Inst.: Data- och informationsteknik
Kursnamn: Logic in Computer Science
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Kurs: DIT201/DAT060
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All answers and solutions must be carefully motivated!
total 60; ≥28: 3, ≥38: 4, ≥50: 5
total 60; ≥28: G, ≥42: VG

All answers **must** be carefully motivated.

1. Give proofs in natural deduction of the following sequents:

   (a) $p \to q, q \to r, p \to s \vdash p \to (r \wedge s)$ (3p)
   Solution:

1.	$p \to q$	premise
2.	$q \to r$	premise
3.	$p \to s$	premise
4.	p	assumption
5.	q	$\to$e$(1,4)$
6.	r	$\to$e$(2,5)$
7.	s	$\to$e$(3,4)$
8.	$r \wedge s$	$\wedge$i$(6,7)$
9.	$p \to r \wedge s$	$\to$i$(4-8)$

   (b) $\neg(p \vee q) \vdash \neg p \wedge \neg q$ (3p)
   Solution:

1.	$\neg(p \vee q)$	premise
2.	p	assumption
3.	$p \vee q$	$\vee$i$_1(2)$
4.	$\bot$	$\to$e$(1,3)$
5.	$\neg p$	$\to$i$(2-4)$
6.	q	assumption
7.	$p \vee q$	$\vee$i$_2(6)$
8.	$\bot$	$\to$e$(1,7)$
9.	$\neg q$	$\to$i$(6-8)$
10.	$\neg p \wedge \neg q$	$\wedge$i$(5,9)$

   (c) $p \vee q \vdash \neg q \to p$ (3p)
   Solution:

1.	$p \vee q$	premise
2.	$\neg q$	assumption
3.	p	assumption
4.	q	assumption
5.	$\bot$	$\to$e$(2, 4)$
6.	p	$\bot$e(5)
7.	p	$\vee$e$(1, 3-3, 4-6)$
8.	$\neg q \to p$	$\to$i$(2-7)$

2. Explain why the following LTL formula

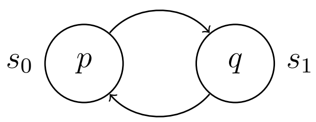
$$((\mathrm{F}\, p) \wedge \mathrm{F}\, q) \to \mathrm{F}(p \wedge q)$$

is *not* valid (2p) and why the following formula is valid (3p)

$$((\mathrm{F}\, p) \wedge \mathrm{F}\, q) \to (\mathrm{F}(p \wedge \mathrm{F}\, q)) \vee \mathrm{F}(q \wedge \mathrm{F}\, p)$$

Solution:

(a) Consider the following transition system $\mathcal{M}$:



The path $\pi = s_0 \to s_1 \to s_0 \to s_1 \to \ldots$ alternating between s_0 and s_1 satisfies $\mathcal{M}, \pi \models \mathrm{F}\, p$ (since $\mathcal{M}, \pi^0 \models p$) and $\mathcal{M}, \pi \models \mathrm{F}\, q$ (since $\mathcal{M}, \pi^1 \models q$) but no state contains both p and q, hence $\mathcal{M}, \pi \not\models \mathrm{F}(p \wedge q)$.

(b) Let $\mathcal{M}$ be a transition system and π a path in $\mathcal{M}$ such that $\mathcal{M}, \pi \models \mathrm{F}\, p \wedge \mathrm{F}\, q$. So there are $i \geq 0$ and $j \geq 0$ with

$$\mathcal{M}, \pi^i \models p \quad \text{and} \quad \mathcal{M}, \pi^j \models q.$$

In case $i \leq j$, we have $\mathcal{M}, \pi^i \models \mathrm{F}\, q$ and hence $\mathcal{M}, \pi \models \mathrm{F}(p \wedge \mathrm{F}\, q)$. Similarly we get $\mathcal{M}, \pi \models \mathrm{F}(q \wedge \mathrm{F}\, p)$ in case $j \leq i$. In either case we obtain

$$\mathcal{M}, \pi \models \mathrm{F}(p \wedge \mathrm{F}\, q) \vee \mathrm{F}(q \wedge \mathrm{F}\, p)$$

what we had to show.

3. We consider the following language: we have one binary predicate symbol R, a unary function symbol f, and a constant c.

(a) Define what a model of this language is. (3p)

(b) Explain why the formula $R(c,c) \to \forall x\, R(x, f(x)))$ is *not* derivable. (2p)

Solution:

A model M of this language consists in a set A, an interpretation R^M which is a subset of $A \times A$, an interpretation f^M which is a function $A \to A$ and an interpretation c^M which is an element of A

For showing that the formula $R(c,c) \to \forall x\, R(x, f(x)))$, it is enough, by *soundness*, to give a model for which this formula is not valid. We can for instance take $A = \{0,1\}$ and $c^M = 0$ and $R^M = \{(0,0)\}$ and $f^M(0) = f^M(1) = 1$. We then have $M \models R(c,c)$ but $M \models \neg\forall x\, R(x, f(x))$.

4. Let P, S and M be unary predicates and R a binary predicate. Decide for each of the sequents below whether they are valid or not, i.e., give a proof in natural deduction or a counter-model. (12p)

(a) $\exists x\,(P(x) \wedge \neg M(x)), \exists y\,(M(y) \wedge \neg S(y)) \vdash \exists z(P(z) \wedge \neg S(z))$

Solution: Not valid. Take the model $\mathcal{M}$ with domain $D = \{0,1\}$ and $P^{\mathcal{M}} = \{0\}, M^{\mathcal{M}} = \{1\}, S^{\mathcal{M}} = \{0\}$.

(b) $\forall x \neg R(x,x) \vdash \forall x\, \forall y\, (R(x,y) \to \neg R(y,x))$

Solution: Not valid. Take the model $\mathcal{M}$ with domain $D = \{0,1\}$ and $R^{\mathcal{M}} = \{(0,1),(1,0)\}$.

(c) $\forall x\, \forall y\, (R(x,y) \to \neg R(y,x)) \vdash \forall z\, \neg R(z,z)$

Solution: Valid.

1.	$\forall x\, \forall y\, (R(x,y) \to \neg R(y,x))$	premise
2.	z_0	assumption
3.	$R(z_0, z_0)$	assumption
4.	$\forall y\, (R(z_0, y) \to \neg R(y, z_0))$	$\forall$e$(1, z_0)$
5.	$R(z_0, z_0) \to \neg R(z_0, z_0)$	$\forall$e$(4, z_0)$
6.	$\neg R(z_0, z_0)$	$\to$e$(5,3)$
7.	$\bot$	$\to$e$(6,3)$
8.	$\neg R(z_0, z_0)$	$\to$i$(3-7)$
9.	$\forall z\, \neg R(z,z)$	$\forall$i$(2-8)$

(d) $\vdash \forall x \exists y\, R(x,y) \vee \forall x \exists y\, \neg R(x,y)$

Solution: Not valid. Take the model $\mathcal{M}$ with domain $D = \{0,1\}$ and $R^{\mathcal{M}} = \{(1,1),(1,0)\}$.

5. Let P, Q, and R be unary predicate symbols, and f a unary function symbol. Give proofs in natural deduction of the following sequents:

(a) $\forall x\,(P(x) \to (Q(x) \vee R(x))), \neg\exists x\,(P(x) \wedge R(x)) \vdash \forall x\,(P(x) \to Q(x))$ (4p)

Solution:

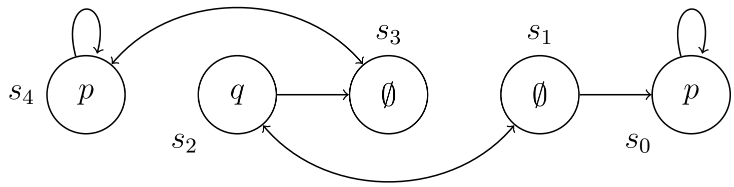
1.	$\forall x\,(P(x) \to (Q(x) \vee R(x)))$	premise
2.	$\neg\exists x\,(P(x) \wedge R(x))$	premise
3.	x_0	assumption
4.	$P(x_0)$	assumption
5.	$P(x_0) \to (Q(x_0) \vee R(x_o))$	$\forall$e$(1, x_0)$
6.	$Q(x_0) \vee R(x_o)$	$\to$e$(5, 4)$
7.	$Q(x_0)$	assumption
8.	$R(x_0)$	assumption
9.	$P(x_0) \wedge R(x_0)$	$\wedge$i$(4, 8)$
10.	$\exists x\,(P(x) \wedge R(x))$	$\exists$i$(9, x_0)$
11.	$\bot$	$\to$e$(2, 10)$
12.	$Q(x_0)$	$\bot$e(11)
13.	$Q(x_0)$	$\vee$e$(6, 7, 8-12)$
14.	$P(x_0) \to Q(x_0)$	$\to$i$(4-13)$
15.	$\forall x\,(P(x) \to Q(x))$	$\forall$i$(3-14)$

(b) $\forall x\,(f(f(x)) = x) \vdash \forall x \exists y\,(x = f(y))$ (4p)

Solution:

1.	$\forall x\,(f(f(x)) = x)$	premise
2.	x_0	assumption
3.	$f(f(x_0)) = x_0$	$\forall$e$(1, x_0)$
4.	$f(f(x_0)) = f(f(x_0))$	$=$i$(f(f(x_0)))$
5.	$x_0 = f(f(x_0))$	$=$e$(3, 4, x = f(f(x_0)))$
6.	$\exists y\,(x_0 = f(y))$	$\exists$i$(5, f(x_0))$
7.	$\forall x \exists y\,(x = f(y))$	$\forall$i$(2-6)$

6. Consider the transition system $\mathcal{M} = (S, \to, L)$ where the states are $S = \{s_0, s_1, s_2, s_3, s_4\}$, the transitions are $s_0 \to s_0, s_1 \to s_0, s_1 \to s_2, s_2 \to s_1, s_2 \to s_3, s_3 \to s_4, s_4 \to s_3, s_4 \to s_4$, and the labeling function is given by $L(s_0) = L(s_4) = \{p\}, L(s_2) = \{q\}$, and $L(s_1) = L(s_3) = \emptyset$.



(a) Do we have $\mathcal{M} \models \mathrm{G}(q \to \mathrm{F}\, p)$? (3p)

Solution: No. Take $\pi := (s_2, s_1, s_2, s_1, \ldots)$ at 0.

(b) Which are the states s that satisfy the CTL formula $\mathrm{AG}(\mathrm{EF}\, p)$ (i.e., where $\mathcal{M}, s \models \mathrm{AG}(\mathrm{EF}\, p)$)? (3p)

Solution: All states because all states satisfy $\mathrm{EF}\, p$.

7. A set of connectives is called *adequate* if for every formula of propositional logic there is an equivalent formula using only connectives from this set. Explain why $\{\wedge, \neg\}$ is adequate. (3p)

Solution:

We can define $p \vee q$ as $\neg(\neg p \wedge \neg q)$ and $p \to q$ as $\neg(p \wedge \neg q)$.

8. We fix a language with a relation symbol R. Give a model which validates all the following formulae (4p)

$$\forall x\ \neg R(x,x) \qquad \forall x\ \forall y\ \forall z\ ((R(x,y) \wedge R(y,z)) \to R(x,z))$$

$$\forall x\ \exists y\ R(x,y) \qquad \forall x\ \exists y\ R(y,x) \qquad \forall x\ \forall y\ (R(x,y) \to \exists z\ (R(x,z) \wedge R(z,y)))$$

Solution:

A model M is given by taking the domain to be the set of *rationals* $\mathbb{Q}$, or the set of *reals* $\mathbb{R}$, and R^M to be the set of (x,y) such that $x < y$

9. Suppose that Γ is a set of sentences (i.e., formulas without free variables) in a given language such that for any natural number $n \geq 0$, Γ has a model whose domain (carrier set) has at least n elements. Show that Γ has a model whose domain is infinite. (*Hint:* Use the Compactness Theorem.) (3p)

Solution: We extend the language by adding constants c_n for each $n \in \mathbb{N}$, and let

$$\Delta := \Gamma \cup \{c_n \neq c_m \mid n, m \in \mathbb{N} \text{ and } n \neq m\}.$$

We now show that any finite subset $\Delta_0 \subseteq \Delta$ has a model. Since Δ_0 is finite we can find an $k \in \mathbb{N}$ such that

$$\Delta_0 \subseteq \Gamma \cup \{c_n \neq c_m \mid n, m \in \mathbb{N} \text{ such that } n < k, m < k, \text{ and } n \neq m\}. \quad (1)$$

By assumption Γ has a model $\mathcal{M}$ with at least k elements; we can make this a model of the extended language by interpreting $c_0, c_1, \ldots, c_{k-1}$ by the k different elements in the carrier of $\mathcal{M}$; all the other constants c_n, $n \geq k$, are interpreted by, say, a fixed element of the carrier. By construction $c_n^{\mathcal{M}} \neq c_m^{\mathcal{M}}$ for $n, m < k$ with $n \neq m$, and hence $\mathcal{M}$ models each formula on the right-hand side in (**??**), and thus also each formula in Δ_0.

So we showed that any finite subset of Δ is satisfiable, and hence by the Compactness Theorem also Δ is satisfiable, say by a model $\mathcal{N}$. Since $\Gamma \subseteq \Delta$, $\mathcal{N}$ is also a model of Γ, and moreover $\{c_n^{\mathcal{N}} \mid n \in \mathbb{N}\}$ is an infinite subset of the carrier of $\mathcal{N}$, because for $n \neq m$ we have $\mathcal{N} \models c_n \neq c_m$, i.e., $c_n^{\mathcal{N}} \neq c_m^{\mathcal{N}}$.

10. We write $\varphi \leftrightarrow \psi$ to mean $(\varphi \to \psi) \wedge (\psi \to \varphi)$. In a language with only one predicate symbol P explain why the following formula is valid in all models (5p)

$$(\forall x \forall y\, (P(x) \leftrightarrow P(y))) \leftrightarrow ((\forall x\, P(x)) \vee (\forall x\, \neg P(x)))$$

Solution: For a model to satisfy $\varphi \leftrightarrow \psi$ it has to satisfy both φ and ψ or none of them.

If a model $\mathcal{M}$ with carrier $\mathcal{A}$ satisfies $((\forall x\, P(x)) \vee (\forall x\, \neg P(x)))$ then the interpretation of P, $P^{\mathcal{M}}$, is either $\mathcal{A}$ itself or the empty set $\emptyset$. In both cases $\mathcal{M}$ satisfies $(\forall x \forall y\, (P(x) \leftrightarrow P(y)))$: If $P^{\mathcal{M}} = A$ then the conclusion of the implication must hold, while if $P^{\mathcal{M}} = \emptyset$ then the antecedent of the implication cannot hold.

If the model $\mathcal{M}$ does not satisfy $((\forall x\, P(x)) \vee (\forall x\, \neg P(x)))$ then $\mathcal{A} = P^{\mathcal{M}} \cup Q$ where both $P^{\mathcal{M}}$ and Q are non-empty and disjoint. And then $\mathcal{M}$ will not satisfy $(\forall x \forall y\, (P(x) \leftrightarrow P(y)))$ either: because otherwise it would follow that if $a \in P^{\mathcal{M}}$ then $b \in P^{\mathcal{M}}$ for any $a, b \in A$, but then we can pick them such that $a \in P^{\mathcal{M}}$ and $b \in Q$ and reach a contradiction.

Good Luck!

Jan and Thierry