1}{(s+1)^6}$$

is connected to the nonlinear function given by $output = sign\{input\}$ in a feedback structure. Investigate the possibility of periodic solutions by use of the describing function method. Also, give estimates of the possible frequency and amplitude of an oscillation.

6 points

5. Consider the nonlinear system:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 + 2x_1^2 \\ x_3 + u \\ x_1 - x_3 \end{bmatrix}, y = x_1$$

Design a state feedback control law such that the output y asymptotically tracks the reference signal $r(t) = \sin(\omega t)$ (where ω is an arbitrary known frequency).

4 points

6. A time-varying automotive system is modelled by the law of motion

$$\frac{d}{dt}\left(m(t) \cdot \frac{dy}{dt}\right) = u(t)$$

where y is the position, m is the mass of the vehicle and u is the applied force. In principle, the total mass is monotonously decreasing due to loss of fuel. However, loss of fuel is a rather **slow process** compared to changes in speed and position. Assume that there is a separate estimation system for the continuous accurate update of the mass and its derivatve. Can a gain scheduled **proportional controller** K_p, using the estimates of m and $\dot{m}$, be used in a system where the position y should track a given reference y_{ref}? The resulting phase margin should be 45 degrees. Motivate carefully!

3 points

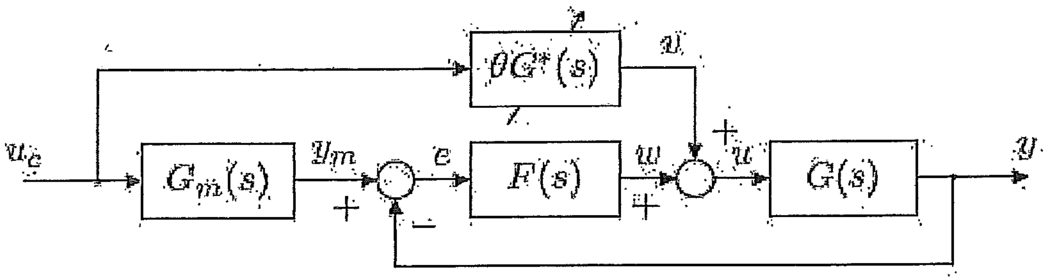
7. In this problem, we shall study an adaptive controller as depicted in the figure below. The controller uses a combination of feedforward and feedback. The process is given by the transfer function

$$G(s) = G_n(s)/\theta_0$$

where $G_n(s)$ is assumed to be known with $G_n(0) = 1$ and $1/\theta_0$ is the unknown steady-state gain of the process. $F(s)$ is a fixed regulator and $G_m(s)$ is a reference model for the desired closed-loop system. The feedforward signal $v(t)$ is given by

$$v(t) = \theta G^*(p)u_c(t) = \theta[G_m(p)/G_n(p)]u_c(t)$$

where $G^* = G_m/G_n$ is a fixed filter and the scalar θ shall be adapted online.



(a) Show that the desired closed-loop transfer function (from u_c to y) is obtained if θ has the correct value θ_0.

2 points

(b) Determine an adaption law for θ using the MIT-rule, and using the criterion $J = w^2$.
(Hint: Use the approximation $FG/(1+FG) \approx 1$.)

2 points

(c) Derive an error model for w and show stability of the adaptation rule in (b) under suitable assumptions.

2 points

1.
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ (2-\cosh x_1)x_2 - \sinh x_1 \end{bmatrix}$$

Stationary points: $x_2 = 0 \Rightarrow \sinh x_1 = 0 \Rightarrow$
$x_1 = 0 \Rightarrow (0,0)$ is the only stat. point.

$$\begin{bmatrix} \Delta\dot{x}_1 \\ \Delta\dot{x}_2 \end{bmatrix} = \left[\begin{bmatrix} 0 & 1 \\ -\sinh x_1 \cdot x_2 - \cosh x_1 & 2-\cosh x_1 \end{bmatrix}\right]_{\substack{x_1=0 \\ x_2=0}} =$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\begin{vmatrix} \lambda & -1 \\ 1 & \lambda-1 \end{vmatrix} = \lambda^2 - \lambda + 1 = 0 \Rightarrow$$

both eigenvalues are in the RHP $\Rightarrow$ <u>unstable!</u>

2. $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_2 - x_1^3 \\ 3x_1 - 2x_2^5 \end{bmatrix}$$

Try $V = a x_1^m + b x_2^n$

$$\dot{V} = -(x_2 + x_1^3)\frac{\partial V}{\partial x_1} + (3x_1 - 2x_2^5)\frac{\partial V}{\partial x_2} =$$

$$= -(x_2 + x_1^3)\, m a x_1^{m-1} + (3x_1 - 2x_2^5)\, n b x_2^{n-1} =$$

$$= -am\, x_1^{m-1} x_2 - am\, x_1^{m+2} +$$
$$+ 3bn\, x_1 x_2^{n-1} - 2bn\, x_2^{n+4}$$

Make $m - 1 = 1 \Rightarrow m = 2$ and

$n - 1 = 1 \Rightarrow n = 2$

$$\dot{V} = \underbrace{(6b - 2a)}_{=0 \;\Rightarrow\; a = 3b} x x_2 - 2a x_1^4 - 4b x_2^6 =$$

$a = 3b$. Let $b = 1/4$

$$= -\frac{3}{2}x_1^4 - x_2^6 = -\left(\frac{3}{2}x_1^4 + x_2^6\right) < 0$$

$\forall\ x_1$ and $x_2 \neq 0$. $\Rightarrow$

$\dot{V}$ is a neg. def. function everywhere

$\Rightarrow$ the origin is globally asympt. stable.

3. Given $G(p)$ a minimal order state-space model is (for example)

$$\dot{x} = \begin{bmatrix} -4 & 1 & 0 & 0 \\ -6 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \\ 3 \\ 8 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$$ (or any other state-space model!)

Consider the Lyapunov equation

$$PA + A^T P = -Q$$

~~K-Y-Lemma states that a necessary condition for finding P and Q both > 0 is that that all eigenvalues in LHP. G(p) is a SPRF~~

$P > 0$ and $Q > 0 \Rightarrow G(p)$ is SPR.

$$G(p) = \frac{p^3 + 2p^2 + 3p + 8}{(p+1)^4}$$

The polynomial $p^3 + 2p^2 + 3p + 8$ does <u>not</u> have all its zeros in the LHP $\Rightarrow$ $G(p)$ is non-minimum phase $\Rightarrow$

$\Rightarrow G(p) = C(pI - A)^{-1}B$ is not SPR $\Rightarrow$

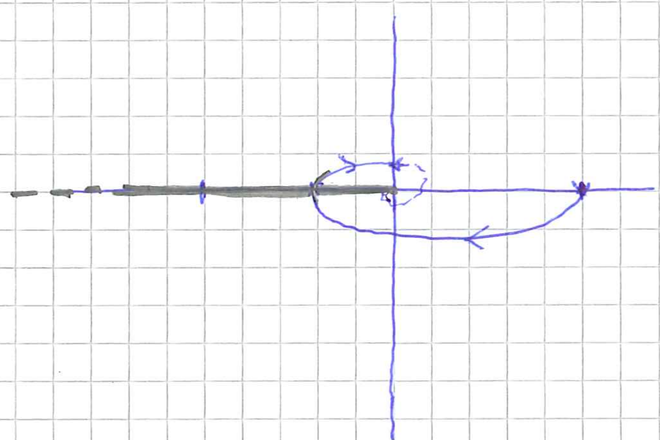
$\Rightarrow$ You cannot find P and $Q > 0$!

4. $G(j\omega) = \frac{1}{(j\omega+1)^6}$

$$|G(j\omega)| = \left(\frac{1}{\sqrt{\omega^2+1}}\right)^6 = \frac{1}{(\omega^2+1)^3}$$

$$\angle G(j\omega) = -6\,\mathrm{atan}(\omega)$$

ω	$\lvert G \rvert$	$\angle G$
0	1	0
$1/\sqrt{3}$	27/64	-180°
1	1/8	-270°
∞	0	-540°



$Y_f(C) = \frac{4}{\pi C} \Rightarrow$

$\Rightarrow -\frac{1}{Y_f(C)} = -\frac{\pi C}{4}$

One possible cross: $-\frac{27}{64} = -\frac{\pi}{4} C$

$\Rightarrow C = \underline{\underline{\frac{27}{16\pi}}}$ ($\approx 0{,}537 \approx 0.54$)

and $\omega = \underline{\underline{1/\sqrt{3}}}$ ($\approx 0.577 \approx 0.58$ rad/sec)

5. $$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} x_2 + 2x_1^2 \\ x_3 + u \\ x_1 - x_3 \end{pmatrix}, \quad y = x_1$$

$$\dot{y} = \dot{x}_1 = x_2 + 2x_1^2$$
$$\ddot{y} = \dot{x}_2 + 4\dot{x}_1 x_1 = 4(x_2 + 2x_1^2)x_1 + x_3 + u$$

Minimum phase? $y(t) \equiv 0 \Rightarrow \dot{x}_3 = -x_3 \Rightarrow$
Minimum phase, as the zerodynamics are clearly stable! Let $e = y - y_R$:

$$\dot{e} = \dot{y} - \dot{y}_R = x_2 + 2x_1^2 - \dot{y}_R$$

$$\ddot{e} = \ddot{y} - \ddot{y}_R = x_3 + u + 4(x_2 + 2x_1^2)x_1 - \ddot{y}_R$$

Take $u = -x_3 - 4(x_2 + 2x_1^2)x_1 + \ddot{y}_R -$
$- k_1 e - k_2 \dot{e} \Rightarrow$

$$\ddot{e} + k_2 \dot{e} + k_1 e = 0$$

which is a stable system for $k_1 > 0, k_2 > 0$

$\Rightarrow e(t) \to 0$ as $t \to \infty$.

5.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 + 2x_1^2 \\ x_3 + u \\ x_1 - x_3 \end{bmatrix} = \overset{f =}{\begin{bmatrix} x_2 + 2x_1^2 \\ x_3 \\ x_1 - x_3 \end{bmatrix}} + \overset{g =}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} u$$

$$y = x_1 \Rightarrow \dot{y} = \dot{x}_1 = x_2 + 2x_1^2 \quad \text{(not degree 1 !)}$$

$$\ddot{y} = \dot{x}_2 + 4x_1\dot{x}_1 = x_3 + u + 4x_1x_2 + 8x_1^3$$

$\Rightarrow$ degree 2 !

From (17.9) we have

$$u = \frac{r - L_f^2 h}{L_g L_f h} \qquad h = x_1 \text{ in this case}$$

$$L_f h = \nabla h \cdot f = f_1 \overbrace{\frac{\partial h}{\partial x_1}}^{=1} + f_2 \overbrace{\frac{\partial h}{\partial x_2}}^{=0} + f_3 \overbrace{\frac{\partial h}{\partial x_3}}^{=0} =$$

$$= x_2 + 2x_1^2$$

$$L_f^2 h = L_f(L_f h) = f_1 \frac{\partial}{\partial x_1}(L_f h) + \ldots + f_3 \frac{\partial}{\partial x_3}(L_f h)$$

$$= (x_2 + 2x_1^2) \cdot 4x_1 + \cancel{(x_3 + u) \cdot} \; x_3 \cdot 1 + (x_1 - x_3) \cdot 0$$

$$= 4x_1x_2 + 8x_1^3 + x_3$$

$$L_g L_f h = \overbrace{g_1}^{=0} \cdot \frac{\partial}{\partial x_1} L_f h + \overbrace{g_2}^{=1} \cdot \underbrace{\frac{\partial}{\partial x_2} L_f h}_{=1} + \overbrace{g_3}^{=0} \frac{\partial}{\partial x_3} L_f h$$

$$u = r - (4x_1x_2 + 8x_1^3 + x_3)$$

6. $\frac{d}{dt}(m\dot{y}) = m\ddot{y} + \dot{m}\dot{y} = u$

$\hat{m} = \alpha$ and $\hat{\dot{m}} = \beta$

α and β can be regarded as constants compared to y and $\dot{y}$!

$$Y(s) = \frac{U(s)}{\alpha s^2 + \beta s}$$

P-~~C~~ontroller $\Rightarrow u = K_p(r-y)$

$$T(s) = \frac{K_p}{\alpha s^2 + \beta s + K_p}$$

$\alpha > 0$ as $m(t) > 0 \ \forall t$
$K_p > 0$ (if chosen so)

But: $\beta \leq 0 \ \forall t$ as
$m(t)$ decreases, then $\dot{m}(t) \leq 0 \ \forall t$

So if $\beta = 0 \Rightarrow T(s) = \frac{K_p}{\alpha s^2 + K_p}$
an oscillator!

~~[illegible]~~ if $\beta < 0 \Rightarrow T(s)$ is unstable

A P-controller cannot stabilize the system!

The Hamiltonian is $H = u^2 + \lambda(bu - cx) = \left(u + \frac{\lambda b}{2}\right)^2 - \frac{\lambda^2 b^2}{4} - \lambda cx$, which is minimized for

$$u = \begin{cases} 0, & -b\lambda/2 \le 0 \\ -b\lambda/2, & 0 < -b\lambda/2 \le u_{\max} \\ u_{\max}, & u_{\max} < -b\lambda/2 \end{cases}$$

We also have the adjoint equation

$$\dot{\lambda} = -H_x^T = c\lambda\,,\ \lambda(T) = -a \quad \Leftrightarrow \quad \lambda(t) = -ae^{c(t-T)} < 0\,.$$

The optimal control strategy thus becomes

$$u = \begin{cases} \frac{ba}{2}e^{c(t-T)}, & \frac{ba}{2}e^{c(t-T)} \le u_{\max} \\ u_{\max}, & \text{otherwise} \end{cases}$$

or with numerical values, $u(t) = \min\left\{44 \cdot e^{0.2(t-7)}, 30\right\}$.

5.

(a) The transfer function is determined by "the direct path divided by [one plus the loop gain]". This leads to

$$\frac{y}{u_c} = \frac{GFG_m + GG^*}{1+GF} = \frac{GFG_m + G\theta G_m/G_n}{1+FG} = \frac{GFG_m + \frac{G_n}{\theta_0}\theta G_m/G_n}{1+FG} = \frac{GF+1}{1+FG}G_m = G_m$$

if $\theta = \theta_0$.

(b) The MIT rule uses an update law moving in the negative direction of the gradient of J with respect to θ. We get $\frac{\partial J}{\partial \theta} = 2w\frac{\partial w}{\partial \theta}$. The update law should thus be

$$\dot{\theta} = -\gamma w \frac{\partial w}{\partial \theta}\,. \tag{1}$$

The only signal that is unclear if we can measure in this update law is $\frac{\partial w}{\partial \theta}$. So try to determine it by first writing w as a function of θ:

$$w = \frac{FG_m - FG\theta G_m/G_n}{1+FG}u_c \Rightarrow \frac{\partial w}{\partial \theta} = \frac{-FGG_m/G_n}{1+FG}u_c \approx -\frac{G_m}{G_n}u_c$$

if we use the proposed approximation $\frac{FG}{1+FG} \approx 1$. So $\dot{\theta} = \gamma w \frac{G_m}{G_n}u_c$ is an approximation of (1) we settle for with the MIT rule.

(c) We want to reach an error model $w = H(s)[\tilde{\theta}^T\varphi]$, where $\tilde{\theta} = \theta - \theta_0$. If $H(s)$ is SPR, the update law $\dot{\tilde{\theta}} = \gamma w \varphi$ can be proven to be asymptotically stable with Lyapunov analysis. From (b) we have

$$w = \frac{FG_m - FG\theta G_m/G_n}{1+FG}u_c = \frac{FG}{1+FG}\left[\tilde{\theta}\frac{G_m}{G_n}u_c\right],$$

so if the "closed-loop system" $FG/(1+FG)$ is SPR, stability of the update law in (b) can be proven.

Note. For full points in (c), a brief analysis (or referring to the correct details/equations in the course material) for showing stability under the SPR assumption is needed.