Svar till tentamen i EMI190 Elektriska kretsar och signaler del B 990410

1)

a) Det elektriska ljuset kommer att sampla den roterande skivan med nätfrekvensen. Varvtalet regleras så att korrekt varvtal viks ner till nollfrekvens. Solljuset är kontinuerligt varför något vikningsfenomen inte förekommer i detta fall.

b) Addera två sinussignaler med frekvenserna 60 Hz resp. 90 Hz. Detta ger en periodisk signal med periodtiden 1/30 s, dock med amplituden noll på grundfrekvensen.

c) Laplace-transformera $h(t)$: $H(s) = \frac{1}{s} + \frac{1}{s^2+1} = \frac{s^2+s+1}{s(s^2+1)}$.
$Y(s) = H(s)X(s) \Rightarrow (s^3+s)Y(s) = (s^2+s+1)X(s) \Rightarrow y^{(3)}(t) + y'(t) = x''(t) + x'(t) + x(t)$

d) $X(z) = \frac{1+z^{-1}}{1-0.8z^{-1}+0.15z^{-2}} = \{\text{partialbråksuppdelning}\} = \frac{7.5}{1-0.5z^{-1}} - \frac{6.5}{1-0.3z^{-1}} \Rightarrow h[n] = (7.5 \cdot 0.5^n - 6.5 \cdot 0.3^n)u[n]$

2) $x(t) = e^{-50t}u(t)$, which gives us that $X(e^{j\omega}) = \frac{1}{j\omega+50}$ and $|X(e^{j\omega})|^2 = X(e^{j\omega})X^*(e^{j\omega}) = \frac{1}{\omega^2+2500}$. Thus we will, by Parseval's relation get the energy of the whole signal as $\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\omega^2+2500} d\omega = \frac{1}{2\pi}[\frac{1}{\sqrt{2500}}\arctan(\frac{\omega}{\sqrt{2500}})]_{-\infty}^{\infty} = \frac{1}{100\pi}(\pi/2 - (-\pi/2)) = \frac{1}{100}$. We pass this signal through a low-pass filter with cut-off frequency $f_c = 30Hz$ which corresponds to $\omega_c = 60\pi$ in rad/s. Thus the energy of the signal at the output of the filter will be $\frac{1}{2\pi}\int_{-60\pi}^{60\pi} \frac{1}{\omega^2+2500} d\omega = \frac{1}{2\pi}[\frac{1}{\sqrt{2500}}\arctan(\frac{\omega}{\sqrt{2500}})]_{-60\pi}^{60\pi} = \frac{2}{100\pi}\arctan(6\pi/5) = \frac{1}{100}0.8349$. Thus 83.5% of the energy passes through the filter.

3)

a) $\Omega_b = \omega_b T = 10 \cdot 0.1 = 1$. Pre-warping: $\omega_b' = \frac{2}{T}\tan\frac{\Omega_b}{2} \approx 10.93$. Prototyp-filter: $H(s) = \frac{\omega_b'}{s+\omega_b'}$. Bilinjär transform: $H(z) = H(s)|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{\omega_b' T + \omega_b' T z^{-1}}{2+\omega_b' T + (\omega_b' T - 2)z^{-1}} \approx \frac{1.09+1.09z^{-1}}{3.09-0.91z^{-1}}$

b) $Y(z) = H(z)X(z) \Rightarrow 3.09Y(z) - 0.91z^{-1}Y(z) = 1.09X(z) + 1.09z^{-1}X(z) \Rightarrow 3.09y[n] - 0.91y[n-1] = 1.09x[n] + 1.09x[n-1] \Rightarrow y[n] - 0.29y[n-1] = 0.35x[n] + 0.35x[n-1]$

4)

a) We have a periodic input signal $x(t)$. To be able to write it in the frequency domain we calculate the Fourier series of the signal. We have a period of 2, thus $\omega_0 = \pi$. We get the Fourier series coefficients a_k as $a_k = \frac{1}{2}\int_0^2 x(t)e^{-jk\pi t}dt = \frac{1}{2}\int_{3/2}^2 e^{-jk\pi t}dt = \frac{1}{-2jk\pi}\left[e^{-jk\pi t}\right]_{3/2}^2 = \frac{e^{-j3\pi k/2-1}}{2jk\pi} = \frac{j^k-1}{2jk\pi}$. Thus the signal can be written as $x(t) = \sum_{n=-\infty}^{\infty} a_k e^{jk\pi t}$ and the corresponding frequency representation yields $X(j\omega) = \sum_{n=-\infty}^{\infty} a_k \delta(w - n\pi)$. We see that the Dirac pulses lie with a distance of π from each other. Since the filter cut off all frequencies above 8 (and below -8) only five Dirac pulses will remain after the filter, corresponding to $a_{-2}, a_{-1}, a_0, a_1, a_2$. a_0

is the average value of the signal over one period and equals $\frac{1}{4}$. Thus the output of the filter will be

$$\begin{aligned} y(t) &= a_0 + a_1|H(j\pi)|e^{j\arg(H(j\pi))}e^{j\pi t} + a_{-1}|H(j(-\pi))|e^{j\arg(H(j(-\pi)))}e^{-j\pi t} \\ &+ a_2|H(j2\pi)|e^{j\arg(H(j2\pi))}e^{j2\pi t} + a_{-2}|H(j(-2\pi))|e^{j\arg(H(j(-2\pi)))}e^{-j2\pi t} \end{aligned}$$

From the calculations above we get that

$$a_1 = \frac{1+j}{2\pi} = \frac{e^{j\pi/4}}{\sqrt{2}\pi},\ a_{-1} = \frac{1-j}{2\pi} = \frac{e^{-j\pi/4}}{\sqrt{2}\pi},\ a_2 = \frac{j}{2\pi} = \frac{e^{j\pi/2}}{2\pi},\ a_{-2} = \frac{-j}{2\pi} = \frac{e^{-j\pi/2}}{2\pi}$$

and from the figure of the frequency response of the filter we get that

$$\begin{aligned} |H(j\pi)| &= |H(j(-\pi))| = \frac{8-\pi}{8},\ \arg(H(j\pi)) = \frac{-\pi}{16},\ \arg(H(j(-\pi))) = \frac{\pi}{16} \\ |H(j2\pi)| &= |H(j(-2\pi))| = \frac{4-\pi}{4},\ \arg(H(j2\pi)) = \frac{-\pi}{8},\ \arg(H(j(-2\pi))) = \frac{\pi}{8} \end{aligned}$$

We get $y(t)$ as

$$\begin{aligned} y(t) &= \frac{1}{4} + \frac{e^{j\pi/4}}{\sqrt{2}\pi}\frac{(8-\pi)}{8}e^{-j\pi/16}e^{j\pi t} + \frac{e^{-j\pi/4}}{\sqrt{2}\pi}\frac{(8-\pi)}{8}e^{j\pi/16}e^{+j\pi t} \\ &+ \frac{e^{j\pi/2}}{2\pi}\frac{(4-\pi)}{4}e^{-j\pi/8}e^{j2\pi t} + \frac{e^{-j\pi/2}}{2\pi}\frac{(4-\pi)}{4}e^{j\pi/8}e^{+j2\pi t} \\ &= \frac{1}{4} + \frac{(8-\pi)}{4\sqrt{2}\pi}\frac{e^{j(\pi t-\pi/16+\pi/4)} + e^{-j(\pi t-\pi/16+\pi/4)}}{2} \\ &+ \frac{(4-\pi)}{4\pi}\frac{e^{j(2\pi t-\pi/8+\pi/2)} + e^{-j(2\pi t-\pi/8+\pi/2)}}{2} \\ &= \frac{1}{4} + \frac{(8-\pi)}{4\sqrt{2}\pi}\cos(\pi t - \pi/16 + \pi/4) + \frac{(4-\pi)}{4\pi}\cos(2\pi t - \pi/8 + \pi/2) \\ &= \frac{1}{4} + \frac{(8-\pi)}{4\sqrt{2}\pi}\cos(\pi(t + 3/16)) + \frac{(4-\pi)}{4\pi}\cos(2\pi(t+3/16)) \end{aligned}$$

b) Let b_k be the Fourier series coefficients of the output $y(t)$, from a) we know that

$$b_0 = \frac{1}{4}, b_1 = \frac{(8-\pi)}{8\sqrt{2}\pi}e^{j3\pi/16}, b_{-1} = \frac{(8-\pi)}{8\sqrt{2}\pi}e^{-j3\pi/16}, b_2 = \frac{(4-\pi)}{8\pi}e^{j3\pi/8}, b_{-2} = \frac{(4-\pi)}{8\pi}e^{-j3\pi/8}$$

We see that we have sampling frequency (in rad/s) 8, thus all frequencies above 4 will be aliased. This includes the Dirac pulses corresponding to b_2 and b_{-2}. If we only look at the interval $[-\pi, \pi]$ we see that the frequency lying at π in continuous time will become $(\Omega = \omega T, T = \pi/4)$ $\frac{\pi}{4}\pi$. In the same way $-\pi$ corresponds to $-\pi^2/4$. The frequency at 2π will be folded to $-\frac{4\pi-\pi^2}{2}$ and the frequency at -2π ends up at $\frac{4\pi-\pi^2}{2}$. Thus in the interval $[-\pi, \pi]$ $Y(e^{-\Omega})$ becomes

$$b_0\delta(\Omega) + b_1\delta(\Omega - \frac{\pi^2}{4}) + b_{-1}\delta(\Omega + \frac{\pi^2}{4}) + b_2\delta(\Omega + \frac{4\pi-\pi^2}{2}) + b_{-2}\delta(\Omega - \frac{4\pi-\pi^2}{2})$$

and for the whole spectrum, it is periodic with period 2π thus

$$\begin{aligned} Y(e^{-\Omega}) &= \sum_{k=-\infty}^{\infty} b_0\delta(\Omega + 2\pi k) + b_1\delta(\Omega - \frac{\pi^2}{4} + 2\pi k) + b_{-1}\delta(\Omega + \frac{\pi^2}{4} + 2\pi k) \\ &+ b_2\delta(\Omega + \frac{4\pi - \pi^2}{2} + 2\pi k) + b_{-2}\delta(\Omega - \frac{4\pi - \pi^2}{2} + 2\pi k) \end{aligned}$$

Observe that in the aliasing formula we have

$$Y(e^{j\Omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} Y_c(j(\frac{\Omega - 2\pi k}{T}))$$

where $Y_c(j\omega)$ represents the continuous-time transform. In general we would need to scale the discrete-time transform by $\frac{1}{T}$ but this cancels out sine when we scale Dirac-pulses we have to multiply by the inversion of the scaling, in this case $\frac{1}{1/T} = T$ thus this cancels the original multiplication with $\frac{1}{T}$

5) 1-b, 2-c, 3-d, 4-a, Motivation: The Hanning window has a sharper peak in the frequency domain than the rectangular window, ie, the side lobes decay faster. The number of samples in the time- and frequency domain of a window is the same.