CHALMERS UNIVERSITY OF TECHNOLOGY
Department of Signals and Systems
Division of Automatic Control, Automation and Mechatronics

# EXAMINATION IN NONLINEAR AND ADAPTIVE CONTROL

(Course ESS076)

Friday August 26, 2011

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| Time and place: | 08:30 - 12:30 at Maskin |
| Teacher: | Torsten Wik (5146 or 0739 870570) |

**The following items are allowed:**

1. *Control Theory* (Glad Ljung) or *Applied Nonlinear Control* (Slotine/Li)
2. ESS076 Supplement
3. Mathematical handbooks of tables such as Beta Mathematics Handbook.

**Notes, calculator, mobile telephones, laptops or palmtops, are not allowed!** Reasonable notes in the textbook are allowed but no solved problems.

The total points achievable are 30 with the following scales for grading

| | |
|---|---|
| Grade 3: | at least 12 points |
| Grade 4: | at least 18 points |
| Grade 5: | at least 24 points |

Incorrect solutions with significant errors, unrealistic results or solutions that are difficult to follow result in 0 points.

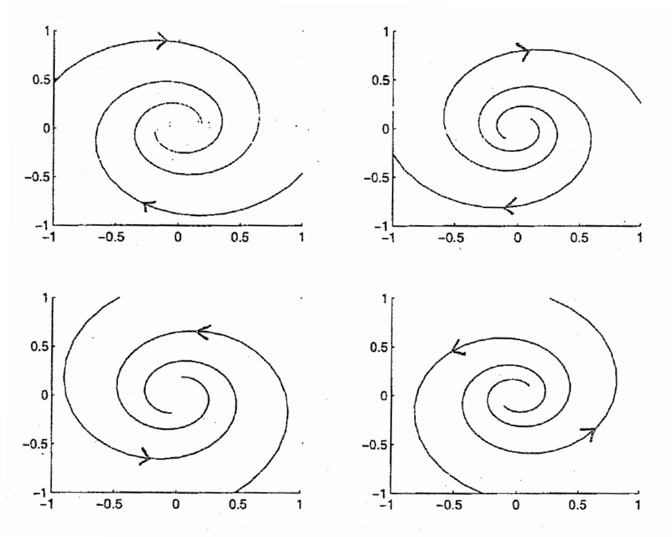
Grading results are posted not later than September 12. Review of the grading is offered on Monday September 12 at 12:30-13:30. If you cannot attend at this occasion, any objections concerning the grading must be filed in written form not later than two weeks after the regular review occasion.

Good Luck!

1. (a) Consider the linear system

$$\frac{d}{dt}x(t) = \begin{bmatrix} -0.2 & 1 \\ -1 & -0.2 \end{bmatrix} x(t)$$

which of the following phase portraits can qualitatively describe the behavior of this system?




2 p.

(b) Consider the first order system

$$G(s) = \frac{1}{1+s}$$

having input $u$ and output $y$.

Show that of all controllers having bounded control signal, $|u(t)| \leq 1$, the controller

$$u(t) = \begin{cases} -1, & y(t) > 0 \\ 1, & y(t) < 0 \end{cases}$$

is the controller that takes $y$ to zero from an arbitrary initial value in the shortest time.

3 p.

2. Equations on the form

$$\ddot{y} + h(y)\dot{y} + g(y) = 0$$

are called Lienard equations and have been intensively studied during the development of radio and vacuum tube technology.

(a) Let $x_1 = y$, $x_2 = \dot{y}$ and give the corresponding state space model. What conditions are required for the origin to be the only stationary point? 2 p.

(b) Use the Lyapunov function candidate

$$V(x) = \int_0^{x_1} g(y)dy + \frac{1}{2}x_2^2$$

to derive necessary conditions on $g$ and $h$ for the origin to be a globally asymptotically stable equilibrium (stationary point). 2 p.

(c) Suppose these conditions are not necessarily fulfilled but we have the possibility to control the Lienard equation with a nonlinear feedback using an input signal $u$:

$$\ddot{y} + h(y)\dot{y} + g(y) - k(y)u(y, \dot{y}) = 0$$

Give an example of a control law for $u$ that makes the origin globally asymptotically stable and state the necessary conditions on $k$. 2 p.

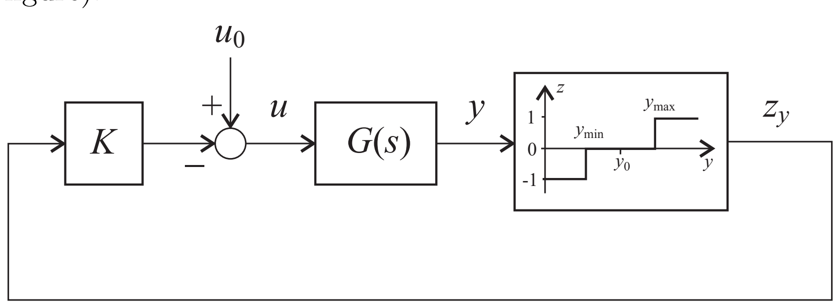
3. The level of the output $y$ from the linear system

$$G(s) = \frac{1}{s(1 + 0.01s)^2}$$

is measured such that

$$z_y = \begin{cases} 1 & , y > y_{\max} \\ 0 & , y \in [y_{\min}, y_{\max}] \\ -1 & , y < y_{\min} \end{cases}$$

The input $u$ is then controlled such that $u = u_0 - Kz_y$, where $u_0$ is the stationary input that gives the desired output level $y_0 = (y_{\min} + y_{\max})/2$ (see figure).




If $K$ is too large the system will exhibit self-oscillations. Determine a value of $K$ that should probably not be exceeded!

Might be useful:

$$\frac{d}{dx}\left(\frac{1}{x}\sqrt{1 - \frac{\alpha^2}{x^2}}\right) = \frac{2\alpha^2 - x^2}{x^3\sqrt{x^2 - \alpha^2}}$$

6 p.

4. To ensure a rapid and unique solution to the optimization carried out online in MPC the optimization problem can be formulated as a quadratic program (QP):

$$\begin{aligned} \min_z \quad & \frac{1}{2} z^T Q z + c^T z \\ \text{subject to} \quad Az & \leq b \\ Ez & = d \end{aligned} \tag{1}$$

Consider the system

$$\begin{aligned} x(k+1) &= 0.8x(k) + u(k) \\ y(k) &= 2x(k) \end{aligned}$$

with the input constraint $-1 \leq u \leq 1$ and output constraints $-2 \leq y \leq 2$.

Assume we want to minimize the quadratic cost criterion

$$V(k) = \sum_{i=1}^{2} (\hat{y}(k+i) - r(k+i))^2 + \sum_{i=0}^{1} q \Delta u(k+i)^2 \tag{2}$$

where $\hat{y}(k+i)$ is the output signal at $k+i$, as predicted at $k$, $r(k+i)$ is the reference value, and $\Delta u(k+i) = u(k+i) - u(k+i-1)$ is the control increment at $k+i$.

This optimization problem can now be reformulated as a QP as follows:

(a) Reformulate the state space model to the form

$$\begin{aligned} \xi(k+1) &= \Phi \xi(k) + \Gamma \Delta u(k) \\ y(k) &= C \xi(k) \end{aligned} \tag{3}$$

where $\xi(k) = \begin{bmatrix} x(k) & u(k-1) \end{bmatrix}^T$.

1 p.

(b) Let

$$z = \begin{bmatrix} \hat{\xi}^T(k|k) & \hat{\xi}^T(k+1|k) & \cdots & \hat{\xi}^T(k+2|k) & \Delta u(k) & \Delta u(k+1) \end{bmatrix}^T$$

where the predictions are determined using (3).

Rewrite the constraints on $u$ and $y$ to fit the QP formulation (i.e. determine $A$ and $b$ in (1).

3 p.

(c) Express the state space model as the equality constraint $Ez = d$.

2 p.

(d) Reformulate the cost (2) to the form in (1) (i.e. determine $Q$ and $c$).

2 p.

5. (Source: K Holmåker) The student Emilia wants to plan her studies to maximize her efficiency. Let $x$ be her level of knowledge and $u$ her working effort intensity. We assume the increase in knowledge is proportional to her effort, which should give $\dot{x} = bu$. However, since Emilia also forget a given fraction of the knowledge she has acquired

$$\dot{x}(t) = bu(t) - cx(t), \quad 0 \leq t \leq T, \quad x(0) = x_0,$$

where $b$ and $c$ are positive constants. The maximum effort Emilia is prepared to spend on the course is $u_{\max}$, i.e.

$$0 \leq u(t) \leq u_{\max}, \quad 0 \leq t \leq T$$

Emilias goal is to achieve maximum knowledge at the time of the exam ($t = T$) with a minimum effort. Therefore she wants to maximize

$$ax(T) - \int_0^T u^2(t)dt$$

where $a$ is a positive constant.

Determine the optimal control strategy for $T = 7$ (weeks), $u_{\max} = 30$ (hours/week), $a = 22$, $b = 4$ and $c = 0.2$.

5 p.

1 a) $\det(\lambda I - A) = \det\begin{bmatrix} \lambda+0.2 & -1 \\ 1 & \lambda+0.2 \end{bmatrix}$

$$= (\lambda + 0.2)^2 + 1 = 0$$

$\Rightarrow \lambda = -0.2 \pm i$ i.e. stable focus

$x_1 = 0$ and $x_2 > 0 \Rightarrow \dot{x}_1 > 0$

$\therefore$ Upper left plot

b) State space model for $G(s)$

$$\begin{cases} \dot{x} = -x + u \\ y = x \end{cases}$$

which is controllable and since

$\Psi(t_f, x(t_f)) = x(t_f) = 0 \Rightarrow \Psi_x = 1$

has full rank Theorem 18.5 and 18.6 $\Rightarrow$ The optimal control is of Bang-bang type with no changes $(n-1=0)$

To reach the origin we must have

$y(t) > 0 \Rightarrow u(t) = -1$

$y(t) < 0 \Rightarrow u(t) = 1$

2) $\ddot{y} + h(y)\dot{y} + g(y) = 0$

a) $\left.\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \end{aligned}\right\} \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -h(x_1)x_2 - g(x_1) \end{cases}$

Equilibrium $0 = x_2$

$0 = h(x_1)x_2 + g(x_1) = g(x_1) = 0$

Unique iff $g(x_1) \neq 0 \;\forall x_1 \neq 0$

b) $V(x) = \int_0^{x_1} g(y)\,dy + \frac{1}{2}x_2^2$

$> 0 \;\forall x \neq 0$ iff $\underline{\underline{\int_0^{x_1} g(y)\,dy > 0 \;\forall x_1 \neq 0}}$

$\dot{V} = g(x_1)\dot{x}_1 + x_2\dot{x}_2$

$= g(x_1)x_2 + x_2(-h(x_1)x_2 - g(x_1))$

$= -h(x_1)x_2^2 < 0 \;\forall x \neq 0$ iff $\underline{\underline{h(x_1) > 0 \;\forall x_1 \neq 0}}$

c) $\dot{x}_2 = -h(x_1)x_2 - g(x_1) + k(x_1)u(x_1, x_2)$

if $\underline{\underline{k(x_1) \neq 0 \;\forall x_1}}$ we can apply

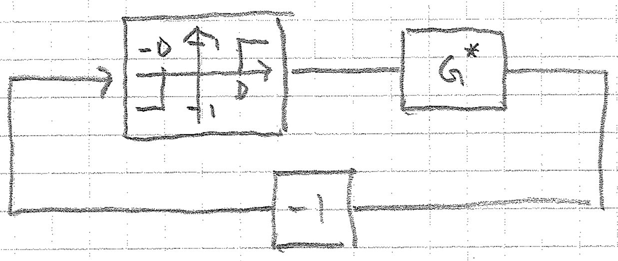
$u = -\frac{1}{k}\left(h(x_1)x_2 + g(x_1) - l_1x_1 - l_2x_2\right)$

$\Rightarrow \dot{x} = \begin{bmatrix} 0 & 1 \\ -l_1 & -l_2 \end{bmatrix} x$

$\det\begin{bmatrix} \lambda & -1 \\ l_1 & \lambda + l_2 \end{bmatrix} = 0 \Rightarrow \lambda^2 + \lambda l_2 + l_1 = 0$

$\lambda_{1,2} \in$ LHP iff $l_1 > 0$ and $l_2 > 0$

3) The problem is equivalent to




where

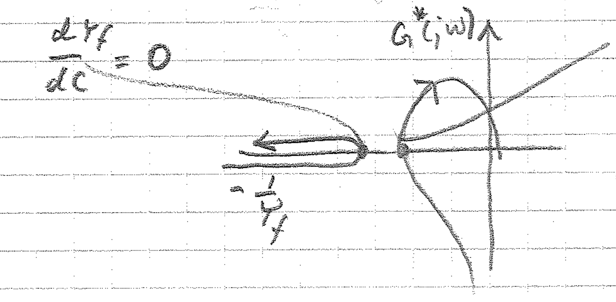
$$G^* = \frac{K}{s(1+0.01s)^2}$$

and $$D = \frac{y_{max} - y_{min}}{2}$$

The describing fcn for the 'relay' is (GL p 405)

$$Y_f(C) = \frac{4}{\pi C}\sqrt{1 - \frac{D^2}{C^2}} \quad , \quad C > D$$

Selfoscillation is avoided if $-\frac{1}{Y_f}$ 'turns' to the left of the point where $G^*(j\omega)$ crosses the negative real axis




$$\angle G^*(j\omega) = -90^\circ - 2\,\text{atan}\, 0.01\omega = -180^\circ$$

$$\Rightarrow \omega = 100$$

$$\frac{dY_f}{dC} = \frac{4}{\pi}\left(\frac{2D^2 - C^2}{C^3\sqrt{C^2 - D^2}}\right) = 0 \Rightarrow C = D\sqrt{2}$$

$$\Rightarrow Y_f = \frac{4}{\pi D\sqrt{2}}\sqrt{1 - \frac{1}{2}} = \frac{2}{\pi D}$$

$$-\frac{1}{Y_f} = G^* \Rightarrow \frac{\pi D}{2} = \frac{K}{100 \cdot 2} \Rightarrow \underline{\underline{K = 50\pi\,(y_{max} - y_{min})}}$$

4)

$$x(k+1) = 0.8\,x(k) + u(k)$$

$$y(k) = 2x(k)$$

a) $\xi(k) = [\,x(k)\ u(k-1)\,]^T$

$$\xi(k+1) = \begin{bmatrix} x(k+1) \\ u(k) \end{bmatrix} = \underbrace{\begin{bmatrix} 0.8 & 1 \\ 0 & 1 \end{bmatrix}}_{\Phi} \underbrace{\begin{bmatrix} x(k) \\ u(k-1) \end{bmatrix}}_{\xi(k)} + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\Gamma} \Delta u(k)$$

$$y(k) = \underbrace{[\,2 \quad 0\,]}_{C}\,\xi(k)$$

b) $z(k) = [\,\xi^T(k)\ \ \xi^T(k+1)\ \ \xi^T(k+2)\ \ \Delta u(k)\ \ \Delta u(k+1)\,]$

$$1 \geq u(k) = \Delta u(k) + u(k-1) = \Delta u(k) + \underbrace{[\,0 \ \ 1\,]}_{C_u}\xi(k)$$

$$\Rightarrow [\,C_u \quad 0 \quad 0 \quad 1 \quad 0\,]\,z \leq 1$$

$$-1 \leq u(k) \Rightarrow [\,-C_u \quad 0 \quad 0 \quad -1 \quad 0\,]\,z \leq 1$$

p.s.s. $u(k+1)$

$$-2 \leq y(k) = 2x(k) = \underbrace{[\,2 \quad 0\,]}_{C_\xi}\,\xi(k)$$

$$\Rightarrow [\,-C_\xi \quad 0 \quad 0 \quad 0 \quad 0\,]\,z \leq 2$$

$$2 \geq y(k) \Rightarrow [\,C_\xi \quad 0 \quad 0 \quad 0 \quad 0\,]\,z \leq 2$$

p.s.s $y(k+1)$ and $y(k+2)$

$$\therefore \quad \underbrace{\begin{bmatrix} C_u & 0 & 0 & 1 & 0 \\ -C_u & 0 & 0 & -1 & 0 \\ 0 & C_u & 0 & 0 & 1 \\ 0 & -C_u & 0 & 0 & -1 \\ -C_\xi & 0 & 0 & 0 & 0 \\ C_\xi & 0 & 0 & 0 & 0 \\ 0 & -C_\xi & 0 & 0 & 0 \\ 0 & C_\xi & 0 & 0 & 0 \\ 0 & 0 & -C_\xi & 0 & 0 \\ 0 & 0 & C_\xi & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \xi(k) \\ \xi(k+1) \\ \xi(k+2) \\ \Delta u(k) \\ \Delta u(k+1) \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

$$z \leq b$$

c) Three predictions $\hat{\xi}(k)$, $\hat{\xi}(k+1)$ and $\hat{\xi}(k+2)$ are needed:

$$\xi(k) = \Phi \xi(k-1) + \Gamma \Delta u(k-1) = d_0 \text{ known RH}$$
$$\xi(k+1) = \Phi \xi(k) + \Gamma \Delta u(k)$$
$$\xi(k+2) = \Phi \xi(k+1) + \Gamma \Delta u(k+1)$$

$$\therefore \quad \underbrace{\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ -\Phi & I & 0 & -\Gamma & 0 \\ 0 & -\Phi & I & 0 & -\Gamma \end{bmatrix}}_{E} z = \underbrace{\begin{bmatrix} d_0 \\ 0 \\ 0 \end{bmatrix}}_{d}$$

d) $V(k) = \sum_{1}^{2} \underbrace{(\hat{y}(k+i) - r(k+i))^2}_{\hat{y}^2 - 2r\hat{y} + r^2} + g \sum_{0}^{1} \Delta u^2(k+i)$

Since $r$ is not free minimizing $V$ is the same as minimizing

$$V^* = \sum_{1}^{2} (C\xi(k+i))^2 - 2r(k+i)\,C\xi(k+i) + g\sum_{0}^{1} \Delta u^2(k+i)$$

$$= \underbrace{[\xi(k)\ \ \xi(k+1)\ \ \xi(k+2)\ \ \Delta u(k)\ \ \Delta u(k+1)]}_{z^T} \cdot \underbrace{\begin{bmatrix} 0 & & & & \\ & CC^T & & 0 & \\ & & CC^T & & \\ & 0 & & g & \\ & & & & g \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \xi(k) \\ \vdots \\ \Delta u(k+1) \end{bmatrix}}_{z} -$$

$$- \underbrace{2[0\ \ r(k+1)C\ \ r(k+2)C\ \ 0\ \ 0]}_{c^T} z$$

5. Optimal control problem with

$f(x) = bu - cx$, $L = u^2$, $\Phi(x) = -ax$

min instead of max

Adjoint eqn

$$\dot{\lambda} = -H_x = c\lambda, \quad \lambda(T) = -a$$

which has the solution $\lambda(t) = -ae^{c(t-T)}$

Minimization of the Hamiltonian:

$$H = u^2 + \lambda(bu - cx)$$

$$\Rightarrow \begin{cases} u = -\frac{b\lambda}{2} & \text{if } -\frac{b\lambda}{2} < u_{max} \\ u = u_{max} & \text{if } -\frac{b\lambda}{2} \geq u_{max} \end{cases}$$

i.e. $u = \min\left(44e^{0.2(t-7)}, 30\right)$