CHALMERS UNIVERSITY OF TECHNOLOGY
Department of Signals and Systems
Division of Automatic Control, Automation and Mechatronics

EXAMINATION IN NONLINEAR AND ADAPTIVE CONTROL

(Course ESS076)

Tuesday May 24, 2011

Time and place:	14:00 - 18:00 at Maskin
Teacher:	Torsten Wik (5146 or 0739 870570)

The following items are allowed:

1. *Control Theory* (Glad Ljung) or *Applied Nonlinear Control* (Slotine/Li)
2. ESS076 Supplement
3. Mathematical handbooks of tables such as Beta Mathematics Handbook.

Notes, mobile telephones, laptops or palmtops, are not allowed! Reasonable notes in the textbook are allowed but no solved problems.

The total points achievable are 30 with the following scales for grading

Grade 3:	at least 12 points
Grade 4:	at least 18 points
Grade 5:	at least 24 points

Incorrect solutions with significant errors, unrealistic results or solutions that are difficult to follow result in 0 points.

Grading results are posted not later than June 7. Review of the grading is offered on June 7 at 13:00-14:00. If you cannot attend at this occasion, any objections concerning the grading must be filed in written form not later than two weeks after the regular review occasion.

Good Luck!

1. (a) Determine the local stability properties of the system

$$\begin{aligned}\dot{x}_1 &= (1-x_1)(1+x_2)\\ \dot{x}_2 &= (1+x_1)(1-x_2)\end{aligned}$$

2 p.

(b) Use Lyapunov theory to determine the stability properties of

$$\begin{aligned}\dot{x}_1 &= -x_1^3 - 2x_1x_2^2\\ \dot{x}_2 &= (x_1^2-1)x_2\end{aligned}$$

2 p.

(c) Determine the describing function for the nonlinearity

$$f(x) = x + x^2$$

2 p.

(d) Show that for any nonlinear system

$$\dot{x}(t) = f(x(t))u(t), \qquad f(x) > 0$$

controlled by a state feedback

$$u(t) = -kx(t), \qquad k > 0$$

the origin is globally asymptotically stable.

2 p.

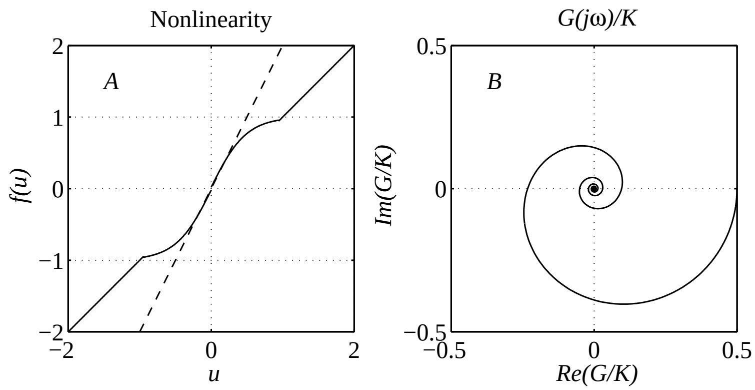
(e) Make an exact linearization of the system

$$\begin{aligned}\dot{x}_1 &= x_3 + \cos(x_2 - x_3)\\ \dot{x}_2 &= x_1^2 + x_3^2 + u\\ \dot{x}_3 &= x_1 + x_3^2 + u\\ y &= x_2 - x_3\end{aligned}$$

4 p.

2. Negative feedback through a static nonlinearity f (see Figure A) is applied to a system having a transfer function

$$G(s) = \frac{Ke^{-2s}}{(s+0.5)(s+4)}$$



(a) Use the small gain theorem to determine for what values of K the system is stable.

(b) Use the circle criterion to determine for what values of K the system is stable.

4 p.

3. An adaptive PI controller can be based on MRAS and pole placement. Assume we have a first order process

$$y(t) = \frac{b}{p+a}u(t)$$

where p is the differential operator.

(a) A non-adaptive feedback PI controller can be defined by

$$u(t) = K(1 + \frac{1}{pT_I})(r(t) - y(t))$$

Show that the PI controller can be expressed as a pole placement controller given by

$$R(p)u(t) = T(p)r(t) - S(p)y(t)$$

with a suitable choice of the polynomials R, S and T.

(b) What is the polynomial identity that has to be satisfied to get a desired characteristic polynomial A_m for the closed loop system?

(c) Show how the process model can be reparametrized, using the polynomial identity, such that the output y can be written as a linear regression, $y = \theta^T \varphi_f$ with realizable filtered input and output signals in φ_f.

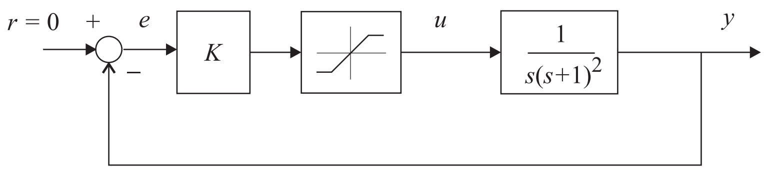
(d) Describe an adaptive scheme based on the estimation error $\epsilon = y - \hat{\theta}^T \varphi_f$ and show that the estimation error converges to zero using the Lyapunov function $V = \frac{1}{2}\tilde{\theta}^T\tilde{\theta}$.

5 p.

4. A P-controlled system has a constraint on the control signal,

$$-1 < u < 1$$

see the figure below. Will the system oscillate? If so how will the amplitude and frequency depend on the gain K?

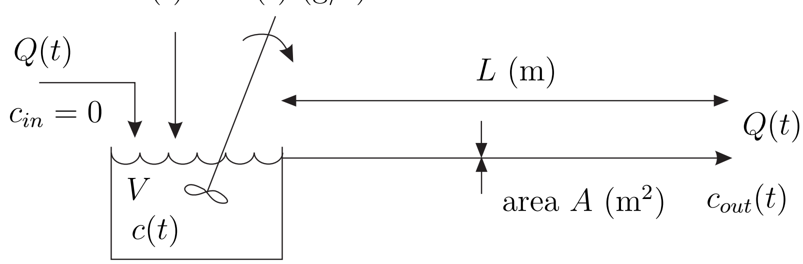


Hint: The function $\frac{2}{\pi}(\arcsin(1/z) + (1/z)\sqrt{1 - z^{-2}})$ is a strictly decreasing function on $z \in (1, \infty)$.

4 p.

5. In a preparation vessel a substance is to be diluted into a desired setpoint for the concentration c_{out} into the next process stage. The concentration is measured at the inlet of that process.

$u(t) = \dot{m}(t)$ (g/s)



(a) The flow through the pipe can be assumed to be a so-called plug flow, i.e. a pure transportation delay. Show that if we assume the flow Q is constant, the volume $V = 1$ m^3 and the pipe has a length $L = 10$ m and a cross-sectional area $A = 0.1$ m^2, the transfer function from u to c_{out} (g/m^3) is

$$G(s) = \frac{1}{Q}\frac{e^{-s/Q}}{1 + s/Q}$$

(b) A PI-controller will be used to control the system, but a complicating circumstance is that depending on the production very different flows Q will be used.

Suggest a gain scheduling strategy for the controller parameters.

5 p.

1 a) $\begin{cases} \dot{x}_1 = (1-x_1)(1+x_2) \\ \dot{x}_2 = (1+x_1)(1-x_2) \end{cases} \Leftrightarrow \dot{x} = f(x)$

$$\frac{df}{dx} = \begin{bmatrix} -1-x_2 & 1-x_1 \\ 1-x_2 & -1-x_1 \end{bmatrix}$$

Eq. pts:

$\begin{cases} \bar{x}_1 = 1 \\ \bar{x}_2 = 1 \end{cases} \Rightarrow A = \left.\frac{df}{dx}\right|_{\bar{x}} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$

$\det(\lambda I - A) = (\lambda+2)^2 = 0 \Rightarrow \lambda_{1,2} = -2 \in LHP$

$\Rightarrow$ locally as. stable

$\begin{cases} \bar{x}_1 = -1 \\ \bar{x}_2 = -1 \end{cases} \Rightarrow A = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

$\det(\lambda I - A) = \lambda^2 - 4 = 0 \Rightarrow \lambda_{1,2} = \pm 2$

Unstable

b) $\dot{x}_1 = -x_1^3 - 2x_1x_2^2$

$\dot{x}_2 = (x_1^2 - 1)x_2$

Let $V = \alpha x_1^2 + \beta x_2^2$, $\alpha > 0,\ \beta > 0$

$\Rightarrow \dot{V} = DVf = 2\alpha x_1(-x_1^3 - 2x_1x_2^2) + 2\beta x_2(x_1^2 - 1)x_2$

$= -2\alpha x_1^4 - 4\alpha x_1^2x_2^2 + 2\beta x_2^2 x_1^2 - 2\beta x_2^2$

$= -2\alpha x_1^4 - 2\beta x_2^2 < 0 \quad \forall\ x \neq 0$

if $\beta = 2\alpha$. Since $V > 0\ \forall x \neq 0$ and $V \to \infty$ as $x \to \infty$ $x = 0$ is globally as. stable

c) $f(x) = x + x^2$

$a(C) = \frac{1}{\pi}\int_0^{2\pi} (C\sin\alpha + C^2\sin^2\alpha)\cos\alpha\, d\alpha = \dots = 0$

$b(C) = \frac{1}{\pi}\int_0^{2\pi} (C\sin\alpha + C^2\sin^2\alpha)\sin\alpha\, d\alpha = \dots$

$= \frac{1}{\pi}\int_0^{2\pi} C\sin^2\alpha\, d\alpha = C$

$\Rightarrow Y_f(C) = \dfrac{b(C)}{C} = 1$

d) $\left.\begin{aligned} \dot{x} &= f(x)u \\ u &= -kx \end{aligned}\right\} \Rightarrow \dot{x} = -f^*(x)x$, $f^*(x) > 0 \quad \forall\ x \neq 0$

$V = x^2 \Rightarrow \dot{V} = 2x\dot{x} = -2f^*(x)x^2 < 0 \quad \forall x \neq 0$

$V \to \infty$ as $x \to \infty \Rightarrow x = 0$ globally as. stable

1 e) $\dot{x}_1 = x_3 + \cos(x_2 - x_3)$

$\dot{x}_2 = x_1^2 + x_3^2 + u$

$\dot{x}_3 = x_1 + x_3^2 + u$

$y = x_2 - x_3$

$\Rightarrow \dot{y} = \dot{x}_2 - \dot{x}_3 = x_1^2 - x_1$

$\ddot{y} = (2x_1 - 1)(x_3 + \cos(x_2 - x_3))$

$\dddot{y} = 2\dot{x}_1^2 + (2x_1 - 1)(x_1 + x_3^2 + u + (x_1^2 - x_1)\sin(x_2 - x_3))$

$= r$

$$\Rightarrow u = -x_1 - x_3^2 - (x_1^2 - x_1)\sin(x_2 - x_3) + \frac{1}{2x_1 - 1}(r - 2(x_3 + \cos(x_2 - x_3))^2$$

i.e with $z_1 = y$, $z_2 = \dot{y}$ and $z_3 = \ddot{y}$

$$\Rightarrow \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

2 a) From plot A we see that $|f(x)| \le 2|x|$.
Hence

$$\|f(x)\|_2^2 = \int_{-\infty}^{\infty} f(x(t))^2 \, dt \le \int_{-\infty}^{\infty} 4\, x(t)^2 \, dt$$

$$= 4 \|x\|_2^2$$

where equality holds for $x \equiv 2$ for ex.
Hence the gain of the nonlinearity is 2.

Small gain theorem gives that the feedback system is stable if

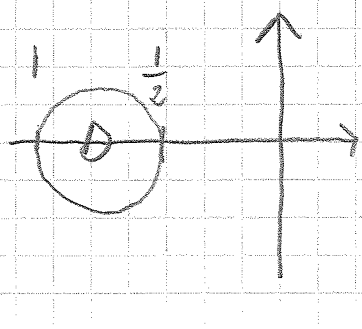
$$\|G\| \le \frac{1}{2}$$

From plot B or the t.f. directly we see

$$\frac{1}{K} \|G\| = \frac{1}{K} \sup_{\omega} |G(j\omega)| = 0.5$$

Hence, system stable if $\underline{\underline{K < 1}}$

b) We have $1 \le \frac{f(x)}{x} \le 2$ and G has no unstable poles. System is then i/o-stable if $G(j\omega)$ passes outside and to the right of the disc D. Since



$\mathrm{Re}\{\frac{G}{K}\} \ge -0.25$ the system is stable if $K < 2$

(less conservative than SGT!)

3 a) $u = K\left(1 + \frac{1}{pT_I}\right)(r-y) = \frac{K}{pT_I}(1+pT_I)(r-y)$

$$\underbrace{p}_{R(p)}u = \underbrace{\left(Kp + \frac{K}{T_I}\right)}_{\substack{T(p) = t_0 p + t_1 \\ = s_0 p + s_1}} r - \underbrace{\left(Kp + \frac{K}{T_I}\right)}_{S(p) = s_0 p + s_1} y$$

b) Process $y(t) = \frac{b}{p+a}u(t) \equiv \frac{B(p)}{A(p)}u(t)$

$\Rightarrow n = 1,\ m = 0$

$\Rightarrow$ Observer polynomial obsolete ($\deg A_0 = n - m - 1$)

Hence $y(t) = \frac{B_m}{A_m}r(t) = \frac{BT}{AR+BS}r$

$\Rightarrow$ polynomial identity: $A_m = p(p+a) + b\left(Kp + \frac{K}{T_i}\right)$
$\equiv p^2 + a_1 p + a_2$

c) $A_m y = [p(p+a) + b(s_0 p + s_1)]y$

$= pbu + b(s_0 p + s_1)y = \bar{R}u + \bar{S}y$

$\Rightarrow y(t) = \frac{pb}{A_m}u(t) + \frac{bs_0 p}{A_m}y(t) + \frac{bs_1}{A_m}y(t)$

$$= \underbrace{\begin{bmatrix} b & bs_0 & bs_1 \end{bmatrix}}_{\theta^T} \underbrace{\begin{bmatrix} \frac{p}{A_m}u & \frac{p}{A_m}y & \frac{1}{A_m}y \end{bmatrix}^T}_{\varphi_f}$$

d) $\varepsilon(t) = y - \hat{\theta}^T\varphi_f = \theta^T\varphi_f - \hat{\theta}^T\varphi_f = -\tilde{\theta}^T\varphi_f$

$\dot{\hat{\theta}} = \gamma\varphi_f\varepsilon = -\gamma\varphi_f\tilde{\theta}^T\varphi_f \Rightarrow \hat{\bar{R}}\ \&\ \hat{\bar{S}}$

Control law $\hat{\bar{R}}u = -\hat{\bar{S}}y + \hat{\bar{S}}r$

$V = \frac{1}{2}\tilde{\theta}^T\tilde{\theta} \Rightarrow \dot{V} = \tilde{\theta}^T(-\gamma\varphi_f\tilde{\theta}^T\varphi_f) = -\gamma(\tilde{\theta}^T\varphi_f)^2 = -\varepsilon^2 \Rightarrow \varepsilon \xrightarrow[t\to\infty]{} 0$

4) The describing function for the saturation is

$$Y_f(c) = \begin{cases} \frac{2}{\pi}\left(\arcsin\frac{1}{c} + \frac{1}{c}\sqrt{1-c^{-2}}\right) & c>1 \\ 1 & c \le 1 \end{cases}$$

$-\frac{1}{Y_f}$ goes from -1 to $-\infty$ as c increases

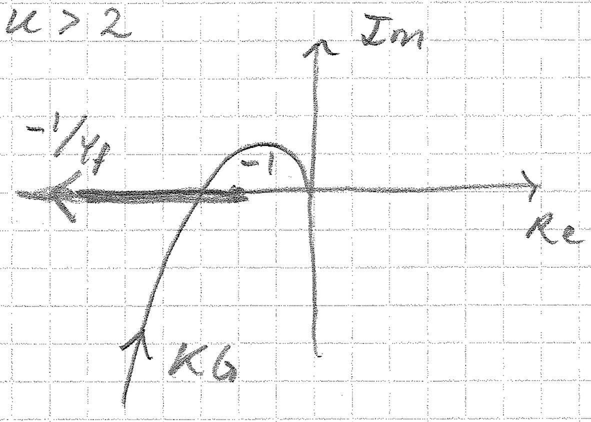
To ensure stability $KG(j\omega)$ should therefore pass to the right of -1

$$\angle G(j\omega_\pi) = -90 - 2\operatorname{atan}\omega = -180 \Rightarrow \omega_\pi = 1$$

$$|G(j\omega_\pi)| = \frac{1}{1\cdot\sqrt{2}\cdot\sqrt{2}} = \frac{1}{2}$$

Hence, stable if $K < 2$

If $K > 2$



This indicates stable oscillations since increasing amplitude will move the intersection towards larger amplitudes leaving KG to the right ($\Rightarrow$ stable)

Frequency will be ω_π $\forall K$

Amplitude is given by $\frac{1}{Y_f(c)} = K|G(j\omega_\pi)| = \frac{K}{2}$

5 a) MB over tank

$$\frac{d}{dt}(Vc) = u - Qc \iff G_0(s) = \frac{1/Q}{1 + s\frac{V}{Q}}$$

Volume in pipe : $L \cdot A = 1 \ m^3$

$\Rightarrow$ Transport delay : $\frac{LA}{Q} = \frac{1}{Q}$ s

$$\Rightarrow C_{out}(s) = \underbrace{\frac{1/Q \ e^{-s/Q}}{1 + s/Q}}_{G(s)} U(s) \quad \text{for } V = 1$$

b) Design a PI-controller

$$F(\tilde{s}) = K\left(1 + \frac{1}{\tilde{s}T}\right), \quad \text{where } K = \tilde{K}Q$$

for

$$G(\tilde{s}) = \frac{1/Q \ e^{-\tilde{s}}}{1 + \tilde{s}}, \quad \text{where } \tilde{s} = \frac{s}{Q}$$

Then $\tilde{K}$ and T will be the same for all Q.

$$F(\tilde{s}) = \tilde{K}Q\left(1 + \frac{1}{s\frac{T}{Q}}\right) = K_p\left(1 + \frac{1}{sT_i}\right)$$

Hence, the schedule is to set the 'normal' PI parameters to

$$K_p = \tilde{K}Q \quad \text{and} \quad T_i = \frac{T}{Q}$$